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Mappings on pairwise para-lindelőf bitopological spaces

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Abstract The aim of this paper is to study and present the effect of some types of mapping on pairwise paralindelőf spaces, pairwise nearly paralindelőf spaces and pairwise almost paralindelőf spaces. The main results are that the paralindelőf property is not preserved under closed mappings. But it is preserved under perfect mappings in bitopological settings.

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1. Introduction

A bitopological space (X, τ_1, τ_2) is known as a set X together with two arbitrary topologies τ_1 and τ_2 which are defined on X (see [1]). The notions of mappings and continuity stand among the most essential concepts in topology. Some topological properties are preserved under some types of mappings. For example, covering properties as compact, lindelőf and paracompact spaces are preserved under closed mappings and perfect mappings. But paralindelőf spaces are preserved under L-perfect (quasi perfect) mappings but not under closed mappings [6]. A L-perfect (quasi perfect) mappings are a continuous, closed and surjection with Lindelőf point inverses [7,5]. In this work, we extend the result for effecting the closed and L-perfect mappings on paralindelőf spaces in topological space to bitopological space. Also, we study two classes of paralindelőf spaces such that nearly paralindelőf and almost paralindelőf spaces under some kind of mappings as almost continuous mappings in bitopological settings, see [10,11].

In Section 3, we concentrate on some separation axioms. We shall introduce another definition of collectionwise Hausdorff bitopological spaces, pairwise collectionwise Hausdorff. Also, we shall extend the notion of collectionwise normal spaces to bitopological settings.

In Section 4, we will study the effect some mappings on two kinds of paralindelőf bitopological spaces, pairwise paralindelőf spaces. First, we will show the relation between pairwise paralindelőf and pairwise CwH. We prove by example that pairwise paralindelőf spaces are not preserved by closed mappings. In addition, we show the effect of some kinds of L-perfect mappings on paralindelőf property.

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In Section 5, we shall extend some results in [5] to a bitopological settings. We study the notions of pairwise nearly paralindelöf and pairwise almost paralindelöf spaces under some types of mappings.

2. Preliminaries

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) are always meant topological spaces and bitopological spaces, respectively. By $i - int(A)$ and $i - cl(A)$ we shall mean the interior and the closure of a subset A of X with respect to τ_i, respectively, where $i = 1$ or 2.

A subset S of X is said to be (i,j)-regular open (rep. (i,j)-regular closed) if $i - int - (j - cl(S)) = S$ (rep. $i - cl - (j - int(S)) = S$). S is said to be pairwise regular open (resp. pairwise regular closed) if it is both (i,j)-regular open and (j,i)-regular open (resp. (i,j)-regular closed and (j,i)-regular closed). A subset S of X is said to be (i,j)-nearly Lindelöf relative to X if for every family of (i,j)-regular open subsets of X covering S, there exists a countable subfamily covering S.

Definition. A bitopological space X is said to be

(1) (i,j)-almost regular [2] if for each $x \in X$ and each (i,j)-regular open set U containing x, there exists an (i,j)-regular open set V such that $x \in V \subset j - cl(V) \subset U$. X is called pairwise almost regular (p-almost regular) if it is $(1,2)$-almost regular and $(2,1)$-almost regular.
(2) (i,j)-P-space (resp. i-P-space) if countable intersection of i-open sets in X is j-open (resp. i-open). X is called pairwise P-space (resp. P-space) if it is $(1,2)$-P-space and $(2,1)$-P-space (resp. 1-P-space and 2-P-space).

The following definitions is given the concepts of pairwise continuous, pairwise open and pairwise closed functions in the sense of Tallafha et al. [14].

Definition. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) continuous if the functions $f: (X, \tau_1) \to (Y, \sigma_1)$ are both continuous. Equivalently, a function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called i-continuous if the function $f: (X, \tau_I) \to (Y, \sigma_I)$ is continuous. f is said continuous if it is i-continuous for each $i = 1, 2$.
(2) open (resp. closed) if the functions $f: (X, \tau_1) \to (Y, \sigma_1)$ and $f: (X, \tau_2) \to (Y, \sigma_2)$ are both open (resp. closed). Equivalently, a function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called i-open (resp. i-closed) if the function $f: (X, \tau_i) \to (Y, \sigma_i)$ is open (resp. closed). f is said open (resp. closed) if f is i-open (resp. i-closed) for each $i = 1, 2$.

The next lemma is quite similar with the classical results in general topology, so we omit the proof.

Lemma. *If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are bitopological spaces and $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$. Then, f is i-continuous if and only if $f(i - cl(A)) \subseteq i - cl(f(A))$ for every $A \subseteq X$ and $i = 1, 2$.*

in this work, sometimes we shall denote pairwise by $p-$ as p-paralindelöf stand for pairwise paralindelöf. Also, τ_{dis}, τ_{cof} and τ_{coc} are denoted to discrete topology, cofinite topology and cocountable topology respectively.

3. Separation axioms in bitopological spaces

Kelly [1] was the first one who introduced the notion of p-regular spaces and p-normal spaces. Now, we will generalize these concepts to regular bitopological spaces and normal bitopological spaces respectively.

Definition. A bitopological space (X, τ_1, τ_2) is said to be regular (normal) if the topological space (X, τ_1) and (X, τ_2) are both regular (normal). Equivalently, (X, τ_1, τ_2) is regular (normal) space if for each point $x \in X$ and each τ_i-closed set F such that $x \notin F$ (two τ_i-closed subsets F_1 and F_2 such that $F_1 \cap F_2 = \emptyset$), there are two τ_i-open subsets U and V such that $x \in U, F \subset V$ and $U \cap V = \emptyset$ ($F_1 \subset U, F_2 \subset V$ and $U \cap V = \emptyset$) for all $i = 1, 2$.

The notions of collectionwise Hausdorff (CwH) and collectionwise normal spaces have played an increasingly important role in topology. Here, we extend the other concepts of collectionwise Hausdorff and collectionwise normal spaces to bitopological spaces.

Definition. A bitopological space (X, τ_1, τ_2) is said to be (i,j)-CwH space ((i,j)-CwN) if every i-closed discrete collection $D = \{d_\alpha : \alpha \in \Delta\}$ of points (discrete collection $\{F_s : s \in S\}$ of i-closed subsets of X), then there exists a pairwise disjoint collection $\{U_\alpha : \alpha \in \Delta\}$ of j-open sets such that $d_\alpha \in U_\alpha$ for all $\alpha \in \Delta(F_s \subseteq U_s$ for each $s \in S)$.

X is called pairwise CwH (pairwise CwN) if it is both $(1,2)$-CwH and $(2,1)$-CwH ($(1,2)$-CwN and $(2,1)$-CwN).

It is clear that in $i - T_1$ space, every (i,j)-CwN is (i,j)-CwH. In the following example, we can see the relation between these concepts in a clearer point of view.

Example. The space $(\mathcal{R}, \tau_{cof}, \tau_{dis})$ is (τ_{cof}, τ_{dis})-CwN. Also, Since $\mathcal{R}$ is τ_{cof}-T_1, $\mathcal{R}$ is (τ_{cof}, τ_{dis})-CwH.

4. Mappings on pairwise paralindelöf spaces

In this section, we are going to study the behavior for some types of pairwise paralindelöf spaces under several types of combinations of pairwise continuous and pairwise closed functions. We shows that some mappings preserve certain type of pairwise paralindelöf spaces where as others not.

Early in 1969, Fletcher, Hoyle and Patty gave definition of pairwise paracompactness [8]. According to them a bitopological space (X, τ_1, τ_2) is pairwise paracompact if every τ_i-open cover of X has τ_j-open τ_j-locally finite refinement for $i \neq j$ and $i, j = 1, 2$. In sense of Fletcher's definition, we shall generalize it to pairwise paralindelof as following. First, we shall introduce the definition of paralindelöf property in bitopological spaces as a generalization of paracompact which are pairwise paralindelöf spaces.

Definition. A bitopological space (X, τ_1, τ_2) is (i,j)-paralindelöf if for each i-open cover of X, there is j-locally countable

j-open refinement. X is called pairwise paralindelöf if it is $(1,2)$-paralindelöf and $(2,1)$-paralindelöf.

Now, We will state the next Lemma before we show the relation of p-collectionwise Hausdorff spaces with p-paralindelöf spaces.

Lemma. *Let H and K be subsets of a bitopological space (X,τ_1,τ_2). For any $h \in H$ and $k \in K$, let $h*k$ and $k*h$ elements in $H \cup K$. Let $S(x)$ be i-open neighborhood of x for each $x \in H \cup K$.*

Suppose that for each $x \in H \cup K$, there is no element $x' \in H \cup K$ such that $x*x'$ and $x' \in i-cl(S(x))$. Suppose also that for each $x \in H \cup K$, there are only countable many points $x' \in H \cup K$ with $x*x'$ for which $S(x) \cap S(x') \neq \phi$.

Then, each $S(x)$ can be refined to i-open neighborhood $R(x)$ of x. So that the collection $\{R(x) : x \in H \cup K\}$ satisfies the following:

For each $R(x)$, there is no set $R(x')$ such that $x*x'$ and $R(x) \cap R(x') \neq \phi$ for $i = 1, 2$.

Proof. see [3]. □

Theorem. *Every (i,j)-paralindelöf, i-T_1 and i-regular space is (i,j)-CwH.*

Proof. Let $X_0 = \{x_\alpha : \alpha \in \Delta\}$ be a discrete collection of points of X. Since X is i-T_1 space, X_0 is i-closed discrete subset of X. By the regularity of X, for each $\alpha \in \Delta$ let U_α be a i-open neighborhood of x_α such that $i-cl(U_\alpha) \cap X_0 = \{x_\alpha\}$. Then the family $\mathcal{U} = \{U_\alpha ; \alpha \in \Delta\} \cup \{X - X_0\}$ forms i-open cover of X. Since X is (i,j)-paralindelöf, $\mathcal{U}$ has j-locally countable family $\mathcal{V}$ of j-open subsets of X which refines $\mathcal{U}$. For each $\alpha \in \Delta$, let V_α be a j-open neighborhood of x_α belong to $\mathcal{V}$, i.e., $x_\alpha \in V_\alpha \subset \mathcal{V}$ for all $\alpha \in \Delta$. Write $\mathcal{V}_0 = \{V_\alpha : \alpha \in \Delta\}$.

Due to $\mathcal{V}_0 \subset \mathcal{V}, \mathcal{V}_0 = \{V_\alpha : \alpha \in \Delta\}$ is also j-locally countable. So for each $\alpha \in \Delta$, there is j-neighborhood G_α that meets $\mathcal{V}_0$ at most countably many members, i.e., the family $\mathcal{V}_0' = \{V_\alpha \subset \mathcal{V}_0 : V_\alpha \cap G_\alpha\}$ is star countable.

Now, by applying the lemma: let $H = X_0, K = X_0$ and $S(x_\alpha) = V_\alpha' \subset \mathcal{V}_0'$ for each $\alpha \in \Delta$; let $R(x_\alpha) \subset S(x_\alpha)$. Let $W_\alpha = R(x_\alpha)$ for all $\alpha \in \Delta$, then $\mathcal{W}_0 = \{W_\alpha : \alpha \in \Delta\}$ is a collection of disjoint j-open sets with $x_\alpha \in W_\alpha$ for all $\alpha \in \Delta$. Therefore, X is (i,j)-cwH. □

Example. Let $(\mathcal{R}, \tau_{coc}, \tau_{coc})$ be a bitopological space. It is clear that $(\mathcal{R}, \tau_{coc}, \tau_{coc})$ is pairwise paralindelöf space . But it is not pairwise CwH since it is T_1 space but not pairwise Hausdorff space (see [13]).

Now, we will study the closed mapping properties of pairwise paralindelöf spaces. We show that the pairwise paralindelöf spaces are not preserved under closed mappings.

Example. If (X,τ_1,τ_2) is τ_i-normal, on-(τ_i,τ_j)-collectionwise normal and (τ_i,τ_j)-paralindelöf space, there is i-closed mapping $\phi : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ where the space Y does not have the paralindelöf property.

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a discrete collection of τ_i-closed subsets of X. Since X is not (τ_i,τ_j)-collectionwise normal space, $\mathcal{F}$ cannot be separated by τ_j-open sets.

Let define Y as a quotient space obtained from identifying each F_α with a point P_α and let $\phi : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be the corresponding quotient map which is closed. Then Y is σ_i-normal space and so that Y is σ_i-T_1 and σ_i-regular space. It is known that every i-regular (i,j)-paralindelöf spaces are (i,j)-collectionwise Hausdorff. Take the set $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ as σ_i-closed discrete set in Y (since Y is σ_i-T_1). If we suppose that Y is (σ_i,σ_j)-paralindelöf space, the collection $\mathcal{P}$ is separated by σ_j-open sets in Y which makes the collection $\mathcal{F}$ separated by τ_j-open sets in X. This contacts that X is not (i,j)-collectionwise normal space. So Y cannot be (σ_i,σ_j)-collectionwise Hausdorff. Therefore Y must not be (σ_i,σ_j)-paralindelöf space. □

Lemma [9]. *A function $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is i-closed if and only if for each point $y \in y$ and τ_i-open set G in X such that $f^{-1}(y) \subset G$, there exists σ_i-open set H containing y such that $f^{-1}(H) \subset G$.*

Theorem. *Let $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is i-closed, j-continuous and $f^{-1}(y)$ is (τ_j,τ_i)-Lindelof relative to X for each $y \in Y$. Then if Y is (σ_i,σ_j)-paralindelöf, so X is τ_j-paralindelöf.*

Proof. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Gamma\}$ be j-open cover of X. Let $y \in Y$. Since $f^{-1}(y)$ is (j,i)-Lindelof relative to X, there is a countable i-open, j-open subcover such that $f^{-1}(y) \subset \cup_{n \in \mathcal{N}} U_{\alpha_n}$. Since f is i-closed, there is i-open nbd $V(y)$ of y such that $f^{-1}(y) \subset f^{-1}(V(y)) \subset \cup_{n \in \mathcal{N}} U_{\alpha_n}$. Set the family $\mathcal{V} = \{V_y | y \in Y\}$ as i-open cover of Y. $\mathcal{V}$ has j-locally countable family $\mathcal{W} = \{W_y : y \in Y\}$ of j-open sets which refines $\mathcal{V}$. For each $y \in Y, n \in \mathcal{N}$, let $V(y,\alpha_n) = f^{-1}(W_y) \cap U_{\alpha_n}$. Put $\mathcal{V} = \{V(y,\alpha_n) : y \in Y, n \in \mathcal{N}\}$. Thus, V is j-locally countable j-open refinement of U. □

Corollary. *Let $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is closed, continuous and $f^{-1}(y)$ is pairwise Lindelof relative to X for each $y \in Y$. Then if Y is pairwise paralindelöf, so X is paralindelöf.*

5. Mappings on generalizations of paralindelöf bitopological spaces

The idea of nearly paralindelöf spaces has been studied by Daniel Thanapalan [4] by using regular open sets. He continued to study mappings on nearly paralindelöf ([7,5]). In this section, we define the notion of nearly paralindelof spaces in bitopological spaces which will call pairwise nearly paralindelof space. Moreover, we shall study the pairwise some mappings on pairwise nearly paralindelof spaces.

Definition. Let (X,τ_1,τ_2) be a bitopological space. X is (i,j)-nearly paralindelöf space if for every (i,j)-regular open covering of X admits i-open refinement i-locally countable family covering X. X is called pairwise nearly paralindelöf if it is both $(1,2)$-nearly paralindelöf and $(2,1)$-nearly paralindelöf.

Theorem. *Let $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be (i,j)-almost continuous, j-continuous, i-open, i-closed and surjection such that $f^{-1}(y)$ is i-Lindelof relative to X for each $y \in Y$. Then if X is (i,j)-nearly paralindelöf, so is Y.*

Proof. Let $\mathcal{U}$ be (σ_i, σ_j)-regular open cover of Y. Since f is (i,j)-almost continuous [12], $\{f^{-1}(U) : U \in \mathcal{U}\}$ is τ_i-open cover of X. Write $\mathcal{V} = \{\tau_i - int(\tau_j - cl(f^{-1}(U))) : U \in \mathcal{U}\}$. So, $\mathcal{V}$ is (τ_i, τ_j)-regular open cover of X and has τ_i-open refinement $\mathcal{B}$ which is τ_i-locally countable. Let $\mathcal{C} = \{f(B) : B \in \mathcal{B}\}$. Since f is i-open, $\mathcal{C}$ is σ_i-open cover of Y. Now, we shall show that $\mathcal{C}$ is σ_i-locally countable. Since $f^{-1}(y)$ is τ_i-Lindelöf relative to X for each $y \in Y$, there is τ_i-open neighborhood W_x of x which meets only countably many members of $\mathcal{B}$. $\{W_x : x \in f^{-1}(y)\}$ is τ_i-open cover of $f^{-1}(y)$. Then, there is a countable subset $C \subset f^{-1}(y)$ such that $f^{-1}(y) \subset W = \cup\{W_x : x \in C\}$. Moreover, W meets only countably many members of $\mathcal{B}$. If $V = Y - f(X - W)$, then V is σ_i-open in Y. If $V \cap f(B) \neq \emptyset$, then $B \cap W \neq \emptyset$. So, $\mathcal{C}$ is σ_i-locally countable. Since $\mathcal{B}$ is refinement of $\mathcal{V}$, there is $U \in \mathcal{U}$ such that $B \subset \tau_i - int(\tau_j - cl(f^{-1}(U)))$ for each $B \in \mathcal{B}$. Because f is j-continuous and i-open, We have

$$\begin{aligned}\tau_j - cl(f^{-1}(U)) \subset f^{-1}(\sigma_j - cl(U)) \Rightarrow B \\ \subset \tau_i - int(\tau_j - cl(f^{-1}(U))) \\ \subset \tau_i - int(f^{-1}(\sigma_j - cl(U))) \subset f^{-1}(\sigma_j - cl(U)) \\ \Rightarrow f(B) \subset \sigma_j - cl(U) \Rightarrow f(B) \\ \subset \sigma_i - int(\sigma_j - cl(U) = U.\end{aligned}$$

Thus, $\mathcal{C}$ refines $\mathcal{U}$. Therefore, Y is (i,j)-nearly paralindelöf. □

Corollary. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be pairwise almost continuous, continuous, open, closed and surjection such that $f^{-1}(y)$ is Lindelof relative to X for each $y \in Y$. Then if X is pairwise nearly paralindelöf, so is Y.*

Proposition. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be (i,j)-almost continuous, i-continuous, i-open and surjection such that $f^{-1}(y)$ is i-Lindelof relative to X for each $y \in Y$. Then, if X is i-paralindelöf, Y is (i,j)-nearly paralindelöf.*

Proof. Let $\mathcal{U}$ be (σ_i, σ_j)-regular open cover of Y. Since f is (i,j)-almost continuous [12], $\{f^{-1}(U) : U \in \mathcal{U}\}$ is τ_i-open cover of X. So, $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is τ_i-open cover of X and has τ_i-open refinement $\mathcal{B}$ which is τ_i-locally countable. Set $\mathcal{C} = \{f(B) : B \in \mathcal{B}\}$. Since f is i-open and surjection, $\mathcal{C}$ is σ_i-open cover of Y. Now, we shall show that $\mathcal{C}$ is σ_i-locally countable. Since $f^{-1}(y)$ is τ_i-Lindelöf relative to X for each $y \in Y$, there is τ_i-open neighborhood W_x of x which meets only countably many members of $\mathcal{B}$. Then, the family $\{W_x : x \in f^{-1}(y)\}$ is τ_i-open cover of $f^{-1}(y)$. Then, there is a countable subset $C \subset f^{-1}(y)$ such that $f^{-1}(y) \subset W = \cup\{W_x : x \in C\}$. Moreover, W meets only countably many members of $\mathcal{B}$. If we write $V = Y - f(X - W)$, then V is σ_i-open in Y. If $V \cap f(B) \neq \emptyset$, then $B \cap W \neq \emptyset$. So, $\mathcal{C}$ is σ_i-locally countable. Since $\mathcal{B}$ is refinement of $\mathcal{V}$, for each $B \in \mathcal{B}$, there is $U \in \mathcal{U}$ such that $B \subset f^{-1}(U) \Rightarrow f(B) \subset U$. Thus, $\mathcal{C}$ refines $\mathcal{U}$. Therefore, Y is (i,j)-nearly paralindelöf. □

Corollary. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be pairwise almost continuous, continuous, open and surjection such that $f^{-1}(y)$ is Lindelof relative to X for each $y \in Y$. Then, if X is paralindelöf, Y is pairwise nearly paralindelöf.*

P. T. Daniel Thanapalan has studies the concept of almost paralindelöf and its properties (see [4,7,5]). In this section, we shall extend the idea of almost paralindelöf to bitopological spaces show some its behavior under some types of mappings.

Definition. Let (X, τ_1, τ_2) be a bitopological space. X is (i,j)-almost paralindelöf space if for every i-open covering of X admits i-open refinement i-locally countable family $\mathcal{V}$ such that $X = \cup\{j - cl(V) : V \in \mathcal{V}\}$. X is called pairwise almost paralindelöf if it is both $(1,2)$-almost paralindelöf and $(2,1)$-almost paralindelöf.

Theorem. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be continuous, i-open, i-closed and surjection such that $f^{-1}(y)$ is i-Lindelöf relative to X for each $y \in Y$. Then if X is (τ_i, τ_j)-almost paralindelöf, so is Y.*

Proof. Let $\mathcal{U}$ be σ_i-open cover of Y. Then, $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is τ_i-open cover of X and has τ_i-locally countable family $\mathcal{B}$ of τ_i-open subset of X which refines $\mathcal{V}$ and $X = \cup\{\tau_j - cl(B) : B \in \mathcal{B}\}$. Set $\mathcal{C} = \{f(B) : B \in \mathcal{B}\}$ as a collection of σ_i-open subsets of Y. $\mathcal{C}$ refines $\mathcal{U}$ since $\mathcal{B}$ refines $\mathcal{U}$ and for each $B \in \mathcal{B}$, there is a $U \in \mathcal{U}$ such that $B \subset f^{-1}(U)$ so that $f(B) \subset U$. Since f is surjective, we have

$$\begin{aligned}Y = f(X) = f(\cup\{\tau_j - cl(B) : B \in \mathcal{B}\}) = \cup\{f(\tau_j - cl(B)) : B \\ \in \mathcal{B}\} \subset \cup\{\sigma_j - cl(f(B) : B \in \mathcal{B}\}.\end{aligned}$$

Furthermore, since f is i-closed, $f^{-1}(y)$ is τ_i-Lindelöf relative to X and $\mathcal{B}$ is i-locally countable, so that for each $x \in f^{-1}(y)$, there is τ_i-open neighborhood W_x of x which meets only countably many members of $\mathcal{B}$. $\{W_x : x \in f^{-1}(y)\}$ is τ_i-open cover of $f^{-1}(y)$. Thus, there exists a countable subset $C \subset f^{-1}(y)$ such that $f^{-1}(y) \subset W = \cup\{W_x : x \in C\}$. Moreover, W meets only countably many members of $\mathcal{B}$. If $V = Y - f(X - W)$, then V is σ_i-open in Y. So, if $V \cap f(B) \neq \emptyset$, then $B \cap W \neq \emptyset$. Then, $\mathcal{C}$ is σ_i-locally countable. Therefore, Y is (σ_i, σ_j)-almost paralindelöf. □

Corollary. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be continuous, open, closed and surjection such that $f^{-1}(y)$ is lindelöf relative to X for each $y \in Y$. Then if X is pairwise almost paralindelöf, so is Y.*

Theorem. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be i-continuous, i-open, i-closed and surjection such that $f^{-1}(y)$ is Lindelöf relative to X for each $y \in Y$. Then, if σ_i paralindelöf with respect to σ_j, τ_i is paralindelöf with respect to τ_j.*

Proof. Let $\mathcal{U}$ be τ_i-open cover of X. Let $y \in Y$. Since $f^{-1}(y)$ is τ_i-Lindelöf relative to X, there is a countable subset $\Delta(y) = \{\alpha_{n(y)} : n \in \mathbb{N}\}$ of Δ such that $f^{-1}(y) \cup \{U\alpha : \alpha \in \Delta(y)\}$. By closdness of f, there is σ_i-open neighborhood V(y) of y such that $f^{-1}(y) \subset f^{-1}(V(y)) \subset \cup\{U\alpha : \alpha \in \Delta(y)\}$. Set the family $\vartheta = \{V_y : y \in Y\}$ as σ_i-open cover of Y. ϑ has σ_j-locally countable family $\mathcal{W} = \{W_y : y \in Y\}$ of σ_i-open sets which refines ϑ and covers Y. We can think of $\mathcal{W}$ to be precise. For each $y \in Y$, $n \in \mathbb{N}$, let $V(y, \alpha_{n(y)}) = f^{-1}(W_y) \cap U\alpha_{n(y)}$. Put $Q = \{V(y, \alpha_{n(y)}) : y \in Y, n \in \mathbb{N}\}$. Thus, Q is τ_i-locally countable τ_i-open refinement of $\mathcal{U}$. Therefore, τ_i is paralindelöf with respect to τ_j. □

Corollary. *Let* $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ *be pairwise almost continuous, open, closed and surjection such that* $f^{-1}(y)$ *is nearly lindelöf relative to* X *for each* $y\in Y$. *Then if* X *is almost para-lindelöf and pairwise almost regular, so* Y *is pairwise nearly paralindelöf.*

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References

[1] J.C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (3) (1963) 71–89.
[2] A.R. Singal, S.P. Arya, On pairwise almost regular spaces, Glasnik Mat. Ser. III 6 (26) (1971) 335–343.
[3] C.L. Navy, Non paracompactness in paraLindelöf Spaces, The University of Wisconsin – Madison , PhD., 1981.
[4] P.T. Daniel Thanapalan, On nearly paralindeloff spaces, Indian J. Pure Appl. Math. 22 (4) (1991) 303–311.
[5] P.T. Daniel Thanapalan, On mappings of (locally) almost and (locally) nearly paralindelof spaces, Indian J. Pure Appl. Math. 23 (12) (1992) 867–872.
[6] D.K. Burke, Paralindelof spaces and closed mappings, Topology Proc. 5 (1980) 47–57.
[7] P.T. Daniel Thanapalan, Mappings and products of nearly paralindelof, Indian J. Pure Appl. Math. 22 (7) (1991) 567–573.
[8] P. Fletcher, H.B. Hoyle, C.W. Patty, The comparison of topologies, Duke Math. J. 26 (1969) 325–331.
[9] Hend M. Bouseliana, A. Kılıçman, On pairwise super continuous multifunctions, in: International Conference on Mathematical Sciences and Statistics 2013 (ICMSS2013) AIP Conf. Proc., vol. 1557, 2013, pp. 376–379.
[10] A. Kılıçman, Z. Salleh, On pairwise Lindelöf bitopological spaces, Topology Appl. 154 (8) (2007) 1600–1607.
[11] A. Kılıçman, Z. Salleh, On pairwise almost regular-lindeloff spaces, Scientiae Math. Japonicae 70(3) (2009) 285–298 (e2009, pp. 481–494).
[12] S. Bose, D. Sinha, Pairwise almost continuous map and weakly continuous map in bitopological spaces, Bull. Cal. Math. Soc. 74 (1982) 195–206.
[13] L.A. Steen, J.A. Seebach Jr., Counterexamples in Topology, 2nd ed., Springer-Verlag, New York, 1978.
[14] A. Tallafha, A. Al-Bsoul, A. Fora, Countable dense homogeneous bitopological spaces, Tr. J. Math. 23 (1999) 233–242, TÜBİTAK.