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ORIGINAL ARTICLE

Two extensions of coupled coincidence point results for nonlinear contractive mappings

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Abstract In this paper, the notion of mixed f, g monotone mapping is introduced, and the coupled coincidence point theorem for nonlinear contractive mappings in partially ordered complete metric spaces has been proved. Presented theorems are generalizations of the recent fixed point theorems due to Lakshmikantham and Ćirić (2009) [17] and include several recent developments. Also, using the theory of countable extension of t-norm, it has been proved that a common fixed point theorem given in Ćirić (2011) [12] hold for a more general classes of t-norms in fuzzy metric spaces. Their theorem can be used to investigate a large class of problems and has discussed the existence and uniqueness of solution for a periodic boundary value problem.

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1. Introduction and preliminaries

The Banach contraction principle is one of the most important fixed point theorem, end generalized in various directions. For more results, we refer [1–25]. Boyd and Wong [4] extended the Banach contraction principle to the case of nonlinear contraction mappings. Ran and Reurings [23] proved a Banach contraction principle in partially ordered metric spaces. After that, many authors have continued research (see [2,3,17,20,21]). In a recent papers, Bhaskar and Lakshmikantham [3] and Lakshmikantham and Ćirić [17] proved a coupled fixed point results for mixed monotone and contraction mapping in partially ordered metric spaces. Bhaskar and Lakshmikantham [3] noted that their theorem can be used to investigate a large class of problems and has discussed the existence and uniqueness of solution for a periodic boundary value problem.

Definition 1.1. Let $(X, \leqslant)$ be a partially ordered set and $F : X \to X$ is such that for $x, y \in X, x \leqslant y$ implies $F(x) \leqslant F(y)$.

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Then, a mapping F is said to be non-decreasing. Similarly, it is defined as a non-increasing mapping.

Lakshmikantham and Ćirić [17] introduced the following notions of a mixed monotone mapping and a coupled fixed point.

Definition 1.2 [17]. Let $(X, \leqslant)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \leqslant g(x_2) \text{ implies } F(x_1, y) \leqslant F(x_2, y) \qquad (1)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \leqslant g(y_2) \text{ implies } F(x, y_1) \geqslant F(x, y_2). \qquad (2)$$

Definition 1.3 [17]. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), \quad F(y, x) = g(y).$$

Definition 1.4 [17]. Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if

$$g(F(x, y)) = F(g(x), g(y))$$

for all $x, y \in X$.

The main theoretical results of Lakshmikantham and Ćirić in [17] are the following coupled coincidence point theorems.

Theorem 1.5 [17]. *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ and $\lim_{r \to t+} \varphi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \leqslant \varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right) \qquad (3)$$

for all $x, y, u, v \in X$ for which $g(x) \leqslant g(u)$ and $g(y) \geqslant g(v)$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) *F is continuous or*
(b) *X has the following property:*

$$(i) \text{ if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \leqslant x \text{ for all } n, \qquad (4)$$

$$(ii) \text{ if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \leqslant y_n \text{ for all } n. \qquad (5)$$

If there exists $x_0, y_0 \in X$ such that

$$g(x_0) \leqslant F(x_0, y_0) \text{ and } g(y_0) \geqslant F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x),$$

that is, F and g have a coupled coincidence.

Recently, coupled coincidence point results can see in [26–29]. Inspired with Definition 1.3 we introduce in this paper the concept of a mixed fg-monotone mapping and prove a coupled coincidence fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces.

Since the probabilistic metric spaces introduced by Menger [30] are a natural generalization of a metric spaces, Ćirić et al. [12,13] introduced a concept of monotone-generalized contraction in partially ordered probabilistic metric space, and they proved a common fixed point theorem. In [12] Ćirić et al. introduced the concept of mixed monotone-generalized contraction in partially ordered probabilistic metric space, and they proved a coupled coincidence and coupled fixed point theorem where they used a t-norm of H-type. Inspired with that in this paper, we proved that the result in [12] hold for a more general class of t-norms.

Through this paper with Δ^+, we denoted the space of all distribution function, i.e. $\Delta^+ = \{F : \mathbb{R} \cup [0, 1] \to [0, 1] : F$ is left continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$, where the $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of function, i.e. $F \leqslant G$ if and only if $F(x) \leqslant G(x)$ for all $x \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leqslant 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.6. A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (a t-norm) if the following conditions are satisfied:

- $T(a, 1) = a$ for all $a \in [0, 1]$;
- $T(a, b) = T(b, a)$ for all $a, b \in [0, 1]$;
- $a \geqslant b,\ c \geqslant d \Rightarrow \quad T(a, c) \geqslant T(b, d) \quad (a, b, c, d \in [0, 1])$;
- $T(a, T(b, c)) = T(T(a, b), c) \quad (a, b, c \in [0, 1])$.

The following are the four basic t-norms (see [31]):

$$T_M(x, y) = \min(x, y), \quad T_P(x, y) = x \cdot y,$$
$$T_L(x, y) = \max(x + y - 1, 0)$$
$$T_D(x, y) = \begin{cases} \min(x, y) & \text{if} \max(x, y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Some important families of t-norms are given in the following example (see [31]):

Example 1.7.

(i) The Dombi family of t-norms $(T^D_\lambda)_{\lambda \in [0,\ \infty]}$, which is defined by

$$T^D_\lambda(x,\ y) = \begin{cases} T_D(x,\ y), & \lambda = 0 \\ T_M(x,\ y), & \lambda = \infty \\ \frac{1}{1 + \left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{1/\lambda}}, & \lambda \in (0,\ \infty). \end{cases}$$

(ii) The Aczél-Alsina family of t-norms $(T_\lambda^{AA})_{\lambda\in[0,\ \infty]}$, which is defined by

$$T_\lambda^{AA}(x,\ y)=\begin{cases}T_D(x,\ y), & \lambda=0\\ T_M(x,\ y), & \lambda=\infty\\ e^{-((-\log\ x)^\lambda+(-\log\ y)^\lambda)^{1/\lambda}}, & \lambda\in(0,\ \infty).\end{cases}$$

(iii) Sugeno-Weber family of t-norms $(T_\lambda^{SW})_{\lambda\in[-1,\ \infty]}$, which is defined by

$$T_\lambda^{SW}(x,\ y)=\begin{cases}T_D(x,\ y), & \lambda=-1\\ T_P(x,\ y), & \lambda=\infty\\ \max(0,\ \frac{x+y-1+\lambda xy}{1+\lambda}), & \lambda\in(-1,\ \infty).\end{cases}$$

The following class of t-norms, that has proved itself as a highly useful tool in the fixed point theory, was introduced in [15].

Definition 1.8 [15]. Let T be a t-norm, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of t-norms given by the following:

$$T_1(x)=T(x,\ x)\quad\text{and}\quad T_{n+1}(x)=T(T_n(x),\ x).$$

A t-norm T is of the H-type if T is continuous and the sequence $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x=1$.

Remark 1. The family $\{T_n(x)\}_{n\in\mathbb{N}}$ of t-norms is equicontinuous at $x=1$, if for all $\lambda\in(0,\ 1)$, there exists $\delta(\lambda)\in(0,\ 1)$ such that the following implication holds:

$$x>1-\delta(\lambda)\Rightarrow T_n(x)>1-\lambda\quad\text{for all}\quad n\in\mathbb{N}.$$

(see [15]). A trivial example of a t-norm of H-type is $T=T_M$. A non-trivial example can be found in [15].

An arbitrary t-norm T can be extended, due to the associativity, to an n-ary operation on $[0,1]^n$ (see [31]):

$$T(x_1,x_2,\ldots,x_n)=\mathbf{T}_{i=1}^n x_i=T(\mathbf{T}_{i=1}^{n-1}x_i,x_n)\quad\text{and}\quad\mathbf{T}_{i=1}^1 x_i=x_1.$$

Also, a t-norm T can be extend to a countable case as follows:

$$\mathbf{T}_{i=1}^\infty x_i=\lim_{n\to\infty}\mathbf{T}_{i=1}^n x_i,$$

where $(x_n)_{n\in\mathbb{N}}$ is an arbitrary sequence from $[0,1]$. The limit on the right-hand side exists since the sequence $(\mathbf{T}_{i=1}^n x_i)_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

In [16] the following equivalences and proposition proved and that will be used further:

- If $T=T_L$ or $T=T_P$, then

$$\lim_{n\to\infty}\mathbf{T}_{i=n}^\infty x_i=1\Longleftrightarrow\sum_{i=1}^\infty(1-x_i)<\infty.$$

- If $(T_\lambda^*)_{\lambda\in(0,\ \infty)}$ is the Dombi family of t-norms or the Aczél–Alsina family of t-norms and if $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements from $(0,\ 1]$ such that $\lim_{n\to\infty}x_n=1$, then

$$\lim_{n\to\infty}(\mathbf{T}_\lambda^*)_{i=n}^\infty x_i=1\Longleftrightarrow\sum_{i=1}^\infty(1-x_i)^\lambda<\infty.\tag{6}$$

- If $(T_\lambda^{SW})_{\lambda\in(-1,\ \infty]}$ is the Sugeno-Weber family of t-norms and $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements from $(0,\ 1]$ such that $\lim_{n\to\infty}x_n=1$, then

$$\lim_{n\to\infty}(\mathbf{T}_\lambda^{SW})_{i=n}^\infty x_i=1\Longleftrightarrow\sum_{i=1}^\infty(1-x_i)<\infty.\tag{7}$$

Proposition 1.9 [16]. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,\ 1]$ such that $\lim_{n\to\infty}x_n=1$ and t-norm T is of the H-type. Then,*

$$\lim_{n\to\infty}\mathbf{T}_{i=n}^\infty x_i=\lim_{n\to\infty}\mathbf{T}_{i=n}^\infty x_{n+i}=1.$$

Definition 1.10. The ordered triple $(X,\mathcal{F},T)$ is said to be a Menger probabilistic metric space if X is a non-empty set and $\mathcal{F}:X\times X\to\Delta^+$ ($\mathcal{F}(u,v)$ written by $F_{u,v}$ for every $(u,v)\in X\times X$) satisfies the following conditions:

- $F_{u,v}(x)=1$ for every $x>0\Rightarrow u=v\ (u,v\in X)$.
- $F_{u,v}=F_{v,u}$ for every $u,v\in X$.
- $F_{u,v}(x+y)\geqslant T(F_{u,w}(x),F_{w,v}(y))$ for every $u,v,w\in X$ and every $x>0,y>0$.

Definition 1.11. Let $(X,\mathcal{F},T)$ be a Menger probabilistic metric space.

- A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if for every $\varepsilon>0$ and $\lambda>0$ there exists $N\in\mathbb{N}$ such that $F_{x_n,x}(\varepsilon)>1-\lambda$ whenever $n\geqslant N$.
- A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is a Cauchy sequence if and only if for every $\varepsilon>0$ and $\lambda\in(0,\ 1)$ there exists $n_0(\varepsilon,\ \lambda)\in\mathbb{N}$ such that $F_{x_{n+p},x_n}(\varepsilon)>1-\lambda$ for every $n\geqslant n_0(\varepsilon,\ \lambda)$ and every $p\in\mathbb{N}$.
- If a Menger probabilistic metric space $(X,\ \mathcal{F},T)$ is such that every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges in X, then $(X,\ \mathcal{F},T)$ is a complete space.

Definition 1.12. The $(\epsilon,\lambda)-$ topology in Menger probabilistic metric spaces $(X,\mathcal{F},T)$ is introduced by the family of neighbourhoods $\mathcal{U}=\{U_v(\epsilon,\lambda)\}_{(v,\epsilon,\lambda)\in X\times\mathbb{R}_+\times(0,1)}$, where $U_v(\epsilon,\lambda)=\{u;\ F_{u,v}(\epsilon)>1-\lambda\}$. If a t-norm T is such that $\sup_{x<1}T(x,x)=1$, then S is in the (ϵ,λ) topology a metrizable topological space.

In [12] Ćirić et al. proved the following coupled coincidence point theorems.

Theorem 1.13. *Let $(X,\leqslant)$ be a partially ordered set and $(X,\mathcal{F},T)$ be a complete Menger probabilistic metric space under a t-norm T of H-type. Suppose $A:X\times X\to X$ and $f:X\to X$ are two mappings such that A has the f-mixed monotone property on X and for some $k\in(0,1)$,*

$$F_{A(x,y),A(u,v)}(kt)\geqslant\min\{F_{f(x),f(u)}(t),\ F_{f(y),f(v)}(t),\ F_{f(x),A(x,y)}(t),\tag{8}$$

$F_{f(u),A(u,v)}(t),\ F_{f(y),A(y,x)}(t),\ F_{f(v),A(v,u)}(t)\}$

for every $x, y \in X$ *for which* $f(x) \leqslant f(u)$ *and* $f(y) \geqslant f(v)$ *and every* $t > 0$.

Suppose also that $A(X \times X) \subseteq f(X)$, $f(X)$ *is closed and*

(i) if $\{f(x_n)\} \subset X$ *is a non-decreasing sequence with* $\{f(x_n)\} \to f(x)$, *then* $f(x_n) \leqslant f(x)$ *for all* n,

(ii) if $\{f(y_n)\} \subset X$ *is a non-increasing sequence with* $\{f(y_n)\} \to f(y)$, *then* $f(y) \leqslant f(y_n)$ *for all* n.

If there exists $x_0, y_0 \in X$ *such that*

$$f(x_0) \leqslant A(x_0, y_0) \text{ and } f(y_0) \geqslant A(y_0, x_0).$$

Then there exist $p, q \in X$ *such that*

$$f(p) = A(p, q) \text{ and } f(q) = A(q, p),$$

that is, A *and* f *have a coupled coincidence.*

Theorem 1.14. *Let* $(X, \leqslant)$ *be a partially ordered set and* $(X, \mathcal{F}, T)$ *be a complete Menger probabilistic metric space under a t-norm* T *of H-type. Suppose* $A : X \times X \to X$ *and* $f : X \to X$ *are two continuous mappings such that* A *has the f-mixed monotone property on* X *and* f *commutes with* A. *Suppose that for some* $k \in (0, 1)$, *condition (8) hold for every* $x, y \in X$ *for which* $f(x) \leqslant f(u)$ *and* $f(y) \geqslant f(v)$ *and every* $t > 0$.

If there exists $x_0, y_0 \in X$ *such that*

$$f(x_0) \leqslant A(x_0, y_0) \text{ and } f(y_0) \geqslant A(y_0, x_0).$$

Then there exist $p, q \in X$ *such that*

$$f(p) = A(p, q) \text{ and } f(q) = A(q, p),$$

that is, A *and* f *have a coupled coincidence.*

Now, we prove our main results.

2. Main results

Definition 2.1. Let $(X, \leqslant)$ be a partially ordered set and $F : X \times X \to X$, $f : X \to X$ and $g : X \to X$. We say F has the mixed fg-monotone property if F is monotone f-non-decreasing in its first argument and is monotone f-non-increasing in second argument, and also F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad f(x_1) \leqslant f(x_2) \text{ implies } F(x_1, y) \leqslant F(x_2, y), \tag{9}$$

$$y_1, y_2 \in X, \quad f(y_1) \leqslant f(y_2) \text{ implies } F(x, y_1) \geqslant F(x, y_2), \tag{10}$$

and

$$x_1, x_2 \in X, \quad g(x_1) \leqslant g(x_2) \text{ implies } F(x_1, y) \leqslant F(x_2, y), \tag{11}$$

$$y_1, y_2 \in X, \quad g(y_1) \leqslant g(y_2) \text{ implies } F(x, y_1) \geqslant F(x, y_2). \tag{12}$$

Definition 2.2. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $f,\ g : X \to X$ if

$$F(x, y) = f(x) = g(x), \quad F(y, x) = f(y) = g(y). \tag{13}$$

Note that if $f = g$, then Definition 2.2 reduces to Definition 1.3.

Definition 2.3. Let X be a non-empty set, $F : X \times X \to X$ and $f, g : X \to X$. We say F commute with f and g if

$$fg(F(x, y)) = F(fg(x), fg(y))$$

for all $x, y \in X$.

Theorem 2.4. *Let* $(X, \leqslant)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Assume there is a function* $\varphi : [0, +\infty) \to [0, +\infty)$ *with* $\varphi(t) < t$ *and* $\lim_{r \to t+} \varphi(r) < t$ *for each* $t > 0$ *and also suppose* $F : X \times X \to X$ *and* $f, g : X \to X$ *are such that* $fg = gf$, F *has the mixed fg-monotone property and*

$$d(F(x, y), F(u, v)) \leqslant \varphi\left(\frac{d(g(x), g(u)) + d(f(y), f(v))}{2}\right) \tag{14}$$

for all $x, y, u, v \in X$ *for which* $g(x) \leqslant g(u)$ *and* $f(y) \geqslant f(v)$, *and*

$$d(F(x, y), F(u, v)) \leqslant \varphi\left(\frac{d(f(x), f(u)) + d(g(y), g(v))}{2}\right) \tag{15}$$

for all $x, y, u, v \in X$ *for which* $f(x) \leqslant f(u)$ *and* $g(y) \geqslant g(v)$. *Suppose* $F(X \times X) \subseteq f(X) \cap g(X)$, g, f *are continuous and increasing functions and gf commutes with* F *and also suppose either*

(a) *F is continuous or*
(b) *X has the following property:*

$$(i) \text{ if a non-decreasing sequence } \{x_n\} \to x \text{, then } x_n \leqslant x \text{ for all } n, \tag{16}$$

$$(ii) \text{ if a non-increasing sequence } \{y_n\} \to y \text{, then } y \leqslant y_n \text{ for all } n. \tag{17}$$

If there exists $x_0, y_0 \in X$ *such that*

$$g(x_0) \leqslant F(x_0, y_0) = f(x_0) \text{ and } f(y_0) \geqslant F(y_0, x_0) = g(y_0), \tag{18}$$

then there exist $fx, gy \in X$ *such that*

$$f^2(x) = gf(x) = F(f(x), g(y)) \text{ and } g^2(y) = fg(y) = F(g(y), f(x)), \tag{19}$$

that is, F and f, g *have a coupled coincidence.*

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \leqslant F(x_0, y_0) = f(x_0)$ and $f(y_0) \geqslant F(y_0, x_0) = g(y_0)$. Since $F(X \times X) \subseteq f(X) \cap g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0) = f(x_0)$ and $f(y_1) = F(y_0, x_0) = g(y_0)$. Again from $F(X \times X) \subseteq f(X) \cap g(X)$ we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1) = f(x_1)$ and $f(y_2) = F(y_1, x_1) = g(y_1)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = f(x_n) = F(x_n, y_n) \text{ and } f(y_{n+1}) = g(y_n) = F(y_n, x_n) \text{ for all } n \geqslant 0. \tag{20}$$

We shall show that

$$g(x_n) \leqslant g(x_{n+1}), \quad f(x_n) \leqslant f(x_{n+1}) \text{ for all } n \geqslant 0, \tag{21}$$

and

$$f(y_n) \geqslant f(y_{n+1}), \quad g(y_n) \geqslant g(y_{n+1}) \text{ for all } n \geqslant 0. \tag{22}$$

We shall use the mathematical induction. Let $n=0$. Since $g(x_0) \leqslant F(x_0,y_0)$ and $f(y_0) \geqslant F(y_0,x_0)$, and as $g(x_1) = F(x_0,y_0)$ and $f(y_1) = F(y_0,x_0)$, we have

$$g(x_0) \leqslant g(x_1) \tag{23}$$

and

$$f(y_0) \geqslant f(y_1). \tag{24}$$

Based on (10), (11), (23) and (24) we have

$$f(x_0) = F(x_0,y_0) \leqslant F(x_1,y_0) \text{ and } f(x_1) = F(x_1,y_1) \geqslant F(x_1,y_0), \tag{25}$$

which implies that

$$f(x_0) \leqslant f(x_1). \tag{26}$$

Analogous, using (9), (12), (23) and (24) we have

$$g(y_0) = F(y_0,x_0) \geqslant F(y_1,x_0) \text{ and } g(y_1) = F(y_1,x_1) \leqslant F(y_1,x_0). \tag{27}$$

i.e.

$$g(y_0) \geqslant g(y_1). \tag{28}$$

Thus (21) and (22) hold for $n=0$.

Suppose now that (21) and (22) hold for some fixed $n \geqslant 0$. Then, since $f(x_{n-1}) = g(x_n) \leqslant g(x_{n+1}) = f(x_n)$ and $g(y_n) = f(y_{n+1}) \leqslant f(y_n) = g(y_{n-1})$, and as F has the mixed g,f-monotone property, from (20) and (5),

$$f(x_n) = g(x_{n+1}) = F(x_n,y_n) \leqslant F(x_{n+1},y_n) \text{ and } f(x_{n+1}) = g(x_{n+2}) = F(x_{n+1},y_{n+1}) \geqslant F(x_{n+1},y_n), \tag{29}$$

and from (20), (21) and (22)

$$g(y_n) = f(y_{n+1}) = F(y_n,x_n) \geqslant F(y_n,x_{n+1}), \text{ and } g(y_{n+1}) = f(y_{n+2}) = F(y_{n+1},x_{n+1}) \leqslant F(y_n,x_{n+1}). \tag{30}$$

Now from (29) and (30) we get

$$g(x_{n+1}) \leqslant g(x_{n+2}), \quad f(x_n) \leqslant f(x_{n+1})$$

and

$$f(y_{n+1}) \geqslant f(y_{n+2}), \quad g(y_n) \geqslant g(y_{n+1}).$$

Thus by the mathematical induction we conclude that (21) and (22) hold for all $n \geqslant 0$.

Denote

$$\delta_n = d(g(x_n),g(x_{n+1})) + d(f(y_n),f(y_{n+1})) = d(f(x_{n-1}),f(x_n)) + d(g(y_{n-1}),g(y_n)).$$

We show that

$$\delta_n \leqslant 2\varphi\left(\frac{\delta_{n-1}}{2}\right). \tag{31}$$

Since $g(x_{n-1}) \leqslant g(x_n)$ and $f(y_{n-1}) \geqslant f(y_n)$, from (14) and (20) we have

$$d(g(x_n),g(x_{n+1})) = d(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \leqslant \varphi\left(\frac{d(g(x_{n-1}),g(x_n)) + d(f(y_{n-1}),f(y_n))}{2}\right) = \varphi\left(\frac{\delta_{n-1}}{2}\right). \tag{32}$$

Similarly, from (15) and (20), as $f(y_n) \leqslant f(y_{n-1})$ and $g(x_n) \geqslant g(x_{n-1})$,

$$d(f(y_{n+1}),f(y_n)) = d(F(y_n,x_n),F(y_{n-1},x_{n-1})) \leqslant \varphi\left(\frac{d(f(y_{n-1}),f(y_n)) + d(g(x_{n-1}),g(x_n))}{2}\right) = \varphi\left(\frac{\delta_{n-1}}{2}\right). \tag{33}$$

Adding (32) and (33) we obtain (31).

From (31), since $\varphi(t) < t$ for $t > 0$, it follows that a sequence $\{\delta_n\}$ is monotone decreasing. Therefore, there is some $\delta \geqslant 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta + .$$

We show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then, taking the limit as $\delta_n \to \delta+$ of the both sides of (31) and have in mind that we assume that $lim_{r\to t+}\varphi(r) < t$ for all $t > 0$, we have

$$\delta = \lim_{n\to\infty}\delta_n \leqslant 2\lim_{n\to\infty}\varphi\left(\frac{\delta_{n-1}}{2}\right) = 2\lim_{\delta_{n-1}\to\delta+}\varphi\left(\frac{\delta_{n-1}}{2}\right) < 2\frac{\delta}{2} = \delta,$$

a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty}[d(g(x_n),g(x_{n+1})) + d(f(y_n),f(y_{n+1}))] = \lim_{n\to\infty}[d(f(x_{n-1}),f(x_n)) + d(g(y_{n-1}),g(y_n))] = 0. \tag{34}$$

Now, we prove that $\{g(x_n)\} = \{f(x_{n-1})\}$ and $\{f(y_n)\} = \{g(y_{n-1})\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{g(x_n)\}$ or $\{f(y_n)\}$ is not a Cauchy sequence. Then there exist an $\epsilon > 0$ and two subsequences of integers $\{l(k)\}, \{m(k)\}, m(k) > l(k) \geqslant k$ with

$$r_k = d(g(x_{l(k)}),g(x_{m(k)})) + d(f(y_{l(k)}),f(y_{m(k)})) \geqslant \epsilon \text{ for } k \in \{1,2,\ldots\}. \tag{35}$$

We may also assume

$$d(g(x_{l(k)}),g(x_{m(k)-1})) + d(f(y_{l(k)}),f(y_{m(k)-1})) < \epsilon \tag{36}$$

by choosing $m(k)$ to be the smallest number exceeding $l(k)$ for which (35) holds. From (35) and (36) and by the triangle inequality,

$$\begin{aligned}\epsilon \leqslant r_k &\leqslant d(g(x_{l(k)}),g(x_{m(k)-1})) + d(g(x_{m(k)-1}),g(x_{m(k)})) \\ &\quad + d(f(y_{l(k)}),f(y_{m(k)-1})) + d(f(y_{m(k)-1}),f(y_{m(k)})) \\ &= d(g(x_{l(k)}),g(x_{m(k)-1})) + d(f(y_{l(k)}),f(y_{m(k)-1})) + \delta_{m(k)-1} \\ &< \epsilon + \delta_{m(k)-1}.\end{aligned}$$

Taking the limit as $k \to \infty$ we get, by (34),

$$\lim_{k\to\infty} r_k = \epsilon + . \tag{37}$$

By the triangle inequality

$$\begin{aligned} r_k = &\ d(g(x_{l(k)}),g(x_{m(k)})) + d(f(y_{l(k)}),f(y_{m(k)})) \\ \leqslant &\ d(g(x_{l(k)}),g(x_{l(k)+1})) + d(g(x_{l(k)+1}),g(x_{m(k)+1})) \\ &+ d(g(x_{m(k)+1}),g(x_{m(k)})) + d(f(y_{l(k)}),f(y_{l(k)+1})) \\ &+ d(f(y_{l(k)+1}),f(y_{m(k)+1})) + d(f(y_{m(k)+1}),f(y_{m(k)})) \\ = &\ [d(g(x_{l(k)}),g(x_{l(k)+1})) + d(f(y_{l(k)}),f(y_{l(k)+1}))] \\ &+ [d(g(x_{m(k)}),g(x_{m(k)+1})) + d(f(y_{m(k)}),f(y_{m(k)+1}))] \\ &+ [d(g(x_{l(k)+1}),g(x_{m(k)+1})) + d(f(y_{l(k)+1}),f(y_{m(k)+1}))].\end{aligned}$$

Hence

$$r_k \leqslant \delta_{l(k)} + \delta_{m(k)} + d(g(x_{l(k)+1}), g(x_{m(k)+1})) + d(f(y_{l(k)+1}), f(y_{m(k)+1})). \quad (38)$$

Since from (21) and (22) we have $g(x_{l(k)}) \leqslant g(x_{m(k)})$ and $f(y_{l(k)}) \geqslant f(y_{m(k)})$, from (14) and (20)

$$d(g(x_{l(k)+1}), g(x_{m(k)+1})) = d(F(x_{l(k)}, y_{l(k)}), F(x_{m(k)}, y_{m(k)})) \leqslant \varphi\left(\frac{d(g(x_{l(k)}), g(x_{m(k)})) + d(f(y_{l(k)}), f(y_{m(k)}))}{2}\right) = \varphi\left(\frac{r_k}{2}\right), \quad (39)$$

Also from (15) and (20), as $f(y_{m(k)}) \leqslant f(y_{l(k)})$ and $g(x_{m(k)}) \geqslant g(x_{l(k)})$,

$$d(f(y_{m(k)+1}), f(y_{l(k)+1})) = d(F(y_{m(k)}, x_{m(k)}), F(y_{l(k)}, x_{l(k)})) \leqslant \varphi\left(\frac{d(g(x_{l(k)}), g(x_{m(k)})) + d(f(y_{l(k)}), f(y_{m(k)}))}{2}\right) = \varphi\left(\frac{r_k}{2}\right), \quad (40)$$

Inserting (38) and (39) in (40) we obtain

$$r_k \leqslant \delta_{l(k)} + \delta_{m(k)} + 2\varphi\left(\frac{r_k}{2}\right).$$

Letting $k \to \infty$ and using (37) and (40) we get

$$\epsilon \leqslant 2\lim_{k\to\infty} \varphi\left(\frac{r_k}{2}\right) = 2\lim_{r_k\to\epsilon+} \varphi\left(\frac{r_k}{2}\right) < 2\frac{\epsilon}{2} = \epsilon, \quad (41)$$

a contradiction. Thus, our supposition was wrong. Therefore, we proved that $\{g(x_n)\} = \{f(x_{n-1})\}$ and $\{f(y_n)\} = \{g(y_{n-1})\}$ are Cauchy sequences.

Since X complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_{n-1}) = x \text{ and } \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_{n-1}) = y. \quad (42)$$

From (42) and continuity of g, f,

$$\lim_{n\to\infty} f(g(x_n)) = \lim_{n\to\infty} f^2(x_n) = f(x) \text{ and } \lim_{n\to\infty} g(f(y_n)) = \lim_{n\to\infty} g^2(y_n) = g(y). \quad (43)$$

From (20) and commutativity of F and gf,

$$fg(g(x_{n+1})) = fg(F(x_n, y_n)) = F(fg(x_n), fg(y_n)) = F(fg(x_n), gf(y_n)), \quad (44)$$

$$f^2(g(x_{n+1})) = fg(f(x_{n+1})) = fg(F(x_{n+1}, y_{n+1})) = F(fg(x_{n+1}), fg(y_{n+1})) \quad (45)$$

$$= F(fg(x_{n+1}), gf(y_{n+1})).$$

We now show that $f^2(x) = fg(x) = F(fx, gy)$. Suppose that the assumption (a) holds. Taking the limit as $n \to \infty$ in (44) and using the continuity of F we get

$$fg(x) = \lim_{n\to\infty} fg(g(x_{n+1})) = \lim_{n\to\infty} F(fg(x_n), gf(y_n)) = F(\lim_{n\to\infty} fg(x_n), \lim_{n\to\infty} gf(y_n)) = F(fx, gy).$$

Similarly, taking the limit as $n \to \infty$ in (45) and using the continuity of F we get

$$f^2(x) = F(fx, gy).$$

Also, from (20) and commutativity of F and gf, we have

$$gf(f(y_{n+1})) = gf(F(y_n, x_n)) = F(gf(y_n), gf(x_n)) = F(gf(y_n), fg(x_n)). \quad (46)$$

and

$$g^2(f(y_{n+1})) = gf(g(y_{n+1})) = gf(F(y_{n+1}, x_{n+1})) = F(gf(y_{n+1}), gf(x_{n+1})) \quad (47)$$

$$= F(gf(y_{n+1}), fg(x_{n+1})).$$

Analogous, if condition (a) hold, from (46) and (47) we have that

$$gf(y) = g^2(y) = F(gy, fx).$$

Suppose now that (b) holds. Since f and g are increasing mappings we have $f^2(x_{n-1}) = fg(x_n) \leqslant f(x)$ and $g^2(y_{n-1}) = gf(y_n) \geqslant g(y)$ for all n.

Then by the triangle inequality, (15), (43) and (44) we get

$$d(fg(x), F(fx, gy)) \leqslant d(fg(x), fg(g(x_{n+1}))) + d(fg(g(x_{n+1})),$$
$$F(fx, gy)) = d(fg(x), fg(g(x_{n+1}))) + d(F(fg(x_n), fg(y_n)),$$
$$F(fx, gy)) \leqslant d(fg(x), fg(g(x_{n+1})))$$
$$+ \varphi\left(\frac{d(fg(g(x_n)), fg(x)) + d(f(gf(y_n)), fg(y))}{2}\right).$$

So letting $n \to \infty$ yields $d(fg(x), F(fx, gy)) \leqslant 0$. Hence $fg(x) = F(fx, gy)$.

Also,

$$d(f^2(x), F(fx, gy)) \leqslant d(f^2(x), f^2(g(x_n))) + d(f^2(g(x_n)),$$
$$F(fx, gy)) = d(f^2(x), f^2(g(x_n))) + d(fg(fx_n), F(fx, gy)) \leqslant d(f^2(x),$$
$$f^2(g(x_n))) + d(fg(F(x_n, y_n)), F(fx, gy)) \leqslant d(f^2(x),$$
$$f^2(g(x_n))) + \varphi\left(\frac{d(f^2(g(x_n)), f^2(x)) + d(g^2(fy_n), g^2(y))}{2}\right), \quad n \in \mathbb{N}_0.$$

Letting $n \to \infty$ we have $d(f^2(x), F(fx, gy)) \leqslant 0$. Hence $f^2(x) = F(fx, gy)$.

Similarly one can show that $g^2(y) = gf(y) = F(gy, fx)$. Thus we proved that F and g, f have a coupled coincidence point. □

Example 2.5. Let (X, d) be a metric space where $X = [1, +\infty)$. We endow X with the natural ordering of real numbers. Let $f, g : X \to X$ be defined as $f(x) = x^2, g(x) = \sqrt{x}$ for all $x \in X$. Let $F : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} 1, & \text{if } x < y \\ \frac{4+\sqrt{x}-\sqrt{y}}{4}, & \text{if } x \geqslant y. \end{cases}$$

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined as $\varphi(t) = \frac{t}{2}$. Obviously, the mapping F has the mixed $fg-$monotone property, f and g are commuting mappings and F commute with f and g. Now we will show that the mappings F, f and g satisfy the inequality (14). Let $x, y, u, v \in X$ be such that $g(x) \leqslant g(u)$ and $f(y) \geqslant f(v)$. We have considered the following cases.

Case 1: $x \geqslant y$. Since $x \leqslant u$ we have $u \geqslant x \geqslant y \geqslant v$,

$$d(F(x,y),F(u,v)) = \left|\frac{4+\sqrt{x}-\sqrt{y}}{4} - \frac{4+\sqrt{u}-\sqrt{v}}{4}\right|$$
$$\leqslant \frac{|\sqrt{x}-\sqrt{u}|+|\sqrt{y}-\sqrt{v}|}{4} \leqslant \frac{|\sqrt{x}-\sqrt{u}|+|\sqrt{y}-\sqrt{v}||\sqrt{y}+\sqrt{v}||y+v|}{4}$$
$$= \varphi\left(\frac{d(g(x),g(u))+d(f(y),f(v))}{2}\right).$$

Case 2. $x < y, u \geqslant v$ and $x > v$,

$$d(F(x,y),F(u,v)) = \left|1 - \frac{4+\sqrt{u}-\sqrt{v}}{4}\right| = \frac{|\sqrt{u}-\sqrt{v}|}{4}$$
$$\leqslant \frac{|\sqrt{u}-\sqrt{x}|+|\sqrt{x}-\sqrt{v}|}{4}$$
$$\leqslant \frac{|\sqrt{u}-\sqrt{x}|+|\sqrt{y}-\sqrt{v}|}{4}$$
$$\leqslant \frac{|\sqrt{u}-\sqrt{x}|+|y^2-v^2|}{4}$$
$$= \varphi\left(\frac{d(g(x),g(u))+d(f(y),f(v))}{2}\right).$$

Case 3: $x < y, u \geqslant v$ and $x < v$,

$$d(F(x,y),F(u,v)) = \left|1 - \frac{4+\sqrt{u}-\sqrt{v}}{4}\right| = \frac{|\sqrt{u}-\sqrt{v}|}{4}$$
$$\leqslant \frac{|\sqrt{u}-\sqrt{v}+\sqrt{y}-\sqrt{x}|}{4}$$
$$\leqslant \frac{|\sqrt{u}-\sqrt{x}|+|\sqrt{y}-\sqrt{v}|}{4}$$
$$\leqslant \varphi\left(\frac{d(g(x),g(u))+d(f(y),f(v))}{2}\right).$$

Case 4: $x < y, u < v$,

$$d(F(x,y),F(u,v)) = 0 \leqslant \varphi\left(\frac{d(g(x),g(u))+d(f(y),f(v))}{2}\right).$$

Analogy shows that inequality (15) holds for every $x, y, u, v \in X$ for which $f(x) \leqslant f(u)$ and $g(y) \geqslant g(v)$. Then, there exists a couple coincidence point $(1,1)$ for mappings F, f and g.

Now, we prove our main theorem in probabilistic metric spaces.

Theorem 2.6. *Let $(X,\leqslant)$ be a partially ordered set and $(X,\mathcal{F},T)$ be a complete Menger probabilistic metric space, t-norm T is continuous. Suppose $A: X\times X \to X$ and $f: X \to X$ are two mappings such that A has the f-mixed monotone property on X and for some $k \in (0,1)$,*

$$F_{A(x,y),A(u,v)}(kt) \geqslant \min\{F_{f(x),f(u)}(t),\ F_{f(y),f(v)}(t),\ F_{f(x),A(x,y)}(t), \tag{48}$$

$$F_{f(u),A(u,v)}(t),\ F_{f(y),A(y,x)}(t),\ F_{f(v),A(v,u)}(t)\}$$

for every $x, y \in X$ for which $f(x) \leqslant f(y)$ and $f(y) \geqslant f(v)$ and every $t > 0$.

Suppose also that $A(X\times X) \subseteq f(X)$, and (a) or (b) satisfied

(a) *$f(X)$ is closed and*

(i) if a non-decreasing sequence $\{f(x_n)\} \to f(x)$, then $f(x_n) \leqslant f(x)$ for all n, (49)

(ii) if a non-increasing sequence $\{f(y_n)\} \to f(y)$, then $f(y) \leqslant f(y_n)$ for all n. (50)

(b) *A and f are continuous mappings and f commutes with A.*

Let $\mu \in (k,1)$ such that t-norm T satisfies the following condition

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} \min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right), F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\} = 1.$$

If there exists $x_0, y_0 \in X$ such that

$f(x_0) \leqslant A(x_0,y_0)$ and $f(y_0) \geqslant A(y_0,x_0)$,

then there exist $p, q \in X$ such that

$f(p) = A(p,q)$ and $f(q) = A(q,p)$,

that is, A and f have a coupled coincidence.

Proof. Following the proof of Theorem 7. and Theorem 8. in [12] we can construct two sequences $\{x_n\}$ and $\{y_n\}, n \in \mathbb{N}$ such that

$$f(x_{n+1}) = A(x_n,y_n), \quad f(y_{n+1}) = A(y_n,x_n) \tag{51}$$

and

$$f(x_0) \leqslant f(x_1) \leqslant \ldots \leqslant f(x_{n+1}) \leqslant \ldots$$

$$f(y_0) \geqslant f(y_1) \geqslant \ldots \geqslant f(y_{n+1}) \geqslant \ldots.$$

Also, following the proof in [12] we have

$$\min\{F_{f(x_{n+1}),f(x_{n+2})}(t), F_{f(y_{n+1}),f(y_{n+2})}(t)\}$$
$$\geqslant \min\left\{F_{f(x_0),f(x_1)}\left(\frac{t}{k^n}\right), F_{f(y_0),f(y_1)}\left(\frac{t}{k^n}\right)\right\} \tag{52}$$

for all $t > 0$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in (52) we obtain

$$\lim_{n\to\infty} F_{f(x_n),f(x_{n+1})}(t) = 1, \quad t > 0, \tag{53}$$

and

$$\lim_{n\to\infty} F_{f(y_n),f(y_{n+1})}(t) = 1, \quad t > 0. \tag{54}$$

It remains to be proved that the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ and $\{f(y_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences in X.

Let $\sigma = \frac{k}{\mu}$. Since $0 < \sigma < 1$ the series $\sum_{i=1}^{\infty}\sigma^i$ is convergent and there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=n_0}^{\infty}\sigma^i < 1$. Hence for every $n > n_0$ and every $m \in \mathbb{N}$

$$t > t\sum_{i=n_0}^{\infty}\sigma^i > t\sum_{i=n}^{n+m-1}\sigma^i,$$

which implies that

$$F_{f(x_{n+m}),f(x_n)}(t) \geqslant F_{f(x_{n+m}),f(x_n)}\left(t\sum_{i=n}^{n+m-1}\sigma^i\right)$$
$$\geqslant \underbrace{T(T(\ldots T}_{m-times}(F_{f(x_{n+m}),f(x_{n+m-1})}(t\sigma^{n+m-1}),$$
$$F_{f(x_{n+m-1}),f(x_{n+m-2})}(t\sigma^{n+m-2}),\ldots,F_{f(x_{n+1}),f(x_n)}(t\sigma^n))$$
$$\geqslant \underbrace{T(T(\ldots T}_{m-times}(\min\left\{F_{f(x_0),f(x_1)}\left(\frac{t\sigma^{n+m-1}}{k^{n+m-1}}\right),F_{f(y_0),f(y_1)}\left(\frac{t\sigma^{n+m-1}}{k^{n+m-1}}\right)\right\},$$
$$\min\left\{F_{f(x_0),f(x_1)}\left(\frac{t\sigma^{n+m-2}}{k^{n+m-2}}\right),F_{f(y_0),f(y_1)}\left(\frac{t\sigma^{n+m-2}}{k^{n+m-2}}\right)\right\},$$
$$\ldots\min\left\{F_{f(x_0),f(x_1)}\left(\frac{t\sigma^n}{k^n}\right),F_{f(y_0),f(y_1)}\left(\frac{t\sigma^n}{k^n}\right)\right\}$$
$$\geqslant \mathbf{T}_{i=n}^{n+m-1}\min\left\{F_{f(x_0),f(x_1)}\left(\frac{t}{\mu^i}\right),F_{f(y_0),f(y_1)}\left(\frac{t}{\mu^i}\right)\right\}$$
$$\geqslant \mathbf{T}_{i=n}^{\infty}\min\left\{F_{f(x_0),f(x_1)}\left(\frac{t}{\mu^i}\right),F_{f(y_0),f(y_1)}\left(\frac{t}{\mu^i}\right)\right\}.$$

It is obvious that $\lim_{n\to\infty}\mathbf{T}_{i=n}^{\infty}\min\{F_{f(x_0),f(x_1)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),f(y_1)}\left(\frac{1}{\mu^i}\right)\}=1$ implies $\lim_{n\to\infty}\mathbf{T}_{i=m}^{\infty}\min\{F_{f(x_0),f(x_1)}\left(\frac{t}{\mu^i}\right),F_{f(y_0),f(y_1)}\left(\frac{t}{\mu^i}\right)\}=1$ for every $t>0$, and now for every $t>0$ and every $\lambda\in(0,\ 1)$ there exists $n_1(t,\ \lambda)$ such that $F_{f(x_{n+m}),f(x_n)}(t)>1-\lambda$ for every $n\geqslant n1(t,\ \lambda)$ and every $m\in\mathbb{N}$. This means that the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy, and since the space is complete there exists $p\in X$ such that $\lim_{n\to\infty}f(x_n)=f(p)$.

An the same way we can prove that the sequence $f(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and hence there exists $q\in X$ such that $\lim_{n\to\infty}f(y_n)=f(q)$. The rest of the proof is as in [12]. □

Remark 2. Let $(X,\leqslant)$ be a partially ordered set and $(X,\mathcal{F},T)$ be a complete Menger probabilistic metric space, t-norm T is continuous. Let $A:X\times X\to X$ and $f:X\to X$. Suppose that all conditions of Theorem 2.6 are satisfied. If t-norm T is of H-type, then the conclusions of Theorem 2.6 still holds. By Proposition 1.9, all the conditions of the Theorem 2.6 are satisfied. This is in fact result in [12].

Corollary 1. *Let $(X,\leqslant)$ be a partially ordered set and $(X,\mathcal{F},T_\lambda^D)$ for some $\lambda>0$ be a complete Menger probabilistic metric space. Suppose that all conditions of Theorem 2.6 are satisfied. If there exists $x_0,y_0\in X$ such that*

$$f(x_0)\leqslant A(x_0,y_0)\text{ and }f(y_0)\geqslant A(y_0,x_0).$$

and $\mu\in(k,1)$ such that

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\left(1-\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}\right)^{\lambda}<\infty$$

then the conclusions of Theorem 2.6 still holds.

Proof. From equivalence (6) we have

$$\sum_{i=1}^{\infty}\left(1-\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}\right)^{\lambda}<\infty$$
$$\Longleftrightarrow \lim_{n\to\infty}(T_\lambda^D)_{i=n}^{\infty}\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}=1.\quad\square$$

Corollary 2. *Let $(X,\leqslant)$ be a partially ordered set and $(X,\mathcal{F},T_\lambda^{AA})$ for some $\lambda>0$ be a complete Menger probabilistic metric space. Suppose that all conditions of Theorem 2.6 are satisfied. If there exists $x_0,y_0\in X$ such that*

$$f(x_0)\leqslant A(x_0,y_0)\text{ and }f(y_0)\geqslant A(y_0,x_0).$$

and $\mu\in(k,1)$ such that

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\left(1-\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}\right)^{\lambda}<\infty$$

then the conclusions of Theorem 2.6 still holds.

Proof. From equivalence (6) we have

$$\sum_{i=1}^{\infty}\left(1-\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}\right)^{\lambda}<\infty$$
$$\Longleftrightarrow \lim_{n\to\infty}(T_\lambda^{AA})_{i=n}^{\infty}\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}=1.\quad\square$$

Corollary 3. *Let $(X,\leqslant)$ be a partially ordered set and $(X,\mathcal{F},T_\lambda^{SW})$ for some $\lambda>-1$ be a complete Menger probabilistic metric space. Suppose that all conditions of Theorem 2.6 are satisfied. If there exists $x_0,y_0\in X$ such that*

$$f(x_0)\leqslant A(x_0,y_0)\text{ and }f(y_0)\geqslant A(y_0,x_0).$$

and $\mu\in(k,1)$ such that

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\left(1-\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}\right)<\infty$$

then the conclusions of Theorem 2.6 still holds.

Proof. From equivalence (7) we have

$$\sum_{i=1}^{\infty}\left(1-\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}\right)<\infty$$
$$\Longleftrightarrow \lim_{n\to\infty}(T_\lambda^{SW})_{i=n}^{\infty}\min\left\{F_{f(x_0),A(x_0,y_0)}\left(\frac{1}{\mu^i}\right),F_{f(y_0),A(y_0,x_0)}\left(\frac{1}{\mu^i}\right)\right\}=1.\quad\square$$

3. Conclusion

In this paper, we generalized two coupled coincidence point theorems in partially ordered metric spaces and Menger probabilistic metric spaces. An illustrative example in partially ordered metric spaces is given.

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