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ORIGINAL ARTICLE

A common fixed point theorem for weak contractive maps in G_p-metric spaces

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Abstract In this paper, we prove a common fixed point theorem for weak contractive maps by using the concept of G_p-metric spaces which are generalized of G-metric spaces and partial metric spaces. An illustrative example is given to support our results.

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1. Introduction

In 1922, the polish mathematician, Banach [1], proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This principle has many generalizations in different ways which established and introduced by several authors, for convenience we refer the reader to (see; e.g., [2–24]. One such generalizations is a partial metric space which introduced by Matthews [16]. In partial metric spaces, self-distance of an arbitrary point need not to be equal zero.

Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \to R^+, R^+ := [0,\infty)$, such that for all $x, y, z \in X$:

(p^1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
(p^2) $p(x,x) \leqslant p(x,y)$,
(p^3) $p(x,y) = p(y,x)$,
(p^4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

On the other hand, Mustafa and Sims [17] introduced the notation of generalized metric spaces that so-called G-metric spaces and they extended Banach principle in G-metric spaces as follows.

Definition 1.2. Let X be a non-empty set. Suppose that $G : X\times X \times X \to R^+$ satisfies:

(a) $G(x,y,z) = 0$ if $x = y = z$,
(b) $G(x,y,z) > 0, \forall x,y,z \in X, x\neq y$,
(c) $G(x,x,y) \leqslant G(x,y,z), \forall x,y,z \in X, y\neq z$,
(d) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, (symmetry in all three variables),
(e) $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z), \forall x,y,z,a \in X$.

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Then G is called a G-metric on X and (X, G) is called a G-metric space.

Recently, Zand and Nezhad [24] introduced a generalization and unification of both partial metric space and G-metric space, by giving the notation of G_p-metric space in the following way.

Definition 1.3. Let X be a non-empty set. Suppose that $G_p : X \times X \times X \to R^+$ satisfies:

(a) $x = y = z$ if $G_p(x,x,x) = G_p(y,y,y) = G_p(z,z,z)$ $\forall x,y,z \in X$,
(b) $0 \leqslant G_p(x,x,x) \leqslant G_p(x,x,y) \leqslant G_p(x,y,z), \forall x,y,z \in X$,
(c) $G_p(x,y,z) = G_p(x,z,y) = G_p(y,z,x) = \ldots$, (symmetry in all three variables),
(d) $G_p(x,y,z) \leqslant G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a), \forall x,y,z, a \in X$.

Then G_p is called a G_p-metric on X and (X, G_p) is called a G_p-metric space.

Example 1.1 [24]. Let $X = [0, \infty)$ and define $G_p(x,y,z) = \max\{x,y,z\}$ for all $x.y.z \in X$ Then (X, G_p) is a G_p-metric space, Also, one can show that (X, G_p) is not a G-metric space.

Proposition 1.1 [24]. *Let (X, G_p) is a G_p-metric space, then for any $x, y, z \in X$ and $a \in X$,it follows that*

(i) $G_p(x,y,z) \leqslant G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$,
(ii) $G_p(x,y,y) \leqslant 2G_p(x,x,y) - G_p(x,x,x)$,
(iii) $G_p(x,y,z) \leqslant G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) -2G_p\ (a,a,a)$,
(iv) $G_p(x,y,z) \leqslant G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$.

Proposition 1.2 [24]. *Every G_p-metric space (X, G_p) defines a metric space (X, D_{G_p}) where*

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y),$$

for all $x, y \in X$.

Definition 1.4 [24]. Let (X, G_p) be a G_P-metric space a sequence $\{x_n\}$ is called a G_P convergent to $x \in X$ if $\lim_{n,m\to\infty} G_p(x, x_m, x_n) = G_p(x,x,x)$.

A point $x \in X$ is said to be limit point of the sequence $\{x_n\}$ and written $x_n \to x$.

Thus if $x_n \to x$ in a G_p-metric space (X, G_p), then for any $\epsilon > 0$, there exists $l \in N$ such that $|G_p(x, x_n, x_m) - G_p(x,x,x)| < \epsilon$, for all $n, m > l$.

Proposition 1.3 [24]. *Let (X, G_p) is a G_p-metric space, Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent that*

(*i*) $\{x_n\}$ is G_p-*convergent* to x;
(*ii*) $G_p(x_n, x_n, x) \to G_p(x,x,x)$ as $n \to \infty$
(*iii*) $G_p(x_n, x, x) \to G_p(x,x,x)$ as $n \to \infty$.

Definition 1.5 [24]. Let G_p be G_p-metric space.

(i) A sequence $\{x_n\}$ is called a G_P-Cauchy if and only if $\lim_{m,n\to\infty} G_p(x_n, x_m, x_m)$ exists (and is finite).
(ii) A G_P-metric space (X, G_p) is said to be G_P-complete if and only if every GP-Cauchy sequence in X is GP-convergent to $x \in X$ such that $G_p(x,x,x) = \lim_{m,n\to\infty} G_p (x_n, x_m, x_m)$.

Definition 1.6 [17]. The two classes of following mappings are defined $\Psi = \{\psi : \psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $\psi^{-1}(0) = 0\}$, and $\Phi = \{\varphi : \varphi : [0,\infty) \to [0,\infty)$ is lower semi-continuous, nondecreasing and $\varphi^{-1}(0) = 0\}$.

Definition 1.7 [2]. Let $(X, \preceq)$ be a partially ordered set. Two maps $f, g : X \to X$ are said to be weak increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$

Lemma 1.1 [6]. *We note that if (X, G_p) be G_p-metric space, Then*

(*i*) If $G_p(x,y,z) = 0 \Rightarrow x = y = z$,
(*ii*) If $x \neq y$, then $G_p(x,y,y) > 0$.

Abbas, Nazir and Radenovic [2] proved the following result.

Theorem 1.1. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G-metric space X. Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(G(fx, gy, gy)) \leqslant \psi(M(x,y,y)) - \varphi(M(x,y,y)) \tag{1.1}$$

for all comparable $x, y \in X$ where

$$M(x,y,y) = a_1G(x,y,y) + a_2G(x,fx,fx) + a_3G(y,gy,gy) + a_4[G(x,gy,gy) + G(y,fx,fx)]$$

where $a_i > 0$ for $i = \{1,2,3,4\}$ with $a_1 + a_2 + a_3 + 2a_4 \leqslant 1$ .if f or g is continuous or for $\{x_n\}$ a nondecreasing sequence with $x_n \to z$ in X implies $x_n \preceq z$ for all $n \in \mathbb{N}$, then f and g have a common fixed point.

The aim of this paper is to generalize Theorem 1.1 to G_p-metric spaces. Also, in our result, the used contractive condition generalize condition (1.1). Finally, we give an example to support our result.

2. A main result

First we rewrite the continuity of maps in G_p-metric space as follows.

Definition 2.1. Let (X, G_p) be a G_p-metric space, partially ordered and $T : X \to X$ be a given mapping. We say that T is continuous in $x_0 \in X$ if for every sequence x_n in X, we have

(i) x_n converges to x_0 in (X, G_p) implies Tx_n converges to Tx_0 in (X, G_p).
(ii) x_n converges properly to x_0 in (X, G_p) implies Tx_n converges properly to Tx_0 in (X, G_p).

If T is continuous on each point $x_0 \in X$, then we say that T is continuous on (X, G_p).

Now, we state and prove our main result in the following way.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G_p-metric space X. Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(G_p(fx, gy, gy)) \leqslant \psi(M(x, y, y)) - \varphi(M(x, y, y)) \tag{2.1}$$

for all comparable $x, y \in X$ where

$$M(x, y, y) = max\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, gy, gy), [G_p(x, gy, gy) + G_p(y, fx, fx)]/2\}.$$

Suppose that one of the following cases is satisfied:

(i) f or g *is continuous,*
(ii) *if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$.*

Then the maps f and g have a common fixed point.

Proof. Assume that u is a fixed point of f and $G_p(u, gu, gu) > 0$, then from (2.1) with $x = y = u$, we have

$$\psi(G_p(u, gu, gu)) = \psi(G_p(fu, gu, gu)) \leqslant \psi(M(u, u, u)) - \varphi(M(u, u, u)), \tag{2.2}$$

where

$$\begin{aligned} M(u, u, u) &= max\{G_p(u, u, u), G_p(u, fu, fu), G_p(u, gu, gu), [G_p(u, gu, gu) + G_p(u, fu, fu)]/2\} \\ &= max\{G_p(u, u, u), G_p(u, u, u), G_p(u, gu, gu), [G_p(u, gu, gu) + G_p(u, u, u)]/2\} \\ &= max\{G_p(u, u, u), G_p(u, gu, gu)\} = G_p(u, gu, gu). \end{aligned}$$

Hence we get

$$\psi(G_p(u, gu, gu) = \psi(G_p(fu, gu, gu) \leqslant \psi(G_p(u, gu, gu) - \varphi(G_p(u, gu, gu) \Rightarrow \varphi(G_p(u, gu, gu) \leqslant 0.$$

a contradiction. Hence, $G_p(fu, gu, gu) = 0$. So, u is common fixed point of f and g. Similarly, if u is a fixed point of g, then one can deduce that u is also fixed point of f. Now let x_0 be an arbitrary point of X. if $fx_0 = x_0$, then the proof is finished, so we assume that $fx_0 \neq x_0$.

Now, one can construct a sequence $\{x_n\}$ in X as follows:

$$\begin{aligned} &x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2, \\ &x_2 = gx_1 \preceq fgx_1 = fx_2 = x_3, \\ &\vdots \\ &x_n \preceq x_{n+1}. \end{aligned}$$

Now since x_{2n} and x_{2n+1} are comparable so we may assume that $G_p(x_{2n}, x_{2n+1}, x_{2n+1}) > 0$, for every $n \in \mathbb{N}$. If not, then $x_{2n} = x_{2n+1}$ for some n. For all those n, using (2.1), we obtain

$$\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) = \psi(G_p(fx_{2n}, gx_{2n+1}, gx_{2n+1})) \leqslant \psi(M(x_{2n}, x_{2n+1}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1}, x_{2n+1})), \tag{2.3}$$

$$\begin{aligned} M(x_{2n}, x_{2n+1}, x_{2n+1})) = {} & max\{G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n}, fx_{2n}, fx_{2n}), \\ & G_p(x_{2n+1}, gx_{2n+1}, gx_{2n+1}), \\ & [G_p(x_{2n}, gx_{2n+1}, gx_{2n+1}) + G_p(x_{2n+1}, fx_{2n}, fx_{2n})]/2\} \\ = {} & max\{G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n}, x_{2n+1}, x_{2n+1}), \\ & G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ & [G_p(x_{2n}, x_{2n+2}, x_{2n+2}) + G_p(x_{2n+1}, x_{2n+1}, x_{2n+1})]/2\} \\ \leqslant {} & max\{G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}), \frac{1}{2}[G_p(x_{2n}, x_{2n+1}, x_{2n+1}) \\ & + G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) - G_p(x_{2n+1}, x_{2n+1}, x_{2n+1}) \\ & + G_p(x_{2n+1}, x_{2n+1}, x_{2n+1})]\} \\ = {} & G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) \end{aligned}$$

Hence

$$\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) \leqslant \psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) - \varphi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})),$$

implies that $\varphi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) = 0$ and $x_{2n+1} = x_{2n+2}$. Following the similar arguments, we obtain $x_{2n+2} = x_{2n+3}$ and hence x_{2n} becomes a common fixed point of f and g.

Now, by taking $G_p(x_{2n}, x_{2n+1}, x_{2n+1}) > 0$ for $n = 1, 2, 3, \ldots$, consider

$$\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) = \psi(G_p(fx_{2n}, gx_{2n+1}, gx_{2n+1})) \leqslant \psi(M(x_{2n}, x_{2n+1}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1}, x_{2n+1})), \tag{2.4}$$

$$\begin{aligned} M(x_{2n}, x_{2n+1}, x_{2n+1})) = {} & \max\{G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n}, fx_{2n}, fx_{2n}), \\ & G_p(x_{2n+1}, gx_{2n+1}, gx_{2n+1}), \\ & [G_p(x_{2n}, gx_{2n+1}, gx_{2n+1}) + G_p(x_{2n+1}, fx_{2n}, fx_{2n})]/2\} \\ = {} & \max\{G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n}, x_{2n+1}, x_{2n+1}), \\ & G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ & [G_p(x_{2n}, x_{2n+2}, x_{2n+2}) + G_p(x_{2n+1}, x_{2n+1}, x_{2n+1})]/2\} \\ \leqslant {} & max\{G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ & [G_p(x_{2n}, x_{2n+1}, x_{2n+1}) + G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ & - G_p(x_{2n+1}, x_{2n+1}, x_{2n+1}) + G_p(x_{2n+1}, x_{2n+1}, x_{2n+1})]/2\} \\ \leqslant {} & max\{G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ & \frac{1}{2}[G_p(x_{2n}, x_{2n+1}, x_{2n+1}) + G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})]\} \\ = {} & max\{G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})\}. \end{aligned}$$

Now if $G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) \geqslant G_p(x_{2n}, x_{2n+1}, x_{2n+1})$ for some n $= 0,1,2,\ldots$,then $M(x_{2n}, x_{2n+1}, x_{2n+1}) = G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})$ and from (2.4), we have

$$\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) \leqslant \psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) - \varphi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}))$$

implies that $\varphi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) = 0$, a contradiction. Therefore, for all $n \geqslant 0$, $G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leqslant G_p(x_{2n}, x_{2n+1}, x_{2n+1})$. Similarly, we have $G_p(x_{2n}, x_{2n+1}, x_{2n+1}) \leqslant G_p(x_{2n-1}, x_{2n}, x_{2n})$ for all $n \geqslant 0$. Hence for all $n \geqslant 0$

$$G_p(x_{n+1}, x_{n+2}, x_{n+2}) \leqslant G_p(x_n, x_{n+1}, x_{n+1})$$

and $\{G_p(x_{n+1}, x_{n+2}, x_{n+2})\}$ is a non-increasing sequence and so there exists $L \geqslant 0$, such that $\lim_{n\to\infty} G_p(x_{n+1}, x_{n+2}, x_{n+2}) = L$. Then, by the lower semi continuity of φ,

$$\varphi(L) \leqslant \lim_{n\to\infty} \inf \varphi(M(x_n, x_{n+1}, x_{n+1})).$$

We claim that $L = 0$. By lower semi continuity of φ ,taking the upper limit as $n \to \infty$ on either side of

$$\psi(G_p(x_{n+1}, x_{n+2}, x_{n+2})) \leqslant \psi(M(x_n, x_{n+1}, x_{n+1})) - \phi(M(x_n, x_{n+1}, x_{n+1})),$$

we have

$$\psi(L) \leqslant \psi(L) - \lim_{n\to\infty} \inf \varphi(M(x_n, x_{n+1}, x_{n+1})) \leqslant \psi(L) - \varphi(L),$$

i.e. $\varphi(L) \leq 0$. Thus $\varphi(L) = 0$ and we conclude that

$$\lim_{n\to\infty} G_p(x_{n+1}, x_{n+2}, x_{n+2}) = 0. \tag{2.5}$$

Now, we shall show that $\{x_n\}$ is a G_p-Cauchy sequence. For each $n \leqslant m$, and $n, m \in N$ we get

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leqslant G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G_p(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots \\ &\quad + G_p(x_{m-1}, x_m, x_m) - \{G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\quad + \ldots + G_p(x_{m-1}, x_{m-1}, x_{m-1})\} \\ &\leqslant G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G_p(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots \\ &\quad + G_p(x_{m-1}, x_m, x_m). \end{aligned}$$

By taking the limit as $n, m \to \infty$ to both side of the above inequality and from (2.5) we have

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0.$$

It follows that $\{x_n\}$ is a G_p-Cauchy sequence and by G_p-completeness of X, so there exist $z \in X$ such that $\{x_n\}$ converges to z as $n \to \infty$.

Now we will distinguish the cases (i) and (ii) of Theorem 2.1.

(i) Suppose g is continuous, since $x_{2n+1} \to z$, we obtain that $x_{2n+2} = g(x_{2n+1}) = g(z)$. But $x_{2n+2} \to z$. (as a subsequence of $\{x_n\}$) It follows that $g(z) = z$, and from the beginning of the prove we get $g(z) = z = f(z)$.The proof, assuming that f is continuous, is similar to above.
(ii) Suppose that $G_p(z, gz, gz) > 0$ and for $\{x_n\}$ and a nondecreasing sequence with $x_n \to z$ in X implies that $x_{2n+1} \preceq z$ for all $n \in \mathbb{N}$ .Now from (2.1)

$$\begin{aligned} \psi(G_p(x_{2n+1}, gz, gz)) &= \psi(G_p(fx_{2n}, gz, gz)) \\ &\leqslant \psi(M(x_{2n}, z, z)) - \varphi(M(x_{2n}, z, z)), \end{aligned}$$

where

$$\begin{aligned} M(x_{2n}, z, z) &= max\{G_p(x_{2n}, z, z), G_p(x_{2n}, fx_{2n}, fx_{2n}), \\ &\qquad G_p(z, gz, gz), [G_p(x_{2n}, gz, gz) \\ &\qquad + G_p(z, fx_{2n}, fx_{2n})]/2\} \\ &= max\{G_p(x_{2n}, z, z), G_p(x_{2n}, x_{2n+1}, x_{2n+1}), \\ &\qquad G_p(z, gz, gz), [G_p(x_{2n}, gz, gz) \\ &\qquad + G_p(z, x_{2n+1}, x_{2n+1})]/2\} \end{aligned}$$

and on taking limit as $n \to \infty$, implies $\lim_{n\to\infty} M(x_{2n}, z, z) = G_p(z, gz, gz)$. Thus

$$\begin{aligned} \psi(G_p(z, gz, gz)) &= \lim_{n\to\infty} \sup \psi(G_p(fx_{2n}, gz, gz)) \\ &\leqslant \lim_{n\to\infty} \sup[\psi(M(x_{2n}, z, z)) - \varphi(M(x_{2n}, z, z))] \\ &\leqslant \psi(G_p(z, gz, gz)) - \varphi(G_p(z, gz, gz)) \end{aligned}$$

a contradiction. Thus $G_p(z, gz, gz) = 0$ and so $z = fz = gz$. □

Put $\psi(t) = t$ in Theorem 2.1, we obtain the following.

Corollary 2.1. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G_p-metric space X. Assume that there exist $\varphi \in \Phi$ such that*

$$G_p(fx, gy, gy) \leqslant M(x, y, y) - \varphi(M(x, y, y)) \tag{2.6}$$

for all comparable $x, y \in X$ where

$$\begin{aligned} M(x, y, y) = max\{&G_p(x, y, y), G_p(x, fx, fx), G_p(y, gy, gy), \\ &[G_p(x, gy, gy) + G_p(y, fx, fx)]/2\}. \end{aligned}$$

Suppose that one of the following cases is satisfied:

(i) *f* or *g* *is continuous,*
(ii) *if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$.*

Then the maps f and g have a common fixed point.

The following corollary is G_p-metric spaces version of Theorem 1.1.

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G_p-metri space X. Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(G_p(fx, gy, gy)) \leqslant \psi(M(x, y, y)) - \varphi(M(x, y, y)) \tag{2.7}$$

for all comparable $x, y \in X$ where

$$\begin{aligned} M(x, y, y) &= a_1 G_p(x, y, y) + a_2 G_p(x, fx, fx) + a_3 G_p(y, gy, gy) \\ &\quad + a_4[G_p(x, gy, gy) + G_p(y, fx, fx)] \end{aligned}$$

where $a_i > 0$ for $i = \{1, 2, 3, 4\}$ with $a_1 + a_2 + a_3 + a_4 \leqslant 1$.

Then of the following two cases is satisfied:

(i) *f* or *g* *is continuous,*
(ii) *if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$.*

Then the maps f and g have a common fixed point.

If we set $\psi(t) = t$ in Corollary 2.2, we get the following.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G_p-metric space X satisfying*

$$G_p(fx, gy, gy) \leqslant M(x, y, y) - \varphi(M(x, y, y)) \tag{2.8}$$

for all comparable $x, y \in X$ where $\varphi \in \Phi$ and

$$\begin{aligned} M(x, y, y) &= a_1 G_p(x, y, y) + a_2 G_p(x, fx, fx) + a_3 G_p(y, gy, gy) \\ &\quad + a_4[G_p(x, gy, gy) + G_p(y, fx, fx)] \end{aligned}$$

where $a_i > 0$ for i = 1,2,3,4 with $a_1 + a_2 + a_3 + 2a_4 \leqslant 1$.

Suppose that one of the following cases is satisfied:

(i) *f* or *g is continuous,*
(ii) *if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$.*

Then the maps f and g have a common fixed point.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G_p-metric space X satisfying*

$$G_p(fx, gy, gy) \leqslant k max\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, gy, gy), [G_p(x, gy, gy) + G_p(y, fx, fx)]/2\}, \tag{2.9}$$

for all comparable $x, y \in X$.

Suppose that one of the following cases is satisfied:

(i) *f* or *g is continuous,*
(ii) *if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$.Then the maps f and g have a common fixed point.*

Proof. Define $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ for all $t \in [0, \infty)$, where $k \in [0, 1)$. Then it is clear that $\psi \in \Psi$ and $\varphi \in \Phi$. The result follows from Theorem 3.2. □

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G_p-metric space X satisfying*

$$\psi(G_p(fx, gy, gy)) \leqslant \psi(G_p(x, y, y)) - \varphi(G_p(x, y, y)) \tag{2.10}$$

for all comparable $x, y \in X$ where $\psi \in \Psi, \varphi \in \Phi$.

Suppose that one of the following cases is satisfied:

(i) *f* or *g is continuous,*
(ii) *if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$.Then the maps f and g have a common fixed point.*

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and f and g be weakly increasing self mapping on a complete G_p-metric space X satisfying*

$$G_p(fx, gy, gy) \leqslant \frac{G_p(x, y, y)}{1 + G_p(x, y, y)} \tag{2.11}$$

for all comparable $x, y \in X$.

Suppose that one of the following cases is satisfied:

(i) *f* or *g is continuous,*
(ii) *if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$.Then the maps f and g have a common fixed point.*

Example 2.1. Let $X = [0, 1]$ be a set endowed with order $x \preceq y \Longleftrightarrow y \leqslant x$. let $G_p(x, y, z) = max\{x, y, z\}$ be a G_p-metric space on X Define by $f, g : X \to X$ by $f(x) = \frac{x}{12} \forall x \in X$,

$$g(x) = \begin{cases} \frac{x}{6}; & x \in [0, \frac{1}{2}), \\ \frac{x}{3}; & x \in [\frac{1}{2}, 1). \end{cases}$$

it's clear that f is continuous and g is not continuous. and the pair $(f.g)$ is weakly increasing. f, g is commuting at $x = \frac{1}{2}$ $y = \frac{x}{12}, \psi(t) = t^2$ and $\phi(t) = \frac{t^2}{25}, t \in R^+$, then we have from Theorem 2.1

$$\psi(G_p(fx, gy, gy)) \leqslant \psi(M(x, y, y)) - \varphi(M(x, y, y))$$

since $\psi(G_p(fx, gy, gy)) = \psi([max\{fx, gy, gy\}) = \psi(\max(\frac{x}{12}, \frac{y}{6}, \frac{y}{6})$ $= \psi(\frac{x}{12}) = (\frac{x}{12})^2 = \frac{1}{288} = 0.0034$

since $y = \frac{x}{12}$

$$\begin{aligned} M(x, y, y) = {} & \max\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, gy, gy), \\ & [G_p(x, gy, gy) + G_p(y, fx, fx)]/2\} \\ = {} & \max\{\max\{x, y, y\}, \max\{x, fx, fx\}, \max\{y.gy.gy\}, \\ & \frac{1}{2}[\max\{x, gy, gy\} + \max\{y, fx, fx\}\} \\ = {} & \max\left\{x, x, y, \frac{1}{2}\left[x + \frac{x}{12}\right]\right\} = x \end{aligned}$$

Therefor

$$\begin{aligned} \psi(M(x, y, y)) - \varphi(M(x, y, y)) &= \psi(x) - \varphi(x) \\ &= (x^2) - \left(\frac{x^2}{25}\right) \\ &= \frac{24}{25}x^2 \\ &= \left(\frac{24}{25}\right)\left(\frac{1}{4}\right) = 0.24 \end{aligned}$$

then $\psi(G_p(fx, gy, gy)) = \frac{1}{288} = 0.0034 \leqslant \psi(M(x, y, y)) - \varphi (M(x, y, y)) = 0.24$ Hence all the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the common fixed point.

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