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ORIGINAL ARTICLE

A new multivalued contraction and stability of its fixed point sets

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Abstract In this paper we obtain some stability results for fixed point sets associated with a sequence of multivalued mappings. We define multivalued α–ψ contractions and multivalued α-admissible mappings. We use Hausdorff distance in our definition. We show that the fixed point sets of uniformly convergent sequences of multivalued α–ψ contractions which are also assumed to be multivalued α-admissible, are stable under certain conditions. The multivalued mappings we define here are not necessarily continuous. We present two illustrative examples and one open problem.

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1. Introduction and preliminaries

Stability is a concept in dynamical systems related to limiting behaviors. There are various notions of stability both in discrete and continuous dynamical systems [1,2]. In this paper we consider such a problem of stability related to a sequence of multivalued mappings on metric spaces. The limiting behaviors of sequences of mappings have been considered in a large number of papers in recent times as, for instances, in [3,4]. Particularly, stability of fixed point sets for multivalued mapping has been considered in [5–7].

Specially, we are interested in the limit of fixed point sets for a convergent sequence of multivalued mappings, that is, how they are related, in the limit, to the fixed point set of the function to which the sequence converges. We say that the fixed point sets are stable when they converge in the Hausdorff metric to the set of fixed points of the limiting function. More often than not, in the above mentioned problem of stability, sequences of multivalued mappings are considered. One of the reasons behind this is that multivalued mappings often have more fixed points than their singlevalued counterparts. For instance, in the theorem of Nadler [3], which is a multivalued generalization of the Banach contraction principle, and, incidentally, which is also the first work appearing on multivalued contractive fixed point studies, the fixed point is not unique in contrast to the case of Banach's contraction. In those situations the fixed point set becomes larger and, hence, more interesting for the study of stability. In this paper

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we consider the case of α–ψ contractions [8] which is a newly introduced generalization of the Banach's contraction. It should be mentioned that Banach contraction mapping principle [9,10] plays an important role in nonlinear analysis. There has been a large number of generalizations of this result over the years [11–16].

We introduce a multivalued version of α–ψ contraction. We show that for such a multivalued mapping on a complete metric space, the fixed point set is nonempty. We then show that a uniformly convergent sequence of such mappings on a complete metric space has stable fixed point sets, that is, the fixed point sets converge to the fixed point set of the limiting function with respect to the Hausdorff metric.

Throughout this paper $CL(X)$ denotes the family of all nonempty closed subsets of a metric space (X, d) and $P(X)$ denotes the family of all nonempty subsets of X.

The Hausdorff metric H is defined on $CL(X)$ by

$$H(A, B) = \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in B} d(x, B)\right\},$$

where $A, B \in CL(X)$ and $d(x, A) = \inf_{y\in A} d(x, y)$.

H is a metric when it is restricted to the set $CB(X)$, the set of all closed and bounded subsets of X. Otherwise, on $CL(X)$, the set of all closed subsets of X, all the properties of the metric function is satisfied except that $H(A, B)$ can be infinite when either A or B is unbounded. The following is the well known definition of fixed point for multivalued mappings.

Let $T : X \to P(X)$ be a multivalued mapping, a point $z \in X$ is a fixed point of T whenever $z \in Tz$.

Asl et al. [17] introduce the following definition;

Definition 1.1 [17]. Let (X, d) be a metric space; $\alpha : X \times X \to [0, \infty)$ be a mapping and $T : X \to 2^X$ be a closed valued multifunction, where 2^X = collection of all nonempty subsets of X. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and continuous function with $\sum \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$. We say that T is an α_*–ψ contractive multifunction whenever

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leqslant \psi(d(x, y)),\ \textit{for}\ x, y \in X, \tag{1.1}$$

where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$.

In the following we introduce the concept of multivalued α–ψ contraction and multivalued α-admissible.

Definition 1.2 (*Multivalued α–ψ contraction*). Let (X, d) be a metric space, and $\alpha : X \times X \to [0, \infty)$, $\psi : [0, \infty) \to [0, \infty)$ be two mappings such that ψ is a nondecreasing and continuous function with $\sum \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$. $T : X \to CL(X)$ be a multivalued mapping. We say that T is a multivalued α–ψ contraction if

$$\alpha(x, y)H(Tx, Ty) \leqslant \psi(d(x, y)),\ \textit{for all}\ x, y \in X. \tag{1.2}$$

Remark 1.1. In (1.2) of our Definition 1.2 we consider $\alpha(x, y)$ instead of $\alpha_*(Tx, Ty)$ which has been considered in (1.1) of Definition 1.1. $\alpha_*(Tx, Ty)$ is defined as

$$\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\},\ \text{for}\ x, y \in X.$$

From the definition it is clear that $\alpha_*(Tx, Ty)$ is not necessarily equal to $\alpha(x, y)$, and also we cannot compare $\alpha(x, y)$ with $\alpha_*(Tx, Ty)$. Therefore Definition 1.2 is new and independent of Definition 1.1.

Remark 1.2. If T is singlevalued in Definition 1.2, then it is an α–ψ contraction as in [8].

Definition 1.3 (*Multivalued α-admissible*). Let X be any nonempty set. $T : X \to P(X)$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. We say that T is multivalued α-admissible if, for $x, y \in X$,

$$\alpha(x, y) > 1 \Rightarrow \alpha(a, b) > 1,\ \textit{for all}\ a \in Tx\ \textit{and for all}\ b \in Ty.$$

Example 1.1. Let $X = \mathbb{R}$, $\alpha : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$. We define

$$\alpha(x, y) = x^2 + y^2,\ \text{where}\ x, y \in \mathbb{R}.$$

Define $T : \mathbb{R} \longrightarrow P(\mathbb{R})$ by,

$$Tx = \left\{\sqrt{|x|}, -\sqrt{|x|}\right\}.$$

Then T is multivalued α-admissible.

2. Main Result

We first prove that multivalued α–ψ contractions on complete metric spaces have nonempty fixed point sets. In the proof of the following theorem we make use of Lemma 8.1.3(c) of [18].

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a multivalued α–ψ contraction. Also T satisfies the following:*

(i) T is multivalued α-admissible;
(ii) For some $x_0 \in X$, $\alpha(x_0, a) > 1$ holds for all $a \in Tx_0$;
(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) > 1$ for all n, where $x_{n+1} \in Tx_n$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) > 1$ for all n.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be as in the statement of the theorem. By (ii), we have $x_1 \in Tx_0$ such that, $\alpha(x_0, x_1) > 1$. Then, since $x_1 \in Tx_0$, we can choose $x_2 \in Tx_1$ such that, $d(x_1, x_2) \leqslant \alpha(x_0, x_1)H(Tx_0, Tx_1)$. So, by (1.2), we have

$$d(x_1, x_2) \leqslant \alpha(x_0, x_1)H(Tx_0, Tx_1) \leqslant \psi(d(x_0, x_1)). \tag{2.1}$$

Since $x_1 \in Tx_0, x_2 \in Tx_1$ and $\alpha(x_0, x_1) > 1$, by (i), we have $\alpha(x_1, x_2) > 1$.

Again, for $x_2 \in Tx_1$, we can choose $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leqslant \alpha(x_1, x_2)H(Tx_1, Tx_2).$$

Therefore, by (1.2), we have,

$$d(x_2, x_3) \leqslant \alpha(x_1, x_2)H(Tx_1, Tx_2) \leqslant \psi(d(x_1, x_2))$$

$$\leqslant \psi^2(d(x_0, x_1)) \qquad (\text{by (2.1)}). \tag{2.2}$$

Also, since $\alpha(x_1, x_2) > 1, x_2 \in Tx_1$ and $x_3 \in Tx_2$ we have that $\alpha(x_2, x_3) > 1$. Continuing this process we can construct a sequence $\{x_n\}$ such that for all $n \geqslant 1$, $x_{n+1} \in Tx_n$, $\alpha(x_n, x_{n+1}) > 1$ and

$$d(x_n, x_{n+1}) \leqslant \alpha(x_{n-1}, x_n)H(Tx_{n-1}, Tx_n) \leqslant \psi^n(d(x_0, x_1)). \quad (2.3)$$

Now, we have

$$\sum_{k=1}^{\infty} d(x_k, x_{k+1}) \leqslant \sum_{k=1}^{\infty} \psi^k(d(x_0, x_1)) = \Phi(d(x_0, x_1)) < \infty \quad \text{(by an assumption of the theorem).}$$

This implies that, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\{x_n\} \to z$ as $n \to \infty$.

Now we prove that $z \in Tz$.

For all $n \geqslant 1$, $x_{n+1} \in Tx_n$. Therefore $d(x_{n+1}, Tz) \leqslant H(Tx_n, Tz)$. By (iii), $\alpha(x_n, z) > 1$ for all $n \geqslant 1$. Hence we have for all $n \geqslant 1$, $d(x_{n+1}, Tz) \leqslant \alpha(x_n, z)H(Tx_n, Tz) \leqslant \psi(d(x_n, z))$.

Letting $n \to \infty$ we have $d(z, Tz) \leqslant \psi(0)$. Since $\psi(t) \geqslant 0$ and $\psi(t) < t$, for all $t \geqslant 0$, we have that $\psi(0) = 0$.

Therefore, we get $d(z, Tz) = 0$. Since $Tz \in CL(X)$, it follows that $z \in Tz$.

Hence T has a fixed point. □

Example 2.1. Let $X = \mathbb{R}$. $d(x, y) = |x - y|$. Define $T : \mathbb{R} \to CL(\mathbb{R})$ by

$$Tx = \begin{cases} \{1, \frac{1}{4x}\}, & \text{if } x > 1; \\ \{0, \frac{x}{16}\}, & \text{if } 0 \leqslant x \leqslant 1; \\ \{2, 3\}, & \text{otherwise.} \end{cases}$$

$$H(T0, Tx) = \max\{2, 3\} = 3 > 1.$$

Hence, we observe that, the Nadler's multivalued contraction principle [3] cannot be applied here.

Now, we define the mapping, $\alpha : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by,

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x, y \in [0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

and let $\psi : [0, \infty) \to [0, \infty)$ be that

$$\psi(t) = \frac{1}{2}t.$$

Then T is multivalued α–ψ contraction as well as multivalued α-admissible.

Now, for $x, y \in [0, 1]$ we have,

$$d(0, Ty) = \inf\left\{0, \frac{y}{16}\right\} = 0;$$

$$d\left(\frac{x}{16}, Ty\right) = \inf\left\{\left|0 - \frac{x}{16}\right|, \left|\frac{x}{16} - \frac{y}{16}\right|\right\}.$$

$$d(0, Tx) = \inf\left\{0, \frac{x}{16}\right\} = 0;$$

$$d\left(\frac{y}{16}, Tx\right) = \inf\left\{\left|0 - \frac{y}{16}\right|, \left|\frac{x}{16} - \frac{y}{16}\right|\right\}.$$

$$\begin{aligned} H(Tx, Ty) &= \max\left\{\sup_{x \in Tx} d(x, Ty), \sup_{y \in Ty} d(y, Tx)\right\} \\ &= \max\left\{\inf\left\{\left|\frac{x}{16}\right|, \left|\frac{x}{16} - \frac{y}{16}\right|\right\}, \inf\left\{\left|\frac{y}{16}\right|, \left|\frac{y}{16} - \frac{x}{16}\right|\right\}\right\} \\ &= \left|\frac{x}{16} - \frac{y}{16}\right|. \end{aligned}$$

Now,

$$\begin{aligned} \alpha(x, y)H(Tx, Ty) &= 2 \times \left|\frac{x}{16} - \frac{y}{16}\right| = \frac{1}{8}|x - y| \leqslant \frac{|x - y|}{2} \\ &= \psi(d(x, y)). \end{aligned}$$

Hence we observe that T satisfies all the condition of the above theorem, and T has fixed point at $x = 0$.

Theorem 2.2. *Let X be a complete metric space and $F(T_1), F(T_2)$ are the fixed point sets of T_1, T_2 respectively where $T_i : X \to CL(X), i = 1, 2$. Each T_i is multivalued α–ψ contraction as defined in Theorem 2.1 with the same α and ψ. Also each T_i satisfies the following:*

(i) for any $x \in F(T_1)$, we have $\alpha(x, y) > 1$ whenever $y \in T_2x$, and for any $x \in F(T_2)$, we have $\alpha(x, y) > 1$ whenever $y \in T_1x$;
(ii) Each T_i is multivalued α-admissible;
(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) > 1$ for all $n \geqslant 1$ where $x_{n+1} \in T_ix_n, i = 1, 2$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) > 1$ for all $n \geqslant 1$.

Then $H(F(T_1), F(T_2)) \leqslant \Phi(K)$, where $K = \sup_{x \in X} H(T_1x, T_2x)$.

Proof. By Theorem 2.1, $F(T_1)$ and $F(T_2)$ are nonempty. Let $q > 1$ be any number. Choose $x_0 \in F(T_1)$. We can find $x_1 \in T_2x_0$ such that $d(x_0, x_1) \leqslant qK$. For any $x_0 \in F(T_1)$, and $x_1 \in T_2x_0$, we have by (i), $\alpha(x_0, x_1) > 1$. Now, for $x_1 \in T_2x_0$, we can find $x_2 \in T_2x_1$ such that,

$$d(x_1, x_2) \leqslant \alpha(x_0, x_1)H(T_2x_0, T_2x_1).$$

Therefore, by (1.2) and Theorem 2.1, we have,

$$d(x_1, x_2) \leqslant \alpha(x_0, x_1)H(T_2x_0, T_2x_1) \leqslant \psi(d(x_0, x_1)) \leqslant \psi(qK).$$

Since $\alpha(x_0, x_1) > 1, x_1 \in T_2x_0$ and $x_2 \in T_2x_1$, we have by (ii), $\alpha(x_1, x_2) > 1$. For $x_2 \in T_2x_1$ we can choose $x_3 \in T_2x_2$ such that, $d(x_2, x_3) \leqslant \alpha(x_1, x_2)H(T_2x_1, T_2x_2)$. Therefore, by (1.2), we have,

$$\begin{aligned} d(x_2, x_3) &\leqslant \alpha(x_1, x_2)H(T_2x_1, T_2x_2) \leqslant \psi(d(x_1, x_2)) \\ &\leqslant \psi^2(d(x_0, x_1)) \leqslant \psi^2(qK). \end{aligned}$$

Since $x_2 \in T_2x_1$, $x_3 \in T_2x_2$ and since $\alpha(x_1, x_2) > 1$, then we have, by (ii), that $\alpha(x_2, x_3) > 1$.

Continuing this process, we can construct a sequence $\{x_n\}$, such that, $x_{n+1} \in T_2x_n$ for all $n \geqslant 1$. We have for all $n \geqslant 1, \alpha(x_{n-1}, x_n) > 1$, and also that,

$$\begin{aligned} d(x_n, x_{n+1}) &< \alpha(x_{n-1}, x_n)H(T_2x_{n-1}, T_2x_n) \leqslant \psi(d(x_{n-1}, x_n)) \\ &\leqslant \psi^n(d(x_0, x_1)) \leqslant \psi^n(qK). \end{aligned}$$

Now,

$$\sum_{k=1}^{\infty} d(x_k, x_{k+1}) \leqslant \sum_{k=1}^{\infty} \psi^k(d(x_0, x_1)) \leqslant \sum_{k=1}^{\infty} \psi^k(qK) = \Phi(qK) < \infty \quad \text{(by the assumption of the Theorem 2.1)}.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since(X, d) is complete, $\{x_n\}$ converges to z.

Now, we prove that $z \in T_2 z$.

For all $n \geqslant 1$, $x_{n+1} \in T_2 x_n$. Therefore, $d(x_{n+1}, T_2 z) \leqslant H(T_2 x_n, T_2 z)$. By (iii), $\alpha(x_n, z) > 1$ for all $n \geqslant 1$. Hence we have for all $n \geqslant 1$, $d(x_{n+1}, T_2 z) \leqslant \alpha(x_n, z) H(T_2 x_n, T_2 z) \leqslant \psi(d(x_n, z))$.

Letting $n \to \infty$, we get $d(z, T_2 z) \leqslant \psi(0)$. Since $\psi(t) \geqslant 0$ and $\psi(t) < t$ for all $t \geqslant 0$, therefore we have $\psi(0) = 0$. Hence $d(z, T_2 z) = 0$ implies that $z \in T_2 z$.

So $z \in F(T_2)$.

Now using triangular inequality,

$$d(x_0, z) \leqslant \sum_{i=0}^{n} d(x_i, x_{i+1}) + d(x_{n+1}, z) \leqslant \sum_{i=0}^{\infty} d(x_i, x_{i+1}) \leqslant \sum_{i=0}^{\infty} \psi^i(d(x_0, x_1)) \leqslant \sum_{i=0}^{\infty} \psi^i(qK) = \Phi(qK) < \infty.$$

Thus, given arbitrary $x_0 \in F(T_1)$, we can find $z \in F(T_2)$ for which

$$d(x_0, z) \leqslant \Phi(qK).$$

Reversing the roles of T_1 and T_2 we also conclude that for each $y_0 \in F(T_2)$, there exists $y_1 \in T_1 y_0$ and $w \in F(T_1)$ such that, $d(y_0, w) \leqslant \Phi(qK)$.

Hence $H(F(T_1), F(T_2)) \leqslant \Phi(qK)$.

Letting $q \to 1$ we get the result. □

Lemma 2.1. *Let (X, d) be a complete metric space. If $\{T_n\}$ is a sequence of multivalued α–ψ contractions uniformly convergent to T, then T is multivalued α–ψ contraction with the same α and ψ.*

Proof. Since each T_n is multivalued α–ψ contraction, for all $n \geqslant 1$, each T_n satisfies

$$\alpha(x, y) H(T_n x, T_n y) \leqslant \psi(d(x, y)), \text{ for all } x, y \in X.$$

Taking limit $n \to \infty$, we get

$$\alpha(x, y) H(Tx, Ty) \leqslant \psi(d(x, y)), \text{ for all } x, y \in X.$$

Hence T is multivalued α–ψ contraction. □

Theorem 2.3. *Let (X, d) be a complete metric space. $\{T_n\}$ is a sequence of multivalued α–ψ contractions which are also α−admissible, and is uniformly convergent to T. Let T be multivalued α-admissible with the same α. Further let the following condition hold.*

For all $n \geqslant 1$, for any $x \in F(T_n)$, we have $\alpha(x, y) > 1$ whenever $y \in Tx$ and for any $x \in F(T)$, we have $\alpha(x, y) > 1$ whenever $y \in T_n x$.

Then

$$H(F(T_n), F(T)) \to 0 \quad \text{as } n \to \infty,$$

that is, the fixed point sets of T_n are stable.

Proof. By Lemma 2.1, T is multivalued α–ψ contraction. Let $K_n = \sup_{x \in X} H(T_n x, Tx)$. Therefore,

$$\lim_{n\to\infty} K_n = \lim_{n\to\infty} \sup_{x \in X} H(T_n x, Tx) = 0,$$

(since $\{T_n\}$ converges to T uniformly on X).

Therefore, from Theorem 2.2 we get

$$H(F(T_n), F(T)) \leqslant \Phi(K_n) \to 0, \quad \text{as } n \to \infty$$

(since $\Phi(t) \to 0$ as $t \to 0$).

This proves the theorem. □

Lemma 2.2. *Let (X, d) be a complete metric space. If $\{T_n\}$ is a sequence of multivalued α-admissible with the same α and is uniformly convergent to T, then T is multivalued α-admissible if the following condition is satisfied.*

$$\alpha(x_n, y_n) > 1 \Rightarrow \alpha(x, y) > 1, \quad \text{whenever } \{x_n\} \to x \text{ and} \{y_n\} \to y \text{ as } n \to \infty. \tag{2.4}$$

Proof. Let $\alpha(x, y) > 1$, for some $x, y \in X$. Let $a \in Tx$ and $b \in Ty$ be arbitrary. Now, $T_n \to T$ uniformly, which implies that, there exist two sequences $\{x_n \in T_n x\}$ and $\{y_n \in T_n y\}$ such that $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Each T_n is α-admissible. Since $\alpha(x, y) > 1$, it follows that $\alpha(x_n, y_n) > 1$ for all n. Hence by the assumption of (2.4), $\alpha(a, b) > 1$. Thus we have,

$$\alpha(x, y) > 1 \Rightarrow \alpha(a, b) > 1 \text{ for all } a \in Tx \text{ and for all } b \in Ty.$$

Hence, T is multivalued α-admissible. Hence the result. □

Theorem 2.4. *Let (X, d) be a complete metric space. If $\{T_n\}$ is a sequence of multivalued α–ψ contractions which are also multivalued α-admissible with the same α and ψ and is uniformly convergent to T. Let α be such that*

$$\alpha(x_n, y_n) > 1 \Rightarrow \alpha(x, y) > 1, \quad \text{whenever } \{x_n\} \to x \text{ and } \{y_n\} \to y \text{ as } n \to \infty.$$

Further let the following condition hold. For all $n \geqslant 1$, for any $x \in F(T_n)$, we have $\alpha(x, y) > 1$ whenever $y \in T(x)$, and for any $x \in F(T)$, we have $\alpha(x, y) > 1$ whenever $y \in T_n x$. Then

$$H(F(T_n), F(T)) \to 0 \quad \text{as } n \to \infty,$$

that is, the fixed point sets of T_n are stable.

Proof. By Lemmas 2.1 and 2.2, it follows that T is multivalued α–ψ contraction and multivalued α-admissible. Then the theorem follows by an application of Theorem 2.3. □

Example 2.2. Let $X = \mathbb{R}$. $d(x,y) = |x-y|$. Define $T : \mathbb{R} \to CL(\mathbb{R})$ by

$$T_n x = \begin{cases} \{1+\frac{1}{n}, \frac{1}{4x}+\frac{1}{n}\}, & \text{if } x > 1; \\ \{\frac{1}{n}, \frac{1}{n}+\frac{x}{16}\}, & \text{if } 0 < x \leqslant 1; \\ \{0\}, & \text{if } x = 0; \\ \{2,3\}, & \text{otherwise.} \end{cases}$$

Let the mapping $\alpha : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ be given by

$$\alpha(x,y) = \begin{cases} 2, & \text{if } x,y \in (0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

Each T_n is multivalued α-admissible. $T_n \to T$ as $n \to \infty$. The T is given by

$$Tx = \begin{cases} \{1, \frac{1}{4x}\}, & \text{if } x > 1; \\ \{0, \frac{x}{16}\}, & \text{if } 0 < x \leqslant 1; \\ \{2,3\}, & \text{otherwise.} \end{cases}$$

T is multivalued α-admissible. We define $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{1}{2}t.$$

Each T_n is multivalued α–ψ contraction, and T is also multivalued α–ψ contraction. Let $x,y \in (0,1]$;

$$\begin{aligned} H(T_n x, T_n y) &= \max\left\{\sup_{x\in Tx} d(x,Ty), \sup_{y\in Ty} d(y,Tx)\right\} \\ &= \max\left\{\inf\left\{\left|\frac{x}{16}\right|, \left|\frac{x}{16}-\frac{y}{16}\right|\right\}, \inf\left\{\left|\frac{y}{16}\right|, \left|\frac{y}{16}-\frac{x}{16}\right|\right\}\right\} \\ &= \left|\frac{x}{16}-\frac{y}{16}\right|. \end{aligned}$$

Therefore $\alpha(x,y)H(T_n x, T_n y) \leqslant \psi(d(x,y))$.

We observe that all the conditions of Theorem 2.3 are satisfied. $F(T_1) = \{0,1\}$ and $F(T_n) = \{0\}$ for $n \geqslant 2$. $F(T) = \{0\}$. Hence

$$H(F(T_n), F(T)) \to 0 \quad \text{as } n \to \infty.$$

Remark 2.1. There is no assumption of continuity on the mapping we consider in this paper. In fact, Example 2.2 is a case where the mapping is not continuous.

Open problem: A multivalued version of α–ψ contraction was introduced in [17]. The definition of multivalued α–ψ contraction we introduce here is different from that in the above mentioned work. It remains to be seen whether α–ψ contractions can be extended to the multivalued case in some other ways also and in those cases whether the stability of fixed point sets still holds.

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