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Weak and strong mixed vector equilibrium problems on non-compact domain

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Abstract In this paper, we consider two mixed vector equilibrium problems i.e., a weak mixed vector equilibrium problem and a strong mixed vector equilibrium problem which are combinations of a vector equilibrium problem and a vector variational inequality problem. We prove existence results for both the problems in non-compact setting.

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1. Introduction

Many problems of practical interest in optimization, economics and engineering involve equilibrium in their description. The equilibrium problem was first introduced and studied by Blum and Oettli [1] as a generalization of variational inequality problem. It has been shown that the equilibrium problem provides a natural, novel and unified framework to study a wide class of problems arising in nonlinear analysis, optimization, economics, finance and game theory. The equilibrium problem includes many mathematical problems as particular cases such as mathematical programming problems, complementarity problems, variational inequality problems, fixed point problems, minimax inequality problems, and Nash equilibrium problems in noncooperative games. see [1–4].

Let X be a Hausdorff topological vector space, K be a subset of X, and $f : K \times K \to \mathbb{R}$ be a mapping with $f(x,x) = 0$. The classical, scalar-valued equilibrium problem deals with the existence of $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geqslant 0; \quad \forall y \in K.$$

Moreover, in the case of vector valued mappings, let Y be a another Hausdorff topological vector space, $C \subset Y$ a cone. Given a vector mapping $f : K \times K \to Y$, then the problem of finding $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -intC; \quad \forall y \in K,$$

is called weak equilibrium problem and the point $\bar{x} \in K$ is called weak equilibrium point, where $intC$ denotes the interior of the cone C in Y.

In this paper, we consider two types of mixed vector equilibrium problems which are combinations of a vector equilibrium problem and a vector variational inequality problem. Let X and Y be two Hausdorff topological vector spaces. Let K be a nonempty convex closed subset of X and $C \subseteq Y$ a

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pointed closed convex cone with nonempty interior i.e., $intC \neq \emptyset$. The partial order "$\leq_C$" on Y induced by C is defined by $x \leq_C y$ if and only if $y - x \in C$. Let $f: K \times K \to Y$ and $T: K \to L(X, Y)$ be two mappings, where $L(X, Y)$ is the space of all linear continuous mappings from X to Y. Here $\langle T(x), y \rangle$ denotes the evaluation of the linear mapping $T(x)$ at y. Now, we consider the following two problems:

Find $\bar{x} \in K$ such that

$$f(\bar{x}, y) + \langle T(\bar{x}), y - \bar{x} \rangle \notin -intC; \quad \forall y \in K. \tag{1.1}$$

and

$$f(\bar{x}, y) + \langle T(\bar{x}), y - \bar{x} \rangle \notin -C \setminus \{0\}; \quad \forall y \in K. \tag{1.2}$$

We call problem (1.1) as weak mixed vector equilibrium problem and problem (1.2) as strong mixed vector equilibrium problem. Problems (1.1) and (1.2) are unified models of several known problems used in applied sciences, for instance, vector variational inequality problem, vector complementarity problem, vector optimization problem and vector saddle point problem, see e.g. [3,5–9] and references therein. For a more comprehensive bibliography on vector equilibrium problems, vector variational inequality problems and their generalizations, we refer to volume edited by Giannessi [3]. Our results generalize the results obtained by Blum and Oettli [1] and therefore the results of Fan [10] for vector valued mappings. For more details, we refer to [5,11,12]. As the underlying set K is non-compact, therefore we use only a very weak coercivity condition i.e., coercing family.

2. Preliminaries

The following definitions and results are needed in the sequel.

Definition 2.1. Let $g: K \to Y$ be a mapping. Then g is said to be C-convex, if for all $x, y \in K$ and $\lambda \in [0, 1]$

$$g(\lambda x + (1 - \lambda)y) \leq_C \lambda g(x) + (1 - \lambda)g(y),$$

which implies that

$$g(\lambda x + (1 - \lambda)y) \in \lambda g(x) + (1 - \lambda)g(y) - C.$$

Definition 2.2. A mapping $g: K \to Y$ is said to be

(i) lower semicontinuous with respect to C at a point $x_0 \in K$, if for any neighborhood V of $g(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that

$$g(U \cap K) \subseteq V + C;$$

(ii) upper semicontinuous with respect to C at a point $x_0 \in K$, if

$$g(U \cap K) \subseteq V - C;$$

(iii) continuous with respect to C at a point $x_0 \in K$, if it is lower semicontinuous and upper semicontinuous with respect to C at that point.

Remark 2.1. If g is lower semicontinuous, upper semicontinuous and continuous with respect to C at any point of K, then g is lower semicontinuous, upper semicontinuous and continuous with respect to C on K, respectively.

Definition 2.3. A mapping $f: K \times K \to Y$ is said to be C-monotone, if for all $x, y \in K$

$$f(x, y) + f(y, x) \in -C.$$

Lemma 2.1 [9]. *If g is a lower semicontinuous mapping with respect to C, then the set $\{x \in K : g(x) \notin intC\}$ is closed in K.*

Lemma 2.2 [13]. *Let (Y, C) be an ordered topological vector space with a pointed closed convex cone C. Then for all $x, y \in Y$, we have*

(i) *$y - x \in intC$ and $y \notin intC$ imply $x \notin intC$;*
(ii) *$y - x \in C$ and $y \notin intC$ imply $x \notin intC$;*
(iii) *$y - x \in -intC$ and $y \notin -intC$ imply $x \notin -intC$;*
(iv) *$y - x \in -C$ and $y \notin -intC$ imply $x \notin -intC$.*

Definition 2.4 [14]. Consider a subset K of a topological vector space and a topological space Y. A family $\{(C_i, Z_i)\}_{i \in I}$ of pair of sets is said to be coercing for a mapping $F: K \to 2^Y$ if and only if

(i) for each $i \in I, C_i$ is contained in a compact convex subset of K and Z_i is a compact subset of Y;
(ii) for each $i, j \in I$, there exists $k \in I$ such that $C_i \cup C_j \subseteq C_k$;
(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} F(x) \subseteq Z_i$.

Definition 2.5. Let K be a nonempty convex subset of a topological vector space X. A multivalued mapping $G: K \to 2^X$ is said to be KKM mapping, if for every finite subset $\{x_i\}_{i \in I}$ of K,

$$Co\{x_i : i \in I\} \subseteq \bigcup_{i \in I} F(x_i),$$

where $Co\{x_i : i \in I\}$ denotes the convex hull of $\{x_i\}_{i \in I}$.

Theorem 2.1 [14]. *Let X be a Hausdorff topological vector space, Y a convex subset of X, K a nonempty subset of Y and $F: K \to 2^Y$ a KKM mapping with compactly closed values in Y (i.e., for all $x \in K$, $F(x) \cap Z$ is closed for every compact set Z of Y). If F admits a coercing family, then*

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Condition(C): We say that the cone C satisfies *Condition*(C), if there is a pointed convex closed cone $\widetilde{C}$ such that $C \setminus \{0\} \subseteq int\widetilde{C}$.

3. Existence results

In this section, we prove the following existence results for weak and strong mixed vector equilibrium problems (1.1) and (1.2) for non-compact domain.

Theorem 3.1. *Let K be a nonempty closed convex subset of a Hausdorff topological vector space X, Y a Hausdorff topological vector space and C a closed convex pointed cone in Y with $intC \neq \emptyset$. Let $f: K \times K \to Y$ and $T: K \to L(X, Y)$ be two mappings satisfying the following conditions:*

(i) *f is C-monotone;*
(ii) $f(x,x)=0$, *for all* $x \in K$;
(iii) *for any fixed* $x, y \in K$, $t \in [0,1] \mapsto f(ty+(1-t)x, y) \in Y$ *is upper semicontinuous with respect to C at* $t = 0$;
(iv) *for any fixed* $x \in K$, $f(x,\cdot): K \to Y$ *is C-convex, lower semicontinuous with respect to C on K;*
(v) *T is upper semicontinuous with respect to C with non-empty closed values;*
(vi) *there exists a family* $\{(C_i, Z_i)\}_{i \in I}$ *satisfying conditions (i) and (ii) of Definition 2.4 and the following condition: For each* $i \in I$, *there exists* $k \in I$ *such that*

$$\{x \in K : f(y,x) - \langle T(x), y-x \rangle \notin intC, \forall y \in C_k\} \subset Z_i.$$

Then, there exists a point $\bar{x} \in K$ *such that*

$$f(\bar{x}, y) + \langle T(\bar{x}), y-\bar{x} \rangle \notin -intC; \quad \forall y \in K.$$

For the proof of Theorem 3.1, we need the following proposition, for which the assumptions remain same as in Theorem 3.1.

Proposition 3.1. *The following two problems are equivalent:*

(i) *Find* $\bar{x} \in K$ *such that* $f(y,\bar{x}) - \langle T(\bar{x}), y-\bar{x} \rangle \notin intC$; $\forall y \in K$;
(ii) *Find* $\bar{x} \in K$ *such that* $f(\bar{x},y) + \langle T(\bar{x}), y-\bar{x} \rangle \notin -intC$; $\forall y \in K$.

Proof. Suppose (I) holds. Then for fixed $y \in K$, set $x_t = ty + (1-t)\bar{x}$, for $t \in [0,1]$. It is clear that $x_t \in K$, for all $t \in [0,1]$ and hence

$$f(x_t, \bar{x}) - \langle T(\bar{x}), x_t - \bar{x} \rangle \notin intC. \tag{3.1}$$

Since $f(x,x)=0$ and $f(x,\cdot)$ is C-convex, we have

$$0 = f(x_t, x_t) \leq_C tf(x_t, y) + (1-t)f(x_t, \bar{x})$$
$$\Rightarrow tf(x_t, y) + (1-t)f(x_t, \bar{x}) \in C. \tag{3.2}$$

Also,

$$\langle T(\bar{x}), x_t - \bar{x} \rangle = t\langle T(\bar{x}), y-\bar{x} \rangle \Rightarrow (1-t)t\langle T(\bar{x}), y-\bar{x} \rangle - (1-t)\langle T(\bar{x}), x_t - \bar{x} \rangle = 0. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$tf(x_t,y) + (1-t)\{f(x_t,\bar{x}) - \langle T(\bar{x}), x_t-\bar{x} \rangle\} + (1-t)t\langle T(\bar{x}), y-\bar{x} \rangle \in C, \tag{3.4}$$

for all $t \in [0,1]$.

Using (3.1) and (3.4) and (ii) of Lemma 2.2, we have

$$tf(x_t,y) + (1-t)t\langle T(\bar{x}), y-\bar{x} \rangle \notin -intC$$
$$\Rightarrow f(x_t,y) + (1-t)\langle T(\bar{x}), y-\bar{x} \rangle \notin -intC, \forall t \in (0,1]. \tag{3.5}$$

By condition (iii) of Theorem 3.1 as $t \mapsto f(ty+(1-t)x, y)$ is upper semicontinuous with respect to C at $t=0$, therefore from (3.5) we have

$$f(\bar{x}, y) + \langle T(\bar{x}), y-\bar{x} \rangle \notin -intC,$$

and hence (II) holds.

Conversely, Assume that (II) holds for all $y \in K$. In order to prove (I), on contrary suppose that there exists a point $\bar{y} \in K$ such that

$$f(\bar{y},\bar{x}) - \langle T(\bar{x}), \bar{y}-\bar{x} \rangle \in intC \Rightarrow f(\bar{y},\bar{x}) = \langle T(\bar{x}), \bar{y}-\bar{x} \rangle + w; \tag{3.6}$$

for some $w \in intC$.

On the other hand, since f is C-monotone, we have

$$f(\bar{x},\bar{y}) + f(\bar{y},\bar{x}) \in -C \Rightarrow f(\bar{y},\bar{x}) = -f(\bar{x},\bar{y}) - v; \tag{3.7}$$

for some $v \in C$.

Combining (3.6) and (3.7), we have

$$f(\bar{x},\bar{y}) + \langle T(\bar{x}), \bar{y}-\bar{x} \rangle = -w - v \in -intC;$$

which contradicts assumption (II). Therefore (I) holds. □

Proof of Theorem 3.1. For each $y \in K$, consider the set

$$F(y) = \{x \in K : f(y,x) - \langle T(x), y-x \rangle \notin intC\}.$$

By Lemma 2.1, $F(y)$ is closed in K and hence F has compactly closed values in K.

Now, we show that F is a KKM map. For this, let $\{y_i : i \in I\}$ be a finite subset of K and $u \in Co\{y_i : i \in I\}$.

We claim that

$$Co\{y_i : i \in I\} \subseteq \bigcup_{i \in I} F(y_i).$$

In contrary, suppose that $u \notin \bigcup_{i \in I} F(y_i)$. As $u \in Co\{y_i : i \in I\}$, we have $u = \sum_{i \in I} \lambda_i y_i$ with $\lambda_i \geqslant 0$ and $\sum_{i \in I} \lambda_i = 1$.

This follows that

$$f(y_i, u) - \langle T(u), y_i - u \rangle \in intC.$$

Since $intC$ in convex, therefore

$$\sum_{i \in I} \lambda_i \{f(y_i,u) - \langle T(u), y_i - u \rangle\} \in intC. \tag{3.8}$$

Since $f(x,\cdot)$ is C-convex and C-monotone, we have

$$\sum_{i \in I} \lambda_i f(y_i, u) \leq_C \sum_{i,j \in I} \lambda_i \lambda_j f(y_i, y_j)$$
$$= \frac{1}{2} \sum_{i,j \in I} \lambda_i \lambda_j \{f(y_i,y_j) + f(y_j,y_i)\} \leq_C 0. \tag{3.9}$$

Furthermore,

$$0 = \langle T(u), u-u \rangle = \left\langle T(u), \sum_{i \in I} \lambda_i y_i - \sum_{i \in I} \lambda_i u \right\rangle$$
$$= \left\langle T(u), \sum_{i \in I} \lambda_i (y_i - u) \right\rangle = \sum_{i \in I} \lambda_i \langle T(u), y_i - u \rangle. \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\sum_{i \in I} \lambda_i \langle T(u), y_i - u \rangle - \sum_{i \in I} \lambda_i f(y_i, u) \in C$$
$$\Rightarrow \sum_{i \in I} \lambda_i \{f(y_i,u) - \langle T(u), y_i - u \rangle\} \in -C. \tag{3.11}$$

From (3.8) and (3.11), we conclude that

$$\sum_{i \in I} \lambda_i \{f(y_i,u) - \langle T(u), y_i - u \rangle\} \in intC \cap (-C) = \varnothing,$$

which is a contradiction. This follows that $u \in \bigcup_{i \in I} F(y_i)$ and hence $Co\{y_i : i \in I\} \subseteq \bigcup_{i \in I} F(y_i)$. Thus, F is a KKM mapping.

From the assumption (*vi*), we can see that the family $\{(C_i, Z_i)\}_{i \in I}$ satisfies the condition which is for all $i \in I$, there exists $k \in I$ such that

$$\bigcap_{y \in C_k} F(y) \subset Z_i;$$

and therefore it is a coercing family for F.

We deduce that F satisfies all the hypothesis of Theorem 2.1. Therefore, we have

$$\bigcap_{y \in K} F(y) \neq \emptyset.$$

Hence, there exists $\bar{x} \in K$ such that for any $y \in K$

$$f(y, \bar{x}) - \langle T(\bar{x}, y - \bar{x}) \rangle \notin intC.$$

Now applying Proposition 3.1, we obtain that there exists $\bar{x} \in K$ such that for all $y \in K$

$$f(\bar{x}, y) + \langle T(\bar{x}, y - \bar{x}) \rangle \notin -intC.$$

Hence problem (1.1) admits a solution. This completes the proof. □

Corollary 3.1. *Let K, C, $\{(C_i, Z_i)\}_{i \in I}$, f and T satisfy all the assumptions of Theorem* 3.1. *In addition, if C satisfies Condition(C), then the problem (1.2) is solvable i.e., there exists $\bar{x} \in K$ such that for any $y \in K$*

$$f(\bar{x}, y) + \langle T(\bar{x}), y - \bar{x} \rangle \notin -(C \setminus \{0\}).$$

Proof. Let us suppose that C satisfies *Condition*(C). Then there is a pointed convex and closed cone $\widetilde{C}$ in Y such that

$$C \setminus \{0\} \subseteq int\widetilde{C}.$$

Therefore we can easily see that K, C, $\{(C_i, Z_i)\}_{i \in I}$, f and T satisfy all the assumptions of Theorem 3.1. Therefore by Theorem 3.1, we get

$$f(\bar{x}, y) + \langle T(\bar{x}), y - \bar{x} \rangle \notin -int\widetilde{C}; \quad \forall y \in K. \tag{3.12}$$

Since $-(C \setminus \{0\}) \subseteq -int\widetilde{C}$, it follows from (3.12) that there exists $\bar{x} \in K$ such that

$$f(\bar{x}, y) + \langle T(\bar{x}), y - \bar{x} \rangle \notin -(C \setminus \{0\}); \quad \forall y \in K.$$

Hence problem (1.2) admits a solution. This completes the proof. □

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