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ORIGINAL ARTICLE

A viscosity Cesàro mean approximation method for split generalized vector equilibrium problem and fixed point problem

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Abstract In this paper, we introduce and study an explicit iterative method to approximate a common solution of split generalized vector equilibrium problem and fixed point problem for a finite family of nonexpansive mappings in real Hilbert spaces using the viscosity Cesàro mean approximation. We prove a strong convergence theorem for the sequences generated by the proposed iterative scheme. Further we give a numerical example to justify our main result. The results presented in this paper generalize, improve and unify the previously known results in this area.

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1. Introduction

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let Y be a Hausdorff topological space and P be a pointed, proper, closed and convex cone of Y with $\mathrm{int}P \neq \emptyset$.

In 1994, Blum and Oettli [1] introduced and studied the following equilibrium problem (in short, EP): Find $x \in C$ such that

$$F_1(x,y) \geqslant 0, \quad \forall y \in C, \tag{1.1}$$

where $F_1 : C \times C \to \mathbb{R}$ is a bifunction. We denote the solution set of EP(1.1) by sol(EP(1.1)).

In the last two decades, EP(1.1) has been generalized and extensively studied in many directions due to its importance; see for example [2–10] for the literature on the existence and iterative approximation of solution of the various generalizations of EP(1.1). Recently, Kazmi and Rizvi [11] considered the following pair of equilibrium problems in different spaces, which is called *split equilibrium problem* (in short, SEP): Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and let $A : H_1 \to H_2$ be a bounded linear operator then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

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$$F_1(x^*, x) \geqslant 0, \quad \forall x \in C, \tag{1.2}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves } F_2(y^*, y) \geqslant 0, \quad \forall y \in Q. \tag{1.3}$$

They introduced and studied some iterative methods for finding the common solution of SEP(1.2) and (1.3), variational inequality and fixed point problems. We denote the solution set of SEP(1.2) and (1.3) by sol(SEP(1.2) and (1.3)) $:= \{p \in \text{sol}(EP(1.2)) : Ap \in \text{sol}(EP(1.3))\}$. For related work, see [12,14].

In this paper, we introduce and study the following class of split generalized vector equilibrium problems (in short, SGVEP):

Let $F_1 : C \times C \to Y$ and $F_2 : Q \times Q \to Y$ be nonlinear bimappings and let $\phi_1 : C \to Y$, $\phi_2 : Q \to Y$ be nonlinear mappings, then SGVEP is to find $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x) - \phi_1(x^*) \in P, \quad \forall x \in C, \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves } F_2(y^*, y) + \phi_2(y) - \phi_2(y^*) \in P, \quad \forall y \in Q. \tag{1.5}$$

When looked separately, (1.4) is the generalized vector equilibrium problem (GVEP) and we denote its solution set by sol(GVEP(1.4)). The SGVEP(1.4) and (1.5) constitutes a pair of generalized vector equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the GVEP(1.4) in H_1 is the solution of another GVEP(1.5) in another space H_2, we denote the solution set of GVEP(1.5) by sol(GVEP(1.5)). The solution set of SGVEP(1.4) and (1.5) is denoted by $\Gamma = \{p \in \text{sol}(\text{GVEP}(1.4)) : Ap \in \text{sol}(\text{GVEP}(1.5))\}$. GVEP(1.4) has been studied by Kazmi and Farid [19] in Banach spaces.

SGVEP(1.4) and (1.5) generalize multiple-sets split feasibility problem. It also includes as special case, the split variational inequality problem [15] which is the generalization of split zero problems and split feasibility problems, see for detail [33,34,15–17].

If $\phi_1 = \phi_2 = 0$, then SGVEP(1.4) and (1.5) reduces to the split vector equilibrium problem (in short, SVEP): Find $x^* \in C$ such that

$$F_1(x^*, x) \in P, \quad \forall x \in C, \tag{1.6}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves } F_2(y^*, y) \in P, \quad \forall y \in Q, \tag{1.7}$$

which appears to be new and is the vector version of SEP(1.2) and (1.3) [11]. Further, if $H_1 = H_2$, $C = Q$, and $F_1 = F_2$, then SVEP(1.6) and (1.7) reduces to the strong vector equilibrium problem (in short, VEP) of finding $x^* \in C$ such that

$$F_1(x^*, x) \in P, \quad \forall x \in C, \tag{1.8}$$

which has been studied by Kazmi and Khan [18]. In recent years, the vector equilibrium problem has been intensively studied by many authors (see, for example [2–4,18] and the references therein).

Next, we recall that a mapping $T : C \to C$ is said to be contraction if there exists a constant $\alpha \in (0, 1)$ such that $\|Tx - Ty\| \leqslant \alpha\|x - y\|$, $\forall x, y \in C$. If $\alpha = 1$, T is called nonexpansive on C.

The *fixed point problem* (in short, FPP) for a nonexpansive mapping T is:

$$\text{Find } x \in C \quad \text{such that } x \in \text{Fix}(T), \tag{1.9}$$

where Fix(T) is the fixed point set of the nonexpansive mapping T. It is well known that Fix(T) is closed and convex.

In 1997, using Cesàro mean approximation, Shimizu and Takahashi [20] established a strong convergence theorem for a finite family of nonexpansive mappings $\{T^i\}$ $(i = 0, 1, 2, \ldots, N)$ in a real Hilbert space. For further related work, see [21].

Very recently, Colao et al. [23] introduced and studied the following iterative method to obtain a strong convergence theorem for FPP(1.9) of a nonexpansive semigroup $\{T(s) : 0 \leqslant s < \infty\}$ in the presence of the error sequence $\{e_n\}$ in Hilbert space:

$$\begin{cases} x_0 \in C; \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B)T(s)x_n + e_n, \end{cases}$$

where $f : H_1 \to H_1$ is a contraction mapping with constant α; $T : C \to C$ is a nonexpansive mapping, and $B : H_1 \to H_1$ is a strongly positive linear bounded operator, i.e., if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Bx, x\rangle \geqslant \bar{\gamma}\|x\|^2, \quad \forall x \in H_1,$$

with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $t \in (0, 1)$ and proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z\rangle \geqslant 0, \quad \forall x \in \text{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \text{Fix}(T)} \frac{1}{2}\langle Bx, x\rangle - h(x),$$

where h is the potential function for γf.

We note that in spite of the fact that the fixed point iterative methods are designed for numerical purposes, and hence the consideration of errors is of both theoretical and practical importance, however, the condition which implies the errors tend to zero, is not suitable for the randomness of the occurrence of errors in practical computations, see [24].

Motivated by the work of Shimizu and Takahashi [20], Colao et al. [23], Shan and Haung [26] and Kazmi and Rizvi [11,12,14] and by the on going research in this direction, we introduce and study the strong convergence of an explicit iterative method for approximating a common solution of SGVEP(1.4) and (1.5) and FPP(1.9) for a finite family of nonexpansive mappings in real Hilbert spaces using viscosity Cesàro mean approximation in Hilbert spaces. The results presented in this paper generalize, improve and unify many previously known results in this research area, see instance [5,10–13,22,23].

2. Preliminaries

We recall some concepts and results which are needed in sequel.

For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

P_C is called the *metric projection* of H_1 onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \leqslant 0. \tag{2.2}$$

Further, it is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)),\ T(y) - T(x)\rangle \leqslant (1/2)\|(T(x) - x) - (T(y) - y)\|^2, \tag{2.3}$$

and therefore, we get, for all $(x, y) \in H_1 \times \mathrm{Fix}(T)$,

$$\langle x - T(x),\ y - T(x)\rangle \leqslant (1/2)\|T(x) - x\|^2, \tag{2.4}$$

see, e.g. [27, Theorem 3.1].

It is also known that H_1 satisfies Opial's condition [28], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\lim \inf_{n\to\infty} \|x_n - x\| < \lim \inf_{n\to\infty} \|x_n - y\| \tag{2.5}$$

holds for every $y \in H_1$ with $y \neq x$.

Definition 2.1. A mapping $T : H_1 \to H_1$ is said to be firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \geqslant \|Tx - Ty\|^2, \quad \forall x, y \in H_1.$$

Definition 2.2. A mapping $T : H_1 \to H_1$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : H_1 \to H_1$ is nonexpansive and I is the identity operator on H_1.

We note that the averaged mappings are nonexpansive. Further, the firmly nonexpansive mappings are averaged. Further for some key properties of averaged operators, see for instance [16].

Lemma 2.1 [29]. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \lim \inf_{n\to\infty} \beta_n \leqslant \lim \sup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, for all integers $n \geqslant 0$ and $\lim \sup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2 [30]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1 - \alpha_n)a_n + \delta_n, \quad n \geqslant 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

$$\text{(i)} \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \text{(ii)} \quad \lim \sup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leqslant 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 [25]. *Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H_1 with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leqslant \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4. *The following inequality hold in real Hilbert space H_1:*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H_1.$$

Definition 2.3. [26,31]. Let X and Y be two Hausdorff topological spaces, and let E be a nonempty, convex subset of X and P be a pointed, proper, closed, convex cone of Y with $\mathrm{int}P \neq \emptyset$. Let 0 be the zero point of $Y, \mathbb{U}(0)$ be the neighborhood set of $0, \mathbb{U}(x_0)$ be the neighborhood set of x_0, and $f : E \to Y$ be a mapping.

(i) If for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + P \text{ (or } f(x) \in f(x_0) + V - P), \quad \forall x \in U \cap E,$$

then f is called upper P-continuous at x_0. If f is upper P-continuous (lower P-continuous) for all $x \in E$, then f is called upper P-continuous (lower P-continuous) on E;

(ii) If for any $x, y \in E$ and $t \in [0, 1]$, the mapping f satisfies

$$f(x) \in f(tx + (1 - t)y) + P \quad \text{or} \quad f(y) \in f(tx + (1 - t)y) + P,$$

then f is called proper P-quasiconvex;

(iii) If for any $x_1, x_2 \in E$ and $t \in [0, 1]$, the mapping f satisfies

$$tf(x_1) + (1 - t)f(x_2) \in f(tx + (1 - t)y) + P,$$

then f is called P-convex.

Lemma 2.5. [26,32]. *Let X and Y be two real Hausdorff topological spaces; let E be a nonempty, compact, convex subset of X, and let P be a pointed, proper, closed and convex cone of Y with $\mathrm{int}P \neq \emptyset$. Assume that $g : E \times E \to Y$ and $\Phi : E \to Y$ are two mappings. Suppose that g and Φ satisfy*

(i) $g(x, x) \in P$, for all $x \in E$, and $g(\cdot, y)$ is lower P-continuous for all $y \in E$;

(ii) Φ is upper P-continuous on E, and $g(x, \cdot) + \Phi(\cdot)$ is proper P-quasiconvex for all $x \in E$.

Then there exists a point $x \in E$ satisfies

$$G(x, y) \in P \setminus \{0\}, \quad \forall y \in E,$$

where

$$G(x, y) = g(x, y) + \Phi(y) - \Phi(x), \quad \forall x, y \in E.$$

Let $F_1 : C \times C \to Y$ and $\phi_1 : C \to Y$ be two mappings. For any $z \in H_1$, define a mapping $G_{1_z} : C \times C \to Y$ as follows:

$$G_{1_z}(x, y) = F_1(x, y) + \phi_1(y) - \phi_1(x) + \frac{e}{r}\langle y - x, x - z\rangle, \tag{2.6}$$

where r is a positive number in R and $e \in P$.

Assumption 2.1. *Let G_{1_z}, F_1, ϕ_1 satisfy the following conditions:*

(i) For all $x \in C$, $F_1(x, x) \in P$; F_1 is P-monotone, i.e., $F_1(x, y) + F_1(y, x) \in -P$ for all $x, y \in C$; $F_1(\cdot, y)$ is continuous for all $y \in$, and $F_1(x, \cdot)$ is weakly continuous and P-convex, i.e.,

$tF_1(x, y_1) + (1-t)F_1(x, y_2) \in F_1(x, ty_1 + (1-t)y_2) + P,$
$\forall x, y_1, y_2 \in C, \quad \forall t \in [0,1];$

(ii) $G_{1_z}(\cdot, y)$ *is lower P-continuous for all* $y \in C$ *and* $z \in H_1$, *and* $G_{1_z}(x, \cdot)$ *is proper P-quasiconvex for all* $x \in C$ *and* $z \in H_1$.
(iii) $\phi_1(\cdot)$ *is P-convex and weakly continuous.*

Lemma 2.6 [26]. *Assume that* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *are nonempty, compact and convex sets. Assume that* F_1, ϕ_1 *and* G_{1_z} *are satisfying Assumption* 2.1. *For* $r > 0$ *and for all* $x \in H_1$, *define a mapping* $T_r^{(F_1,\phi_1)} : H_1 \to C$ *as follows:*

$$T_r^{(F_1,\phi_1)}(x) = \{z \in C : F_1(z, y) + \phi_1(y) - \phi_1(z) + \frac{e}{r}\langle y - z, z - x\rangle \in P, \quad \forall y \in C\}.$$

Then the following hold:

(i) $T_r^{(F_1,\phi_1)}(x)$ *is nonempty for all* $x \in H_1$.
(ii) $T_r^{(F_1,\phi_1)}$ *is single-valued and firmly nonexpansive.*
(iii) $\mathrm{Fix}(T_r^{(F_1,\phi_1)}) = \mathrm{sol}(\mathrm{GVEP}(1.4))$ and $\mathrm{sol}(\mathrm{GVEP}(1.4))$ *is closed and convex.*

Further, assume that $F_2 : Q \times Q \to Y$, $\phi_2 : Q \to Y$ and $G_{2_z} : Q \times Q \to Y$ defined by

$$G_{1_z}(u, v) = F_2(u, v) + \phi_2(v) - \phi_2(u) + \frac{e}{r}\langle v - u, u - w\rangle,$$

are satisfying Assumption 2.1. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{(F_2,\phi_2)} : H_2 \to Q$ as follows:

$$T_s^{(F_2,\phi_2)}(w) = \Big\{u \in Q : F_2(u, v) + \phi_2(v) - \phi_2(u) + \frac{e}{s}\langle v - u, u - w\rangle \in P, \quad \forall v \in Q\Big\}.$$

Then, we easily observe that $T_s^{(F_2,\phi_2)}(w)$ is nonempty for each $w \in H_2$; $T_s^{(F_2,\phi_2)}$ is single-valued and firmly nonexpansive; sol(GVEP(2.7))is closed and convex and $\mathrm{Fix}(T_s^{(F_2,\phi_2)}) = \mathrm{sol}(\mathrm{GVEP}(2.7))$, where sol(GVEP(2.7)) is the solution set of the following GVEP: Find $y^* \in Q$ such that

$$F_2(y^*, y) + \phi_2(y) - \phi_2(x) \in P, \quad \forall y \in Q. \tag{2.7}$$

We observe that $\mathrm{sol}(\mathrm{GVEP}(1.5)) \subset \mathrm{sol}(\mathrm{GVEP}(2.7))$. Further, it is easy to prove that Γ is closed and convex set.

Notation. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H_1$.

3. Main result

In this section, we prove a strong convergence theorem based on the proposed viscosity Cesàro mean approximation method for computing the approximate common solution of SGVEP(1.4) and (1.5) and FPP(1.9) for a finite family of nonexpansive mappings in real Hilbert spaces.

First, we have the following lemma. The proof is similar to the proof given in [26], and hence omitted.

Lemma 3.1. *Let* F_1, ϕ_1 *and* G_{1_z} *satisfy Assumption* 2.1 *and let* $T_r^{(F_1,\phi_1)}$ *be defined as in Lemma* 2.6 *for* $r > 0$. *Let* $x_1, x_2 \in H_1$ *and* $r_1, r_2 > 0$. *Then:*

$$\left\|T_{r_2}^{(F_1,\phi_1)}(x_2) - T_{r_1}^{(F_1,\phi_1)}(x_1)\right\| \leqslant \|x_2 - x_1\| + \frac{|r_2 - r_1|}{r_2}\left\|T_{r_2}^{(F_1,\phi_1)}(x_2) - x_2\right\|.$$

Now, we prove the following main result.

We assume that $\Gamma \neq \emptyset$.

Theorem 3.1. *Let* H_1 *and* H_2 *be two real Hilbert spaces; let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty, compact and convex subsets; let Y be a Hausdorff topological space and let P be a proper, closed and convex cone of Y with* $intP \neq \emptyset$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $F_1 : C \times C \to Y$, $F_2 : Q \times Q \to Y$, $\phi_1 : C \to Y$ *and* $\phi_2 : Q \to Y$ *are nonlinear mappings satisfying Assumption* 2.1 *and* F_2 *is upper semicontinuous in first argument. Let* $T^i : C \to C$ *be a nonexpansive mapping for each* $i = 0, 1, 2, \ldots, n$ *such that* $\Theta = \bigcap_{i=1}^n \mathrm{Fix}(T^i) \cap \Gamma \neq \emptyset$. *Let* $f : H_1 \to H_1$ *be a contraction mapping with constant* $\alpha \in (0,1)$ *and B be a strongly positive bounded linear self adjoint operator on* H_1 *with constant* $\bar{\gamma} > 0$ *such that* $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. *For a given* $x_0 \in C$ *arbitrarily, let the iterative sequences* $\{u_n\}$ *and* $\{x_n\}$ *be generated by*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\phi_1)}\big(x_n + \delta A^*\big(T_{r_n}^{(F_2,\phi_2)} - I\big)Ax_n\big); \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)\frac{1}{n+1}\sum_{i=0}^{n} T^i u_n + \gamma_n e_n, \end{cases} \tag{3.1}$$

where $\{e_n\}$ *is an bounded error sequence in* H_1, $\delta \in (0, 1/L)$, L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are the sequences in* $(0,1)$ *and* $r_n \subset (0,\infty)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;
(iv) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $z \in P_\Theta$, *where* $z = P_\Theta(I - B + \gamma f)z$.

Proof. By using condition (i) and Lemma 2.3, we can observe that there exists a unique element $z \in H_1$ such that $z = P_{\bigcap_{i=1}^n \mathrm{Fix}(T^i)\cap\Gamma}(I - B + \gamma f)(z)$, see [12].

Let $p \in \Theta := \bigcap_{i=0}^n \mathrm{Fix}(T^i) \cap \Gamma$, i.e., $p \in \Gamma$, we have $p = T_{r_n}^{(F_1,\phi_1)}p$ and $Ap = T_{r_n}^{(F_2,\phi_2)}(Ap)$. Using the similar arguments used in proof of Theorem 3.1 [11], we have the following estimates:

$$\|u_n - p\|^2 \leqslant \|x_n - p\|^2 + \delta(L\delta - 1)\left\|\big(T_{r_n}^{(F_2,\phi_2)} - I\big)Ax_n\right\|^2. \tag{3.2}$$

Since, $\delta \in \left(0, \frac{1}{L}\right)$, we obtain

$$\|u_n - p\|^2 \leqslant \|x_n - p\|^2. \tag{3.3}$$

Now, on setting $t_n := \frac{1}{n+1}\sum_{i=0}^n T^i$, we can easily observe that the mapping t_n is nonexpansive. Since $p \in \Theta$, we have

$$t_n p = \frac{1}{n+1}\sum_{i=0}^{n} T^i p = \frac{1}{n+1}\sum_{i=0}^{n} p = p. \tag{3.4}$$

Since $\{e_n\}$ is bounded, using condition (ii), we obtain that $\left\{\frac{\gamma_n\|e_n\|}{\alpha_n}\right\}$ is bounded. Then, there exists a nonnegative real number K such that

$$\|\gamma f(p) - Bp\| + \frac{\gamma_n\|e_n\|}{\alpha_n} \leqslant K, \quad \text{for all } n \geqslant 0. \tag{3.5}$$

Further, it follows by (3.1), (3.3) and (3.5) that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)t_n u_n+\gamma_n e_n-p\|\\ &\leqslant \alpha_n\|\gamma f(x_n)-Bp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-p\|+\gamma_n\|e_n\|\\ &\leqslant \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Bp\|+\beta_n\|x_n-p\|+\gamma_n\|e_n\|\\ &\quad +(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-p\|\\ &\leqslant \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\|+(1-\alpha_n\bar{\gamma})\|x_n-p\|+\gamma_n\|e_n\|\\ &\leqslant (1-(\bar{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|+\alpha_n K\\ &\leqslant \max\left\{\|x_n-p\|,\frac{K}{\bar{\gamma}-\gamma\alpha}\right\},\quad n\geqslant 0\\ &\ \ \vdots\\ &\leqslant \max\left\{\|x_0-p\|,\frac{K}{\bar{\gamma}-\gamma\alpha}\right\}.z\end{aligned} \tag{3.6}$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}, \{t_n u_n\}$ and $\{f(x_n)\}$ are bounded.

Next, it follows from Lemma 3.1 that

$$\|u_{n+1} - u_n\| \leqslant \|x_{n+1} - x_n\| + \delta\|A\|\sigma_n + \delta_n,$$

where

$$\sigma_n = \left|1 - \frac{r_{n+1}}{r_n}\right| \left\|T_{r_n}^{(F_2,\phi_2)}Ax_n - Ax_n\right\|,$$

$$\delta_n = \left|1 - \frac{r_{n+1}}{r_n}\right| \Big\|T_{r_n}^{(F_1,\phi_1)}\big(x_n + \delta A^*\big(T_{r_n}^{(F_2,\phi_2)} - I\big)Ax_n\big) - \big(x_n + \delta A^*\big(T_{r_n}^{(F_2,\phi_2)} - I\big)Ax_n\big)\Big\|,$$

see [12] for details.

Next, we easily estimate that

$$\|t_{n+1}u_{n+1} - t_n u_n\| \leqslant \|u_{n+1} - u_n\| + \frac{2}{(n+2)}\|u_n - p\| + \frac{2}{(n+2)}\|p\|.$$

It follows from the above two inequalities that

$$\|t_{n+1}u_{n+1} - t_n u_n\| \leqslant \|x_{n+1} - x_n\| + \delta\|A\|\sigma_n + \delta_n + \frac{2}{n+2}\|u_n - p\| + \frac{2}{n+2}\|p\|. \tag{3.7}$$

Setting $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$, then we have

$$l_n = \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n B)t_n u_n + \gamma_n e_n}{1-\beta_n}, \quad \text{and}$$

$$l_{n+1} - l_n = \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\gamma f(x_{n+1}) - Bt_{n+1}u_{n+1} + \frac{\gamma_{n+1}e_{n+1}}{\alpha_{n+1}}\right) + t_{n+1}u_{n+1} - t_n u_n + \frac{\alpha_n}{1-\beta_n}\left(Bt_n u_n - \gamma f(x_n) - \frac{\gamma_n e_n}{\alpha_n}\right).$$

It follows from (3.7) that

$$\begin{aligned}\|l_{n+1}-l_n\| &\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})-Bt_{n+1}u_{n+1}\|+\frac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}\right)\\ &\quad +\|t_{n+1}u_{n+1}-t_n u_n\|+\frac{\alpha_n}{1-\beta_n}\left(\|Bt_n u_n-\gamma f(x_n)\|+\frac{\gamma_n\|e_n\|}{\alpha_n}\right)\\ &\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})-Bt_{n+1}u_{n+1}\|+\frac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}\right)+\|x_{n+1}-x_n\|\\ &\quad +\gamma\|A\|\sigma_n+\delta_n+\frac{2}{n+2}\|u_n-p\|+\frac{2}{n+2}\|p\|\\ &\quad +\frac{\alpha_n}{1-\beta_n}\left(\|Bt_n u_n-\gamma f(x_n)\|+\frac{\gamma_n\|e_n\|}{\alpha_n}\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| &\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})-Bt_{n+1}u_{n+1}\|+\frac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}\right)\\ &\quad +\frac{\alpha_n}{1-\beta_n}\left(\|Bt_n u_n-\gamma f(x_n)+\frac{\gamma_n\|e_n\|}{\alpha_n}\right)\\ &\quad +\gamma\|A\|\sigma_n+\delta_n+\frac{2}{n+2}\|u_n-p\|+\frac{2}{n+2}\|p\|.\end{aligned}$$

Taking $n \to \infty$ and using the conditions (i)–(iv), we obtain

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.8}$$

From Lemma 2.1 and (3.8), we obtain $\lim_{n\to\infty}\|l_n - x_n\| = 0$ and

$$\|x_{n+1} - x_n\| \leqslant \lim_{n\to\infty}(1-\beta_n)\|l_n - x_n\| = 0. \tag{3.9}$$

Since, we can write

$$\begin{aligned}\|x_n - t_n u_n\| &\leqslant \|x_n - x_{n+1}\| + \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)t_n u_n + \gamma_n e_n - t_n u_n\|\\ &\leqslant \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - Bt_n u_n\| + \beta_n\|x_n - t_n u_n\| + \gamma_n\|e_n\|,\end{aligned}$$

and then

$$\|x_n - t_n u_n\| \leqslant \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\left(\|\gamma f(x_n) - Bt_n u_n\| + \frac{\gamma_n\|e_n\|}{\alpha_n}\right).$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\|x_n - t_n u_n\| = 0. \tag{3.10}$$

Again, since $\{x_n\}$ is bounded, we may assume a nonnegative real number K such that $\|x_n - p\| \leqslant M$. It follows from (3.2) and Lemma 2.4 that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(x_n)-Bp)+\beta_n(x_n-t_n u_n)+(1-\alpha_n B)(t_n u_n-p)+\gamma_n e_n\|^2\\ &\leqslant \|(1-\alpha_n B)(t_n u_n-p)+\beta_n(x_n-t_n u_n)\|^2+2\langle\alpha_n\gamma f(x_n)-Bp+\gamma_n e_n, x_{n+1}-p\rangle\\ &\leqslant [\|(1-\alpha_n B)(t_n u_n-p)\|+\beta_n\|x_n-t_n u_n\|]^2+2\alpha_n\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+2\langle\gamma_n e_n, x_{n+1}-p\rangle\\ &\leqslant [(1-\alpha_n\bar{\gamma})\|u_n-p\|+\beta_n\|x_n-t_n u_n\|]^2+2\alpha_n\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+2\gamma_n\|e_n\|M\\ &= (1-\alpha_n\bar{\gamma})^2\|u_n-p\|^2+\beta_n^2\|x_n-t_n u_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\\ &\quad \times\|x_n-t_n u_n\|+2\alpha_n\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+2\gamma_n\|e_n\|M\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\leqslant (1-\alpha_n\bar{\gamma})^2[\|x_n-p\|^2+\delta(L\delta-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2] \\
&\quad +\beta_n^2\|x_n-t_nu_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_nu_n\| \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+2\gamma_n\|e_n\|M \\
&\leqslant \|x_n-p\|^2+(\alpha_n\bar{\gamma})^2)\|x_n-p\|^2 \\
&\quad +(1-\alpha_n\bar{\gamma})^2\delta(L\delta-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2 \\
&\quad +\beta_n^2\|x_n-t_nu_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n \\
&\quad -p\|\|x_n-t_nu_n\| \\
&\quad +2\alpha_n\left(\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+\frac{\gamma_n\|e_n\|M}{\alpha_n}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1-\alpha_n\bar{\gamma})^2\delta(1-L\delta)\|(T_{r_n}^{F_2}-I)Ax_n\|^2
&\leqslant \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-t_nu_n\|^2 \\
&\quad +\alpha_n\bar{\gamma}^2\|x_n-p\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_nu_n\| \\
&\quad +2\alpha_n\left(\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+\frac{\gamma_n\|e_n\|M}{\alpha_n}\right) \\
&\leqslant (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\beta_n^2\|x_n-t_nu_n\|^2 \\
&\quad +\alpha_n\bar{\gamma}^2\|x_n-p\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_nu_n\| \\
&\quad +2\alpha_n\left(\gamma\|f(x_n)\|+\|Bp\|+\frac{\gamma_n\|e_n\|}{\alpha_n}\right)M.
\end{aligned}$$

Since $\delta(L\delta-1)>0$, $\|x_{n+1}-x_n\|\to 0$ and $\|x_n-t_nu_n\|\to 0$ as $n\to\infty$ and from (i) and (ii), we obtain

$$\lim_{n\to\infty}\left\|\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n\right\|^2=0. \tag{3.12}$$

Next, we show that $\|x_n-u_n\|\to 0$ as $n\to\infty$. Since $p\in\Theta$, we can obtain

$$\|u_n-p\|^2\leqslant\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\left\|\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n\right\|,$$

see [11]. It follows from (3.11) and (3.12) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leqslant (1-\alpha_n\bar{\gamma})^2\|u_n-p\|^2+\beta_n^2\|x_n-t_nu_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\| \\
&\quad \times\|x_n-t_nu_n\|+2\alpha_n\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle \\
&\quad +2\gamma_n\|e_n\|M \\
&\leqslant (1-\alpha_n\bar{\gamma})^2\left[\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\left\|\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n\right\|\right] \\
&\quad +\beta_n^2\|x_n-t_nu_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_nu_n\| \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+2\gamma_n\|e_n\|M \\
&\leqslant \|x_n-p\|^2+(\alpha_n\bar{\gamma})^2\|x_n-p\|^2-(1-\alpha_n\bar{\gamma})^2 \\
&\quad \|u_n-x_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n-x_n)\|\left\|\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n\right\| \\
&\quad +\beta_n^2\|x_n-t_nu_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\| \\
&\quad \|x_n-t_nu_n\| \\
&\quad +2\alpha_n\left(\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+\frac{\gamma_n\|e_n\|M}{\alpha_n}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1-\alpha_n\bar{\gamma})^2\|u_n-x_n\|^2 &\leqslant \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-t_nu_n\|^2 \\
&\quad +\alpha_n\bar{\gamma}^2\|x_n-p\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_nu_n\| \\
&\quad +2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n-x_n)\|\left\|\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n\right\| \\
&\quad +2\alpha_n\left(\langle\gamma f(x_n)-Bp, x_{n+1}-p\rangle+\frac{\gamma_n\|e_n\|M}{\alpha_n}\right) \\
&\leqslant (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\beta_n^2\|x_n-t_nu_n\|^2 \\
&\quad +2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n-x_n)\|\left\|\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n\right\| \\
&\quad +\alpha_n\bar{\gamma}^2\|x_n-p\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_nu_n\| \\
&\quad +2\alpha_n\left(\gamma\|f(x_n)\|+\|Bp\|+\frac{\gamma_n\|e_n\|}{\alpha_n}\right)M.
\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$, $\left\|\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n\right\|\to 0$ and $\|x_n-t_nu_n\|\to 0$ as $n\to\infty$ and from (i) and (iv), we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.13}$$

Using (3.10) and (3.13), we obtain

$$\|t_nu_n-u_n\|\leqslant\|t_nu_n-x_n\|+\|x_n-u_n\|\to 0 \quad \text{as } n\to\infty.$$

Next, we show that $\limsup_{n\to\infty}\langle(\gamma f-B)z, x_n-z\rangle\leqslant 0$, where $z=P_\Theta(I-B+\gamma f)z$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f-B)z, u_n-z\rangle=\lim_{i\to\infty}\langle(\gamma f-B)z, u_{n_i}-z\rangle. \tag{3.14}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to some $w\in C$. Without loss of generality, we can assume that $u_{n_i}\rightharpoonup w$. From $\|t_nu_n-u_n\|\to 0$, we obtain $t_nu_{n_i}\rightharpoonup w$.

Now, we prove that $w\in\bigcap_{i=1}^n\mathrm{Fix}(T^i)\cap\Gamma$. Let us first show that $w\in\mathrm{Fix}(t_n)=\frac{1}{n+1}\sum_{i=0}^n\mathrm{Fix}(T^i)$. Assume that $w\notin\frac{1}{n+1}\sum_{i=0}^n\mathrm{Fix}(T^i)$. Since $u_{n_i}\rightharpoonup w$ and $t_nw\neq w$. Form Opial's condition (2.5), we have

$$\begin{aligned}
\liminf_{i\to\infty}\|u_{n_i}-w\| &< \liminf_{i\to\infty}\|u_{n_i}-t_nw\| \\
&\leqslant \liminf_{i\to\infty}\{\|u_{n_i}-t_nu_{n_i}\|+\|t_nu_{n_i}-t_nw\|\} \\
&\leqslant \liminf_{i\to\infty}\|u_{n_i}-w\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $w\in\mathrm{Fix}(t_n)=\frac{1}{n+1}\sum_{i=0}^n\mathrm{Fix}(T^i)$.

Next, we show that $w\in\mathrm{sol}(\mathrm{GVEP}(1.4))$. Since $u_n=T_{r_n}^{(F_1,\phi_1)}d_n$ where $d_n:=x_n+\delta A^*\left(T_{r_n}^{(F_2,\phi_2)}-I\right)Ax_n$, we have

$$F_1(u_n,y)+\phi_1(y)-\phi_1(u_n)+\frac{e}{r_n}\langle y-u_n, u_n-d_n\rangle\in P,$$
$$\forall y\in C, \tag{3.15}$$

which implies that

$$0\in F_1(y,u_n)-(\phi_1(y)-\phi_1(u_n))-\frac{e}{r_n}\langle y-u_n, u_n-d_n\rangle+P,$$
$$\forall y\in C.$$

Let $y_t=(1-t)w+ty$ for all $t\in(0,1]$. Since $y\in C$ and $w\in C$, we get $y_t\in C$ and now (3.15) shows that

$$0\in F_1(y_t,u_{n_i})-(\phi_1(y_t)-\phi_1(u_{n_i}))-e\left\langle y_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}+\delta A^*\left(\frac{\left(T_{r_{n_i}}^{(F_2,\phi_2)}-I\right)Ax_{n_i}}{r_{n_i}}\right)\right\rangle+P. \tag{3.16}$$

Since A^* is bounded linear, it follows from (3.12) and (3.13) and $\liminf r_n>0$ that $\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\to 0$ and $A^*\left(\frac{\left(T_{r_{n_i}}^{(F_2,\phi_2)}-I\right)Ax_{n_i}}{r_{n_i}}\right)\to 0$, and so

$$0\in F_1(y_t,w)-(\phi_1(y_t)-\phi_1(w))+P. \tag{3.17}$$

It follows from Assumption 2.1 (i) and (iii) that

$$\begin{aligned}
&tF_1(y_t,y)+(1-t)F_1(y_t,w)+t\phi_1(y)+(1-t)\phi_1(w)-\phi_1(y_t) \\
&\quad \in F_1(y_t,y_t)+\phi_1(y_t)-\phi_1(y_t)+P=P,
\end{aligned}$$

which implies that

$$-t[F_1(y_t,y)+\phi_1(y)-\phi_1(y_t)]-(1-t)[F_1(y_t,w)+\phi_1(w)-\phi_1(y_t)]\in -P. \quad (3.18)$$

From (3.17) and (3.18), we get

$$-t[F_1(y_t,y)+\phi_1(y)-\phi_1(y_t)]\in (1-t)[F_1(y_t,w)+\phi_1(w)-\phi_1(y_t)]-P\in -P$$

and so

$$-t[F_1(y_t,y)+\phi_1(y)-\phi_1(y_t)]\in -P.$$

It follows that

$$F_1(y_t,y)+\phi_1(y)-\phi_1(y_t)\in P.$$

Letting $t\to 0$, we obtain

$$F_1(w,y)+\phi_1(y)-\phi_1(w)\in P, \forall y\in C.$$

This implies that $w\in \text{sol(GVEP(1.4))}$.

Next, we show that $Aw\in \text{sol(GVEP(1.5))}$. Since $\|u_n-x_n\|\to 0$, $u_n\rightharpoonup w$ as $n\to\infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup w$ and since A is a bounded linear operator so that $Ax_{n_k}\rightharpoonup Aw$.

Now setting $v_{n_k}=Ax_{n_k}-T_{r_{n_k}}^{(F_2,\phi_2)}Ax_{n_k}$. It follows that from (3.12) that $\lim_{k\to\infty}v_{n_k}=0$ and $Ax_{n_k}-v_{n_k}=T_{r_{n_k}}^{(F_2,\phi_2)}Ax_{n_k}$.

Therefore from Lemma 2.6, we have

$$F_2(Ax_{n_k}-v_{n_k},z)+\phi_1(z)-\phi_1(u_{n_k})+\frac{e}{r_{n_k}}\langle z-(Ax_{n_k}-v_{n_k}), (Ax_{n_k}-v_{n_k})-Ax_{n_k}\rangle\in P, \quad \forall z\in Q.$$

Since F_2 is upper semicontinuous in first argument and P is closed, taking lim sup to above inequality as $k\to\infty$ and using condition (iii), we obtain

$$F_2(Aw,z)+\phi_1(z)-\phi_1(u_{n_k})\in P, \quad \forall z\in Q,$$

which means that $Aw\in \text{sol(GVEP(1.5))}$ and hence $w\in\Gamma$.

Next, we claim that $\limsup_{n\to\infty}\langle(\gamma f-B)z,x_n-z\rangle\leqslant 0$, where $z=P_\Theta(I-B+\gamma f)z$. Now from (2.2), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f-B)z,x_n-z\rangle &=\limsup_{n\to\infty}\langle(\gamma f-B)z,t_nu_n-z\rangle\\ &\leqslant \limsup_{i\to\infty}\langle(\gamma f-B)z,t_nu_{n_i}-z\rangle\\ &=\langle(\gamma f-B)z,w-z\rangle\leqslant 0. \end{aligned} \quad (3.19)$$

Finally, we show that $x_n\to z$. It follows from (3.3) that

$$\begin{aligned}\|x_{n+1}-z\|^2 &=\alpha_n\langle\gamma f(x_n)-Bz,x_{n+1}-z\rangle+\beta_n\langle x_n-z,x_{n+1}-z\rangle\\ &\quad+\langle((1-\beta_n)I-\alpha_nB)(t_nu_n-z)+\gamma_ne_n,x_{n+1}-z\rangle\\ &\leqslant \alpha_n(\gamma\langle f(x_n)-f(z),x_{n+1}-z\rangle+\langle\gamma f(z)-Bz,x_{n+1}-z\rangle)\\ &\quad+\beta_n\|x_n-z\|\|x_{n+1}-z\|+\|(1-\beta_n)I-\alpha_nB\|\|t_nu_n\\ &\quad-z\|\|x_{n+1}-z\|\\ &\quad+\gamma_n\|e_n\|M\\ &\leqslant \alpha_n\alpha\gamma\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle\\ &\quad+\beta_n\|x_n-z\|\|x_{n+1}-z\|+(1-\beta_n-\alpha_n\bar{\gamma})\\ &\quad\|x_n-z\|\|x_{n+1}-z\|+\gamma_n\|e_n\|M\\ &=[1-\alpha_n(\bar{\gamma}-\gamma\alpha)]\|x_n-z\|\|x_{n+1}-z\|+\gamma_n\|e_n\|M\\ &\quad+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle\\ &\leqslant\frac{1-\alpha_n(\bar{\gamma}-\gamma\alpha)}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2)\\ &\quad+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle+\gamma_n\|e_n\|M\\ &\leqslant\frac{1-\alpha_n(\bar{\gamma}-\gamma\alpha)}{2}\|x_n-z\|^2+\frac{1}{2}\|x_{n+1}-z\|^2\\ &\quad+\alpha_n\langle\gamma f(z)-Bz,x_{n+1}-z\rangle+\gamma_n\|e_n\|M.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leqslant [1-\alpha_n(\bar{\gamma}-\gamma\alpha)]\|x_n-z\|^2\\ &\quad+2\alpha_n\left(\langle\gamma f(z)-Bz,x_{n+1}-z\rangle+\frac{\gamma_n\|e_n\|M}{\alpha_n}\right)\\ &=[1-\alpha_n(\bar{\gamma}-\gamma\alpha)]\|x_n-z\|^2+2\alpha_nM_n.\end{aligned} \quad (3.20)$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$, it is easy to see that $\limsup_{n\to\infty}M_n\leqslant 0$. Hence, from (3.19) and (3.20) and Lemma 2.2, we deduce that $x_n\to z$, where $z=P_\Theta(I+\gamma f-B)$. This completes the proof. □

Remark 3.1. The method presented in this paper extend, improve and unify the methods considered in [11–14]. Moreover, the algorithm and approach considered in Theorem 3.1 are different from those considered in [15,16].

4. Numerical example

Now, we give a numerical example which justify Theorem 3.1.

Example 4.1. Let $H_1=H_2=\mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x,y\rangle=xy$, $\forall x,y\in\mathbb{R}$, and induced usual norm $|\cdot|$. Let $Y=\mathbb{R}$, then $P=[0,+\infty)$. Let $C=[0,2]$ and $C=[-4,0]$; let $F_1:C\times C\to\mathbb{R}$ and $F_2:Q\times Q\to\mathbb{R}$ be defined by $F_1(x,y)=(x-6)(y-x)$, $\forall x,y\in C$ and $F_2(u,v)=(u+1)(v-u)$, $\forall u,v\in Q$; let $\phi_1:C\to\mathbb{R}$ and $\phi_2:Q\to\mathbb{R}$ be defined by $\phi_1(x)=4x$, $\forall x\in C$ and $\phi_2(u)=3u$, $\forall u\in Q$, respectively, and let for each $x\in\mathbb{R}$, we define $f(x)=\frac{1}{8}x$, $A(x)=-2x$, $B(x)=2x$, $e_n=sin(n)$, $\forall n$ and let, for each $x\in C$, $T(x)=x$. Then there exist unique sequences $\{x_n\}\subset\mathbb{R}$, $\{u_n\}\subset C$, and $\{z_n\}\subset Q$ generated by the iterative schemes

$$z_n=T_{r_n}^{F_2}(Ax_n);\quad u_n=T_{r_n}^{F_1}\left[x_n+\frac{1}{8}A^*(z_n-Ax_n)\right]; \quad (4.1)$$

$$x_{n+1}=\frac{1}{4n}x_n+\left[0.1+\frac{1}{n^2}\right]x_n+\left[\left(1-\left(0.1+\frac{1}{n^2}\right)\right)I-\frac{1}{n^2}B\right]u_n+\frac{1}{n^3}\sin(n), \quad (4.2)$$

where $\alpha_n=\frac{1}{n}$, $\beta_n=0.1+\frac{1}{n^2}$, $\gamma_n=\frac{1}{n^3}$ and $r_n=1$. Then $\{x_n\}$ converges strongly to $2\in \text{Fix}(T)\cap\Gamma$.

Proof. It is easy to prove that the bifunctions F_1 and F_2 and mappings ϕ_1 and ϕ_2 satisfy the Assumption 2.1 and F_2 is upper semicontinuous. A is a bounded linear operator on $\mathbb{R}$ with adjoint operator A^* and $\|A\|=\|A^*\|=2$. Hence $\delta\in(0,\frac{1}{4})$, so we can choose $\delta=\frac{1}{8}$. Further, f is contraction mapping with constant $\alpha=\frac{1}{5}$ and B is a strongly positive bounded linear operator with constant $\bar{\gamma}=1$ on $\mathbb{R}$. Therefore, we can choose $\gamma=2$ which satisfies $0<\gamma<\frac{\bar{\gamma}}{\alpha}<\gamma+\frac{1}{\alpha}$. Furthermore, it is easy

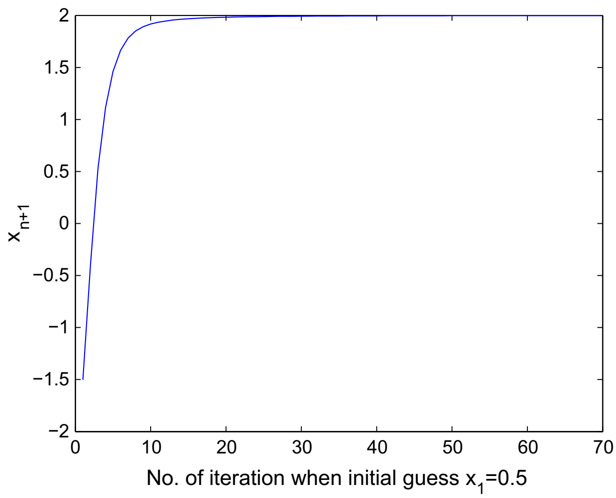

Figure 1 Convergence of $\{x_n\}$.

to observe that $\text{Fix}(T) = (0, \infty)$, $\text{sol}(\text{GVEP}(1.4) = \{2\}$, $\text{sol}(\text{GVEP}(1.5)) = \{-4\}$. Hence $\Gamma := \{2\}$. Consequently, $\text{Fix}(T) \cap \Gamma = \{2\} \neq \emptyset$. After simplification, schemes (4.1) and (4.2) reduce to

$$z_n = -(x_n + 2); \quad u_n = \frac{1}{8}(3x_n + 10); \tag{4.3}$$

$$x_{n+1} = \left[\frac{1}{8n} + \frac{3.5}{8}\right]x_n + \frac{4.5}{4} - \frac{15}{4n} + \frac{1}{n^3}\sin(n). \tag{4.4}$$

Following the proof of Theorem 3.1, we obtain that $\{z_n\}$ converges strongly to $-4 \in \text{sol}(\text{GVEP}(1.5))$ and $\{x_n\}, \{u_n\}$ converge strongly to $w = 2 \in \text{Fix}(T) \cap \Gamma$ as $n \to \infty$.

Next, using the software Matlab 7.0, we have Fig. 1 which shows that $\{x_n\}$ converges strongly to 2.

The proof is completed. □

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