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ORIGINAL ARTICLE

The approximation of common element for maximal monotone operator, generalized mixed equilibrium problem and fixed point problem

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Abstract The purpose of this paper is to get strong convergence theorems for a countable family of relatively quasi-nonexpansive mappings $\{S_n\}_{n=0}^{\infty}$, a maximal monotone operator T, and a generalized mixed equilibrium problem in a uniformly smooth and uniformly convex Banach space lacking condition UARC. Two examples are given to support our results. One is a countable family of uniformly closed relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings. Another is uniformly closed but not satisfies condition UARC. Many recent results in this field have been unified and improved.

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1. Introduction

In an infinite-dimensional Hilbert space, Mann iterative algorithm has only weak covergence, in general, even for non-expansive mappings. Hence in order to have strong convergence, in recent years, the hybrid iteration methods for approximating fixed points of nonlinear mappings have been introduced and studied by various authors [1–6].

Let E be a smooth Banach space. We denote by ϕ the functional on $E \times E$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad \forall\ x,y \in E.$$

A point $p \in C$ is said to be an (strong) asymptotic fixed point of T if there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset C$ such that $(x_n \to p)$ $x_n \rightharpoonup p$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of (strong) asymptotic fixed point is denoted by $(\widetilde{F}(T))$ $\widehat{F}(T)$. Let E be a smooth Banach space, we say that a mapping T is (weak) relatively nonexpansive (see [7–11]) if the following conditions are satisfied:

(i) $F(T) \neq \emptyset$;
(ii) $\phi(p, Tx) \leqslant \phi(p,x),\ \forall x \in C,\ p \in F(T)$;
(iii) $(F(T) = \widetilde{F}(T))$ $F(T) = \widehat{F}(T)$.

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A multivalued operator $T: E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ is called monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i, i = 1, 2$. A monotone operator T is called maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. A method for solving the inclusion $0 \in Tx$ is the proximal point algorithm. This algorithm was first presented by Martinet [12] and generally studied by Rockafellar [13] in a Hilbert space. A mapping $A : C \to E^*$ is called α-inverse-strongly monotone, if there exists an $\alpha > 0$ such that $\langle Ax - Ay, x - y \rangle \geqslant \alpha \|Ax - Ay\|^2, \ \forall x, y \in C$.

It is easy to see that if $A : C \to E^*$ is an α-inverse-strongly monotone mapping, then it is $1/\alpha$-Lipschitzian. Let $T: E \to 2^{E^*}$ be a maximal monotone operator in a smooth Banach space E. We denote the resolvent of T by $J_r := (J + rT)^{-1}J$ for each $r > 0$. Then $J_r : E \to D(T)$ is a single-valued mapping. Also, $T^{-1}0 = F(J_r)$ for each $r > 0$, where $F(J_r)$ is the set of fixed points of J_r. For each $r > 0$, the Yosida approximation of T is defined by $A_r = (J - JJ_r)/r$. It is known that

$$A_r x \in T(J_r x), \quad \forall r > 0 \quad \text{and} \quad x \in E.$$

Let $\varphi : C \to R$ be a real-valued function and $A : C \to E^*$ be a nonlinear mapping and $f : C \times C \to R$ be a bifunction. For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y \in C$, $\lim_{t \to 0} f(tz + (1 - t)x, y) \leqslant f(x, y)$;
(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

The generalized mixed equilibrium problem is to find $u \in C$ [14–16] such that:

$$f(u, y) + \varphi(y) - \varphi(u) + \langle Au, y - u \rangle \geqslant 0, \quad \forall\ y \in C. \tag{1.7}$$

Throughout this paper, we denote $f(u, y) + \varphi(y) - \varphi(u) + \langle Au, y - u \rangle$ by $F(x, y)$. The set of solutions of (1.7) is denoted by $GMEP(F, \varphi)$, i.e.,

$$GMEP(F, \varphi) = \{u \in C : f(u, y) + \varphi(y) - \varphi(u) + \langle Au, y - u \rangle \geqslant 0, \quad \forall\ y \in C\}.$$

If $A = 0$, then problem (1.7) is equivalent to mixed equilibrium problem studied by many authors, which is to find $u \in C$ such that

$$f(u, y) + \varphi(y) - \varphi(u) \geqslant 0, \quad \forall\ y \in C.$$

If $\varphi = 0$, then problem (1.7) is equivalent to generalized equilibrium problem considered by many authors, which is to find $u \in C$ such that

$$f(u, y) + \langle Au, y - u \rangle \geqslant 0, \quad \forall\ y \in C.$$

If $\varphi = 0, A = 0$, then problem (1.7) is reduces to equilibrium problem considered by many authors, which is to find $u \in C$ such that $f(u, y) \geqslant 0, \ \forall y \in C$.

The generalized mixed equilibrium problem includes fixed point problem, optimization problem, variational inequality problem, minimax problem, Nash equilibrium problem as spacial cases [17]. Some methods have been proposed to find its solutions. And, numerous problems in physics, optimation and economics can be reduced to find a solution of generalized equilibrium problem [18].

Algorithms for obtaining fixed point of relatively nonexpansive mappings have been studied widely. For instance, Mann iterative method, Ishikawa-type iterative method, Halpern-type iterative method, hybrid methods, and many other modified methods. Recently, utilizing Nakajo and Takahashi's idea [19], Qin and Su [20] introduced one iterative algorithm for a relatively nonexpansive mapping. By combining Kamimura and Takahashi's idea [21] with Qin and Su [20], Ceng et al. [22] introduced a hybrid proximal-type algorithm for finding an element of fixed point set and zero point set in a uniformly smooth and uniformly convex Banach space. In 2011, Ceng et al. [23] introduced and investigated one hybrid shrinking projection method for a generalized equilibrium problem, a maximal monotone operator and a countable family of relatively nonexpansive mappings. The authors obtained strong convergence theorems.

2. Preliminaries and lemmas

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty closed convex subset of E. It is well known that the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x), \quad \forall\ x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. And it is obvious that

$$(\|x\| - \|y\|)^2 \leqslant \phi(x, y) \leqslant (\|x\| + \|y\|)^2, \quad \forall\ x, y \in E.$$

Next, we recall the notion of generalized f-projection operator and its properties. Let $G : C \times E^* \to R \cup \{+\infty\}$ be a functional defined as following:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.1}$$

where $\xi \in C, \varphi \in E^*, \rho$ is a positive number and $f : C \to R \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed.
(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous relate to ξ when φ is fixed.

We can see that the functional G is a generalization of functional ϕ. That is, functional ϕ is a special case of functional G when $f \equiv 0$.

Definition 2.1 [24]. Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is *a generalized f-projection operator* if for any $\varphi \in E^*$,

$$\Pi_C^f \varphi = \{u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi)\}.$$

For the generalized f-projection operator, Wu and Huang [20] proved the following basic properties:

Lemma 2.2 [24]. *Let E be a real reflexive Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. Then the following statements hold:*

(i) Π_C^f *is a nonempty closed convex subset of C,* $\forall \varphi \in E^*$.
(ii) *If E is smooth, then for all* $\varphi \in E^*, x \in \Pi_C^f \varphi$ *if and only if*

$$\langle x - y, \varphi - Jx \rangle + \rho f(y) - \rho f(x) \geqslant 0, \quad \forall\, y \in C.$$

(iii) *If E is strictly convex and* $f : C \to R \cup \{+\infty\}$ *is positive homogeneous (i.e.,* $f(tx) = tf(x)$ *for all* $t > 0$ *such that* $tx \in C$ *where* $x \in C$*), then* Π_C^f *is a single valued mapping.*

Fan et.al. [25] showed that the condition *f* is positive homogeneous which appeared in Lemma 2.2 can be removed.

Lemma 2.3 [25]. *Let E be a real reflexive Banach space with its dual E^* and C be a nonempty, closed and convex subset of E. Then if E is strictly convex, then* Π_C^f *is a single valued mapping.*

Recall that J is a single valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.1) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx \rangle + \|x\|^2 + 2\rho f(\xi).$$

Notice that whenever $f \equiv 0$, the generalized f-projection operator is equivalent to the generalized projection operator.

Now, we consider the second generalized f-projection operator in a Banach space.

Definition 2.4 [26]. Let E be a real Banach space and C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is *a generalized f-projection operator* if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\}, \quad \forall\, x \in E.$$

Obviously, the definition of relatively quasi-nonexpansive mapping T is equivalent to

(1) $F(T) \neq \emptyset$;
(2) $G(p, JTx) \leqslant G(p, Jx), \ \forall\, x \in C, p \in F(T)$.

Lemma 2.5 [27]. *Let E be a Banach space and* $f : E \to R \cup \{+\infty\}$ *be a lower semi-continuous convex functional. Then there exists* $x^* \in E^*$ *and* $\alpha \in R$ *such that*
$$f(x) \geqslant \langle x, x^* \rangle + \alpha, \quad \forall\, x \in E.$$

Lemma 2.6 [28]. *Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Then the following statements hold:*

(i) Π_C^f *is a nonempty closed and convex subset of C for all* $x \in E$;
(ii) *for all* $x \in E, \hat{x} \in \Pi_C^f x$ *if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(x) \geqslant 0, \quad \forall\, y \in C;$$

(iii) *if E is strictly convex, then* $\Pi_C^f x$ *is a single valued mapping.*

Lemma 2.7 [28]. *Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Let* $x \in E$ *and* $x \in \Pi_C^f$. *Then*

$$\phi(y, x) + G(x, Jx) \leqslant G(y, Jx), \quad \forall\, y \in C.$$

Lemma 2.8 [28]. *Let E be a Banach space and* $y \in E$. *Let* $f : E \to R \cup \{\infty\}$ *be a proper, convex and lower semi-continuous mapping with convex domain* $D(f)$. *If* $\{x_n\}$ *is a sequence in* $D(f)$ *such that* $x_n \rightharpoonup x \in int(D(f))$ *and* $\lim_{n\to\infty} G(x_n, Jy) = G(x, Jy)$, *then* $\lim_{n\to\infty} \|x_n\| = \|x\|$.

The fixed points set $F(T)$ of a relatively quasi-nonexpansive mapping is closed and convex as given in the following lemma.

Lemma 2.9 ([29,30]). *Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let T be a closed relatively quasi-nonexpansive mapping of C into itself. Then F(T) is closed and convex.*

Lemma 2.10 [21]. *Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let* $\{x_n\}_{n=0}^\infty$ *and* $\{y_n\}_{n=0}^\infty$ *be sequences in E such that either* $\{x_n\}_{n=0}^\infty$ *or* $\{y_n\}_{n=0}^\infty$ *is bounded. If* $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

The following result is due to Blum and Oettli [17].

Lemma 2.11 [17]. *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from* $C \times C$ *to R satisfying* $(A1)$–$(A4)$. *Then for* $r > 0$ *and* $x \in E$, *there exists unique z such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geqslant 0, \quad \forall\, y \in C.$$

Lemma 2.12 [31]. *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from* $C \times C$ *to R satisfying* $(A1)$–$(A4)$. *For* $r > 0$ *and* $x \in E$, *define a mapping* $T_r : E \to C$ *as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geqslant 0, \quad \forall\, y \in C\}$$

for all $x \in E$. *Then, the following statements hold.*

(i) T_r *is single-valued.*
(ii) T_r *is a firmly nonexpansive-type mapping, i.e., for all* $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leqslant \langle T_r x - T_r y, Jx - Jy \rangle.$$

(iii) $F(T_r) = \widehat{F}(T_r) = EP(f)$.
(iv) $EP(f)$ *is closed and convex.*

Using Lemma 2.12, one has the following result.

Lemma 2.13 [31]. *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from* $C \times C$ *to R satisfying* $(A1)$–$(A4)$, *and let* $r > 0$. *Then, for* $x \in E$ *and* $q \in F(T_r)$,

$$\phi(q, T_r x) + \phi(T_r x, x) \leqslant \phi(q, x).$$

Utilizing Lemmas 2.11 and 2.12, Yekini Shehu [32] derived the following results.

Proposition 2.14 [32]. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $f: C \times C \to R$ satisfies (A1)–(A4), let $A: C \to E^*$ be a continuous and monotone mapping and $\varphi: C \to R$ be a lower semi-continuous and convex functional. Furthermore, define a mapping $K_r: E \to C$ as follows:*

$$\begin{aligned} K_r(x) = \{u \in C \\ &: f(u,y) + \varphi(y) - \varphi(u) + \langle Au, y-u \rangle + \frac{1}{r}\langle y-u, Ju - Jx\rangle \\ &\geqslant 0, \quad \forall\, y \in C\}, \quad \forall\, x \in E, \end{aligned}$$

then the following properties hold.

(i) *K_r is single-valued,*
(ii) *K_r is a firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*
$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leqslant \langle K_r x - K_r y, Jx - Jy\rangle,$$
(iii) *$F(K_r) = GMEP(F, \varphi)$,*
(iv) *$GMEP(F,\varphi)$ is a closed and convex.*

Since $F(x,y) = f(u,y) + \varphi(y) - \varphi(u) + \langle Au, y-u\rangle$ satisfies conditions (A1)–(A4) (see [26]). We can easily get the following lemma.

Lemma 2.15. *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$. Then, for $x \in E$ and $p \in F(K_r)$,*

$$\phi(p, K_r x) + \phi(K_r x, x) \leqslant \phi(p, x).$$

Moreover, the inequality will be

$$G(p, JK_r x) + \phi(K_r x, x) \leqslant G(p, Jx)$$

in the sense of functional G.

Lemma 2.16 [33]. *Let E be a reflexive, strictly convex, and smooth Banach space and let $T: E \to 2^{E^*}$ be a multivalued operator. For all $r > 0$, then the following statements hold.*

(i) *$T^{-1}0$ is closed and convex if T is maximal monotone such that $T^{-1}0 \neq \emptyset$.*
(ii) *T is maximal monotone if and only if T is monotone with $R(J + rT) = E^*$.*

Lemma 2.17 [34]. *Let E be a reflexive, strictly convex, and smooth Banach space, and let $T: E \to 2^{E^*}$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$. Then the following statements hold.*

(I) *$\phi(z, J_r x) + \phi(J_r x, x) \leqslant \phi(z, x)$ for all $r > 0, z \in T^{-1}0$* and *$x \in E$.*
(II) *$J_r: E \to D(T)$ is a relatively nonexpansive map.*

Definition 2.18. Let E be a Banach space, and C be a nonempty closed convex subset of E. Let $\{S_n\}_{n=0}^{\infty}: C \to E$ be a sequence of mappings of C into E such that $\cap_{n=0}^{\infty} F(S_n)$ is nonempty. $\{S_n\}_{n=0}^{\infty}$ is said to be *uniformly closed*, if $p \in \cap_{n=0}^{\infty} F(S_n)$, whenever $\{x_n\} \to p$ and $\|x_n - S_n x_n\| \to 0$ as $n \to \infty$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of relatively quasi-nonexpansive self-mapping on C which are also uniformly closed mappings. Let $f: E \to R \cup \{\infty\}$ be a proper, convex and lower semi-continuous mapping with convex domain $D(f)$ and $C \subset int(D(f))$. Assume that $T: E \to 2^{E^*}$ is a maximal monotone operator, $A: C \to E^*$ is a continuous and monotone mapping, and $\varphi: C \to R$ is a lower semi-continuous and convex functional. Let $f: C \times C \to R$ be a bifunction satisfying $(A1)-(A4)$. Let $\{x_n\}$ be a sequence generated in the following way:*

$$x_0 \in C_0 \text{ arbitrarily},$$

$$z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n x_n),$$

$$y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n),$$

$$u_n \in C \text{ such that } f(u_n, y) + \varphi(y)$$

$$-\varphi(u_n) + \langle Au_n, y - u_n\rangle$$

$$+\frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geqslant 0, \quad \forall\, y \in C,$$

$$\begin{aligned} C_{n+1} = \{v \in C_n : G(v, Ju_n) &\leqslant \beta_n G(v, Jx_n) + (1-\beta_n)G(v, Jz_n) \\ &\leqslant G(v, Jx_n)\}, \end{aligned}$$

$$x_{n+1} = \Pi^f_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

where $C_0 = C, \{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$. And $\{\alpha_n\}_{n=0}^{\infty}\{\beta_n\}_{n=0}^{\infty}$ are the sequences in $[0,1]$ which satisfy

$$\liminf_{n\to\infty} r_n > 0, \quad \limsup_{n\to\infty} \alpha_n < 1, \quad \limsup_{n\to\infty} \beta_n < 1.$$

Let $\Gamma := GMEP(F,\varphi) \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$, then the sequence $\{x_n\}$ generated above converges strongly to $\Pi^f_{\Gamma} x_0$.

Proof. First, let us show that C_n is a closed and convex subset of C for all $n \geqslant 0$. Indeed, observe that

$$G(v, Ju_n) \leqslant \beta_n G(v, Jx_n) + (1-\beta_n)G(v, Jz_n)$$

$$\Longleftrightarrow 2\langle v, (1-\beta_n)Jz_n + \beta_n Jx_n - Ju_n\rangle$$

$$\leqslant (1-\beta_n)\|z_n\|^2 - \|u_n\|^2 + \beta_n\|x_n\|^2$$

and

$$\beta_n G(v, Jx_n) + (1-\beta_n)G(v, Jz_n) \leqslant G(v, Jx_n)$$

$$\Longleftrightarrow 2\langle v, Jx_n - Jz_n\rangle \leqslant \|x_n\|^2 - \|z_n\|^2.$$

Obviously, C_n is closed and convex for each $n \geqslant 0$.

Second, we show that $\Gamma \subset C_n$ for each $n \geqslant 0$. Indeed, it is clear that $\Gamma \subset C_0 = C$. Suppose that $\Gamma \subset C_n$ for some $n \in N$. Take $w \in \Gamma$ arbitrarily. Then $w \in GMEP(F,\varphi), w \in T^{-1}0$ and $w \in \bigcap_{n=0}^{\infty} F(S_n)$. Since $u_n = K_{r_n} y_n$, applying (3.1) and Proposition 2.14 we have

$$\begin{aligned}
G(w,Ju_n) &= G(w,JK_{r_n}y_n) \leqslant G(w,Jy_n) \\
&= G(w,\beta_n Jx_n + (1-\beta_n)JJ_{r_n}z_n) \\
&= \|w\|^2 - 2\langle (w,\beta_n Jx_n + (1-\beta_n)JJ_{r_n}z_n)\rangle \\
&\quad + \|(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}z_n)\|^2 + 2\rho f(w) \\
&\leqslant \|w\|^2 - 2\beta_n\langle w,Jx_n\rangle - 2(1-\beta_n)\langle w,JJ_{r_n}z_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n}z_n\|^2 + 2\rho f(w) \\
&= \beta_n G(w,Jx_n) + (1-\beta_n)G(w,JJ_{r_n}z_n) \\
&\leqslant \beta_n G(w,Jx_n) + (1-\beta_n)G(w,Jz_n) \\
&= \beta_n G(w,Jx_n) + (1-\beta_n)G(w,\alpha_n Jx_n + (1-\alpha_n)JS_n x_n) \\
&= \beta_n G(w,Jx_n) + (1-\beta_n)[\|w\|^2 \\
&\quad - 2\langle w,\alpha_n Jx_n + (1-\alpha_n)JS_n x_n\rangle + \|\alpha_n Jx_n \\
&\quad + (1-\alpha_n)JS_n x_n\|^2 + 2\rho f(w)] \leqslant \beta_n G(w,Jx_n) \\
&\quad + (1-\beta_n)[\|w\|^2 - 2\alpha_n\langle w,Jx_n\rangle - 2(1-\alpha_n)\langle w,JS_n x_n\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|S_n x_n\|^2 + 2\rho f(w)] \\
&= \beta_n G(w,Jx_n) + (1-\beta_n)[\alpha_n G(w,Jx_n) \\
&\quad + (1-\alpha_n)G(w,JS_n x_n)] \leqslant \beta_n G(w,Jx_n) \\
&\quad + (1-\beta_n)[\alpha_n G(w,Jx_n) + (1-\alpha_n)G(w,Jx_n)] \\
&= G(w,Jx_n).
\end{aligned}$$

This implies that $w \in C_{n+1}$. Therefore, $\Gamma \subset C_n$ for all $n \geqslant 0$. It means that $x_{n+1} = \Pi^f_{C_{n+1}} x_0$ is well defined. Then, by induction, the sequence $\{x_n\}$ generated above is well defined for each integer $n \geqslant 0$.

For showing that $\{x_n\}$ is a Cauchy sequence, we should first show that $\|x_n\|$ and $G(x_n, Jx_0)$ are bounded. From the definition of G and Lemma 2.5, we have

$$\begin{aligned}
G(x_n,Jx_0) &= \|x_n\|^2 - 2\langle x_n,Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\
&\geqslant \|x_n\|^2 - 2\langle x_n,Jx_0\rangle + \|x_0\|^2 + 2\rho\langle x_n,x^*\rangle + 2\rho\alpha \\
&= \|x_n\|^2 - 2\langle x_n,Jx_0 - \rho x^*\rangle + \|x_0\|^2 + 2\rho\alpha \\
&\geqslant \|x_n\|^2 - 2\|x_n\|\|Jx_0 - \rho x^*\| + \|x_0\|^2 + 2\rho\alpha \\
&= (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 \\
&\quad - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha. \qquad (3.2)
\end{aligned}$$

Since $x_n = \Pi^f_{C_n} x_0$, it follows from (3.2) that

$$\begin{aligned}
G(q,Jx_0) &\geqslant G(x_n,Jx_0) \\
&\geqslant (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 \\
&\quad + 2\rho\alpha
\end{aligned}$$

for each $q \in \bigcap_{n=0}^{\infty} F(S_n)$. This implies that $\{x_n\}_{n=0}^{\infty}$ and $\{G(x_n,Jx_0)\}_{n=0}^{\infty}$ are bounded. Note that $C_{n+1} \subset C_n, x_{n+1} = \Pi^f_{C_{n+1}} x_0$. Utilizing Lemma 2.7, we can get

$$\phi(x_{n+1},x_n) + G(x_n,Jx_0) \leqslant G(x_{n+1},Jx_0).$$

Since $\phi(x_{n+1},x_n)$ in nonnegative, we have $G(x_n,Jx_0) \leqslant G(x_{n+1},Jx_0)$. This shows that $\lim_{n\to\infty} G(x_n,Jx_0)$ exists. Similarly, we have $\phi(x_{n+m},x_n) + G(x_n,Jx_0) \leqslant G(x_{n+m},Jx_0)$. Then, we can derive $\lim_{n\to\infty}\phi(x_{n+m},x_n) = 0$. Combining with Lemma 2.10, we get $\lim_{n\to\infty}\|x_{n+m} - x_n\| = 0$, i.e., $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we may assume that $\lim_{n\to\infty} x_n = p$.

Now, we claim that $\|z_n - J_{r_n} z_n\| \to 0$ and $\lim_{n\to\infty}\|x_n - S_n x_n\| = 0$.

Indeed, from the definition of C_{n+1} we have

$$G(x_{n+1},Ju_n) \leqslant G(x_{n+1},Jx_n), \quad \forall\, n \geqslant 0,$$

and

$$G(x_{n+1},Jz_n) \leqslant G(x_{n+1},Jx_n), \quad \forall\, n \geqslant 0,$$

which are equivalent to

$$\phi(x_{n+1},u_n) \leqslant \phi(x_{n+1},x_n), \quad \forall\, n \geqslant 0,$$

and

$$\phi(x_{n+1},z_n) \leqslant \phi(x_{n+1},x_n), \quad \forall\, n \geqslant 0.$$

Since $\phi(x_{n+1},x_n) \to 0$, it follows that $\phi(x_{n+1},u_n) \to 0$ and $\phi(x_{n+1},z_n) \to 0$. Utilizing Lemma 2.10, we conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_{n+1} - u_n\| = \lim_{n\to\infty}\|x_{n+1} - z_n\| = 0,$$

and so

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|u_n - z_n\| = 0, \qquad (3.3)$$

Again since $u_n = K_{r_n} y_n$, as in the proof of the second step, we can derive that

$$\phi(w,u_n) \leqslant \phi(w,y_n) \leqslant \phi(w,x_n), \quad \forall\, w \in \Gamma.$$

Together with Lemma 2.15, we have

$$\begin{aligned}
\phi(u_n,y_n) &= \phi(K_{r_n}y_n,y_n) \leqslant G(w,Jy_n) - G(w,JK_{r_n}y_n) \\
&\leqslant G(w,Jx_n) - G(w,JK_{r_n}y_n) = \phi(w,x_n) - \phi(w,u_n) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle w,Jx_n - Ju_n\rangle \\
&\leqslant (\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|) + 2\|w\|\|Jx_n - Ju_n\| \qquad (3.4)
\end{aligned}$$

Since $\|x_n - u_n\| \to 0$ and J is uniformly norm-to-norm continuous on bounded subsets of E, it follows that $\|Jx_n - Ju_n\| \to 0$ and so $\phi(u_n,y_n) \to 0$. Since E is smooth and uniformly convex, from Lemma 2.10 and (3.4), we have

$$\|u_n - y_n\| \to 0, \text{ and so } \|x_n - y_n\| \to 0. \qquad (3.5)$$

Note that E is uniformly smooth and uniformly convex. Thus J and J^{-1} are uniformly norm-to-norm continuous on bounded subsets of E and E^*, respectively. Hence from (3.1) and (3.5) we can get

$$(1-\beta_n)\|JJ_{r_n}z_n - Jx_n\| = \|Jy_n - Jx_n\| \to 0,$$

and so $\|J_{r_n}z_n - x_n\| \to 0$. This together with $\|x_n - z_n\| \to 0$ which implies that

$$\lim_{n\to\infty}\|z_n - J_{r_n}z_n\| = \lim_{n\to\infty}\|Jz_n - JJ_{r_n}z_n\| = 0. \qquad (3.6)$$

Again from (3.1) and (3.3) we have

$$(1-\alpha_n)\|JS_n x_n - Jx_n\| = \|Jz_n - Jx_n\| \to 0.$$

This implies that $\|JS_n x_n - Jx_n\| \to 0$, and so

$$\lim_{n\to\infty}\|x_n - S_n x_n\| = 0.$$

Since $\{S_n\}_{n=0}^{\infty}$ is a countable family of uniformly closed relatively quasi-nonexpansive mappings, we have $p \in \bigcap_{n=0}^{\infty} F(S_n)$.

Next, let us show that $p \in T^{-1}0$. Since $x_n \to p$, from (3.3) and (3.5) it follows that $z_n \to p$, and $J_{r_n}z_n \to p$. Also, from (3.6) and $\liminf_{n\to\infty} r_n > 0$, we derive

$$\lim_{n\to\infty}\|A_{r_n}z_n\| = \lim_{n\to\infty}\frac{1}{r_n}\|Jz_n - JJ_{r_n}z_n\| = 0.$$

Let $z^* \in Tz$, then it follows from (1.1) and the monotonicity of the operator T that

$$\langle z - J_{r_n} z_n, z^* - A_{r_n} z_n \rangle \geqslant 0.$$

Letting $n \to \infty$, we obtain $\langle z - p, z^* \rangle \geqslant 0$. Then the maximality of the operator T yields $p \in T^{-1}0$.

Now we shall show that $p \in GMEP(F, \varphi)$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, from (3.5) we have $\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0$. From $\liminf_{n\to\infty} r_n > 0$, it follows that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.7}$$

By the definition of $u_n := K_{r_n} y_n$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geqslant 0, \quad \forall\, y \in C, \tag{3.8}$$

where

$$F(u_n, y) = f(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Au_n, y - u_n \rangle.$$

We have from (A2) that

$$\frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geqslant -F(u_n, y) \geqslant F(y, u_n), \quad \forall\, y \in C.$$

Since $y \mapsto f(x, y) + \varphi(y) - \varphi(u_n) + \langle Ax, y - x \rangle$ is convex and lower semi-continuous. Letting $n \to \infty$ in the last inequality, from (3.7) and (A4) we have

$$F(y, p) \leq 0, \quad \forall\, y \in C.$$

For t, with $0 < t < 1$, and $y \in C$, let $y_t = ty + (1 - t)p$. Since $y \in C$ and $p \in C$, then $y_t \in C$ and hence $F(y_t, p) \leq 0$. So, from (A1) we have $0 = F(y_t, y_t) \leqslant tF(y_t, y) + (1 - t)F(y_t, p) \leqslant tF(y_t, y)$. Dividing by t, we have $F(y_t, y) \geqslant 0$, $\forall y \in C$. Letting $t \to 0$, from (A3) we can get $F(p, y) \geqslant 0$, $\forall y \in C$. So, $p \in GMEP(F, \varphi)$. Therefore, we obtain that $p \in \Gamma$.

Finally, we prove that $p = \Pi^f_\Gamma x_0$. In fact, put $\bar{x} = \Pi^f_\Gamma x_0$. From $x_{n+1} = \Pi^f_{C_{n+1}} x_0$ and $\bar{x} \in \Gamma \subset C_{n+1}$, we have $G(x_{n+1}, Jx_0) \leqslant G(\bar{x}, Jx_0), \forall n \geqslant 0$. We know that $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed. This implies that

$$G(p, Jx_0) \leqslant \liminf_{n\to\infty} G(x_{n+1}, Jx_0) \leqslant \limsup_{n\to\infty} G(x_{n+1}, Jx_0) \leqslant G(\bar{x}, Jx_0).$$

Since $\bar{x} = \Pi^f_\Gamma x_0$, so $p = \bar{x}$. Hence, $x_n \to \Pi^f_\Gamma x_0$. □

4. Examples

In this section, two examples are given to support our results.

Example 1. Let $E = l^2$, where

$$l^2 = \{\xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{\frac{1}{2}}, \quad \forall\, \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \qquad \forall \xi = \{\xi_n\},\ \eta = \{\eta_n\} \in l^2, \quad n \in N.$$

It is well known that, l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$x_0 = (1, 0, 0, 0, \ldots), \quad x_1 = (1, 1, 0, 0, \ldots)$$

$$x_2 = (1, 0, 1, 0, 0, \ldots), \quad x_3 = (1, 0, 0, 1, 0, 0, \ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad x_n = (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \cdots)$$

where for all $n \geqslant 1$,

$$\xi_{n,k} = \begin{cases} 1, & \text{if } k = 1,\ n+1; \\ 0, & \text{if } k \neq 1, k \neq n+1. \end{cases}$$

Define a countable family of mappings $S_n : E \to E$ as follows, for all $n \geqslant 0$,

$$S_n(x) = \begin{cases} \frac{n}{n+1} x_n, & \text{if } x = x_n; \\ -x, & \text{if } x \neq x_n. \end{cases}$$

Conclusion 4.1. S_n has a unique fixed point 0, that is $F(S_n) = \{0\} \neq \emptyset, \forall n \geqslant 0$.

Proof. The conclusion is obvious. □

Conclusion 4.2. $\{S_n\}_{n=0}^{\infty}$ is a countable family of relatively quasi-nonexpansive mappings in the sense of functional G.

Proof. We only need to show that $G(0, JS_n x) \leqslant G(0, Jx)$, $\forall x \in E$. Note that $E = l^2$ is a Hilbert space, for any $n \geqslant 0$ we can derive

$$G(0, JS_n x) \leqslant G(0, Jx) \qquad \forall x \in E, \quad \Longleftrightarrow \phi(0, S_n x) \leqslant \phi(0, x),$$

$$\Longleftrightarrow \|0 - S_n x\|^2 \leqslant \|0 - x\|^2, \quad \Longleftrightarrow \|S_n x\|^2 \leqslant \|x\|^2.$$

This imply that Conclusion 4.4 holds. □

Conclusion 4.3. $\{S_n\}_{n=0}^{\infty}$ is not a countable family of relatively nonexpansive mappings in the sense of functional G.

Proof. Obviously, $\{x_n\}$ converges weakly to x_0, and

$$\|x_n - S_n x_n\| = \|\frac{n}{n+1} x_n - x_n\| = \frac{1}{n+1} \|x_n\| \to 0,$$

as $n \to \infty$, so x_0 is an asymptotic fixed point of $\{S_n\}_{n=0}^{\infty}$. Joining with Conclusion 4.3, we can obtain $\bigcap_{n=0}^{\infty} F(S_n) \neq \widehat{F}(\{S_n\}_{n=0}^{\infty})$. □

Conclusion 4.4. $\{S_n\}_{n=0}^{\infty}$ is a countable family of uniformly closed relatively quasi-nonexpansive mappings.

Proof. In fact, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - S_n z_n\| \to 0$ as $n \to \infty$, there exists sufficiently large nature number N such that $z_n \neq x_m$, for any $n, m > N$ (since x_n is not a Cauchy sequence, then it cannot converges to any element in E). Then $S_n z_n = -z_n$ for $n > N$, it follows from $\|z_n - S_n z_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$.

Above all, we prove that $\{S_n\}_{n=0}^{\infty}$ is a countable family of uniformly closed relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings in the sense of functional G. □

Now, we give an example which is a countable family of uniformly closed quasi-nonexpansive mappings but not satisfied condition UARC.

Example 2. Let $X = \Re^2$. For any complex number $x = re^{i\theta} \in X$, define a countable family of nonexpansive mappings as follows,

$$T_n : re^{i\theta} \to re^{i\left(\theta + n\frac{\pi}{2}\right)}, \quad n \in N.$$

Proof. It is easy to see that $\bigcap_{n=1}^{\infty} F(T_n) = \{0\}$.

We first prove that T_n is uniformly closed. In fact, for any strong convergent sequence $\{x_n\} \subset X$ such that $x_n \to x_0$ and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$, there must be $x_0 = 0 \in \bigcap_{n=1}^{\infty} F(T_n)$. Otherwise, if $x_n \to x_0 \neq 0$, and $\|x_{4n+1} - T_{4n+1}x_{4n+1}\| \to 0$, since T_1 is continuous, we have $\|x_{4n+1} - T_{4n+1}x_{4n+1}\| = \|x_{4n+1} - T_1 x_{4n+1}\| \to \|x_0 - T_1 x_0\| \neq 0$. This is a contradiction. Therefore, T_n is uniformly closed.

Besides, take a sequence $x_n = r_n e^{i\theta_n}$. For any given m, by the definition of T_n, we have

$$\|T_n x_n - T_m x_n\| = \{0,\ n = 4km,\ k = 0, \pm 1, \pm 2, \ldots; r_n,\ n = 4km + 1, 4km + 3;\ 2r_n,\ n = 4km + 2.\}$$

So, for any $x_n \nrightarrow 0$, we have $\|T_n x_n - T_m x_n\| \nrightarrow 0$, $as\ n \to \infty$. That is to say T_n does not satisfied condition UARC. □

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