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REVIEW PAPER

Bounded linear operators in quasi-normed linear space ☆

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Abstract In this paper, we define continuity and boundedness of linear operators in quasi-normed linear space. Quasi-norm linear space of bounded linear operators is deduced. Concept of dual space is developed.

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Contents

0. Introduction

It is well known that metric and norm structures play a pivotal role in functional analysis. So in order to develop functional analysis one has to take care of the suitable generalization of these structures. Historically, the problem of generalization of the metric structure came first. Different authors introduced ideas of quasi-metric space [1,2] generalized metric space [3,4], generalized quasi-metric space [5], dislocated metric space [6], fuzzy metric space [7–9], statistical metric space [10], two

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metric space [11], quasi-normed linear space[12], fuzzy normed linear space [13], fuzzy Banach space[14], etc. Many authors [15–21] study the stability of different types of functional equations in different directions.

In [2], Rano introduce the concepts of Cauchy sequence, Convergent sequence, Open set, Closed set, etc. in a quasi-metric space and established some basic theorems such as Cantor's intersection theorem and Baire's category theorem in complete quasi-metric spaces. We give the definition of Contraction mapping and established some fixed point theorem with uniqueness. In [13], some results on finite dimensional quasi-normed linear spaces are established, the idea of equivalent quasi-norm is introduced and Riesz's lemma is proved in this space.

In this paper, we define continuity and boundedness of linear operators in quasi-normed linear space. Quasi-norm linear space of bounded linear operators is deduced. Concept of dual space is developed.

The organization of the paper is as follows:

In Section 1, comprises some preliminary results.

In Section 2, we introduce the concept of continuity and boundedness of linear operators in quasi-normed linear space.

Space of bounded linear operators and dual space are developed in Section 3.

In Section 4, we give some interesting open problems.

Throughout this paper straightforward proofs are omitted.

1. Some preliminary results

Definition 1.1 [12]. Let X be a linear space over the field F and θ the origin of X. Let $|\cdot|_q : X \to [0,\infty)$ satisfying the following conditions:

(QN-1) $|x|_q = 0$ iff $x = \theta$;
(QN-2) $|ex|_q = |e||x|_q$ for $x \in X$ and $e \in F$;
(QN-3) there exists a $K \geqslant 1$ such that

$$|x+y|_q \leqslant K\{|x|_q + |y|_q\} \quad \text{for } x, y \in X.$$

Then $(X, |\cdot|_q)$ is called a quasi-normed linear space (**qnls**) and the least value of the constant $K \geqslant 1$ is called the index of the quasi-norm $|\cdot|_q$.

The quasi-normed linear space $(X, |\cdot|_q)$ is called a strong quasi-normed linear space (**sqnls**) if it satisfies the following additional condition:

(QN-4) There exists $K \geqslant 1$ such that

$$\left|\sum_{i=1}^{n} x_i\right|_q \leqslant K\left\{\sum_{i=1}^{n} |x_i|_q\right\} \quad \forall x_i \in X, \quad \forall n \in N.$$

Note 1.1 [12]. In a quasi-normed linear space $(X, |\cdot|_q)$ with quasi index K,

$$\left|\sum_{i=1}^{n} x_i\right|_q \leqslant K^{n-1}\left\{\sum_{i=1}^{n} |x_i|_q\right\} \quad \forall x_i \in X, \quad \forall n \in N.$$

Note 1.2 [12]. If $K = 1$ then the quasi-norm $|\cdot|_q$ is reduced to a norm on X and $(X, |\cdot|_q)$ a normed linear space.

Note 1.3 [12]. Every normed linear space is a quasi-normed linear space but not conversely, which is justified by the following examples.

Example 1.1 [12]. Let $X = R^2$ be a linear space. For $x = (x_1, x_2) \in X$ define

$$|x|_q = \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2.$$

Then $(X, |\cdot|_q)$ is a quasi-normed linear space but not a normed linear space.

Definition 1.2 [12]. Let $(X, |\cdot|_q)$ be a quasi-normed linear space.

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said
  (a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} |x_n - x|_q = 0$;
  (b) to be a Cauchy sequence if $\lim_{m,n\to\infty} |x_n - x_m|_q = 0$.

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.
(iii) A subset A of X is said to be bounded if there exists a real number $M > 0$ such that $|x|_q \leqslant M \ \forall x \in A$.
(iv) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in A$.
(v) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A.

Proposition 1.1 [12]. *Let $(X, |\cdot|_q)$ be a quasi-normed linear space. Then*

(a) *the limit of a sequence $\{x_n\}$ in X if exists is unique;*
(b) *every subsequence of a convergent sequence converges to the same limit;*
(c) *every convergent sequence in X is a Cauchy sequence.*

Lemma 1.1 [12]. *Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a linearly independent set of vectors in a quasi-normed linear space $(X, |\cdot|_q)$. Then $\exists C > 0$ such that for any choice of scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have*

$$|\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n|_q \geqslant C(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|).$$

Definition 1.3 [12]. Let $(X, |\cdot|_q)$ be a quasi-normed linear space. If X is a finite dimensional linear space then $(X, |\cdot|_q)$ is called a finite dimensional quasi-normed linear space.

2. Bounded linear operators in quasi-normed linear space

In this section we define continuous and bounded linear operators in quasi-normed linear spaces and study some properties in this space.

Definition 2.1. Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be an operator. Then T is said to be continuous at $x \in X_1$ if for any sequence $\{x_n\}$ of

X_1 with $x_n \to x$ i.e. with $\lim_{n\to\infty}|x_n - x|_{q_1} = 0$ implies $T(x_n) \to T(x)$.

i.e. $\lim_{n\to\infty}|T(x_n) - T(x)|_{q_2} = 0$. If T is continuous at each point of X_1, then T is said to be continuous on X_1.

Definition 2.2. Let X_1 and X_2 be any two linear space and $T : X_1 \to X_2$ be an operator. Then T is said to be a linear operator if for any $\lambda_1, \lambda_2 \in F$ and for any $x_1, x_2 \in X_1$

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2).$$

Proposition 2.1. *Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be a linear operator. If T is continuous at a point $x \in X_1$, then T is continuous everywhere on X_1.*

Definition 2.3. Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be an operator. Then T is said to be bounded if $\exists M > 0$ such that

$$|T(x)|_{q_2} \leqslant M|x|_{q_1} \quad \forall x \in X_1.$$

Example 2.1. Let $(X, |\cdot|_q)$ be a quasi-normed linear spaces and $T : X \to X$ be an operator defined by $T(x) = 2x$. Then T is a bounded linear operator.

Theorem 2.1. *Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be a linear operator. Then T is bounded iff T is continuous.*

Proof. Let T be a bounded linear operator. Then $\exists M > 0$ such that

$$|T(x)|_{q_2} \leqslant M|x|_{q_1} \quad \forall x \in X_1.$$

Let $\{x_n\}$ be any sequence in X_1 with $x_n \to x$.

i.e. $\lim_{n\to\infty}|x_n - x|_{q_1} = 0$.

Now

$$\begin{aligned}|T(x_n) - T(x)|_{q_2} &= |T(x_n - x)|_{q_2}\\ &\leqslant M|x_n - x|_{q_1}\\ &\Rightarrow \lim_{n\to\infty}|T(x_n) - T(x)|_{q_2} = 0\\ &\Rightarrow \{T(x_n)\} \to T(x).\end{aligned}$$

So T is continuous.

Conversely, suppose T is a continuous linear operator. We have to prove that T is bounded. If possible suppose T is not bounded linear operator. Then there exists a sequence $\{x_n\}$ in X_1 such that

$$|T(x_n)|_{q_2} > n|x_n|_{q_1}.$$

Clearly $x_n \neq \theta$ for any n. Let

$$x'_n = \frac{x_n}{n|x_n|_{q_1}}.$$

Then

$$|x'_n|_{q_1} = \frac{1}{n};$$
$$\Rightarrow \lim_{n\to\infty}|x'_n|_{q_1} = 0;$$
$$\Rightarrow \lim_{n\to\infty}x'_n = \theta.$$

Since T is continuous, it follows that

$$\lim_{n\to\infty}T(x'_n) = T(\theta);$$
$$\Rightarrow \lim_{n\to\infty}|T(x'_n)|_{q_2} = 0.$$

But $|T(x'_n)|_{q_2} > 1\ \forall n$, which is a contradiction.

Hence T is bounded. □

Theorem 2.2. *Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be a linear operator. If X_1 is of finite dimensional, then T is bounded (so continuous).*

Proof. Let $dim\,X_1 = n$ and $\{x_1, x_2, x_3, \ldots, x_n\}$ be a basis of X_1.

Let $x = \sum_{i=1}^{n}\lambda_i x_i \in X_1$, then

$$|T(x)|_{q_2} = \left|T\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right|_{q_2} = \left|\sum_{i=1}^{n}\lambda_i T(x_i)\right|_{q_2}.$$

By Note 1.1,

$$\left|\sum_{i=1}^{n}\lambda_i T(x_i)\right|_{q_2} \leqslant K^{n-1}\sum_{i=1}^{n}|\lambda_i||T(x_i)|_{q_2}.$$

Let $M = K^{n-1}\mathrm{Max}\{|T(x_1)|_{q_2}, |T(x_2)|_{q_2}, \ldots, |T(x_n)|_{q_2}\}$, then

$$|T(x)|_{q_2} \leqslant M\sum_{i=1}^{n}|\lambda_i|. \tag{2.1}$$

By Lemma 1.1, $\exists C > 0$ such that

$$|x|_{q_1} = \left|\sum_{i=1}^{n}\lambda_i x_i\right|_{q_1} \geqslant C\sum_{i=1}^{n}|\lambda_i|. \tag{2.2}$$

From (2.1) and (2.2) we have

$$|T(x)|_{q_2} \leqslant \frac{M}{C}|x|_{q_1}.$$

Since Lemma 1.1 holds for any arbitrary scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

$$|T(x)|_{q_2} \leqslant \frac{M}{C}|x|_{q_1} \quad \forall x \in X_1.$$

Hence T is bounded (so continuous). □

3. Space of bounded linear operators

Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces. We denote by $L(X_1, X_2)$ the set of all linear operators from $(X_1, |\cdot|_{q_1})$ to $(X_2, |\cdot|_{q_2})$. Then $L(X_1, X_2)$ is a linear space. We shall show that the set of all bounded linear operators from $(X_1, |\cdot|_{q_1})$ to $(X_2, |\cdot|_{q_2})$ is also a linear space.

Theorem 3.1. *Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces. We denote by $B(X_1, X_2)$ the set of all* *bounded linear operators from* $(X_1, |\cdot|_{q_1})$ *to* $(X_2, |\cdot|_{q_2})$*. Then* $B(X_1, X_2)$ *is also a linear space.*

Proof. Let $T_1, T_2 \in B(X_1, X_2)$ and for $x \in X_1$,

$$(T_1 + T_2)(x) = T_1(x) + T_2(x),$$

$$(\lambda\, T_1)(x) = \lambda\, T_1(x).$$

Since T_1 and T_2 are bounded, $\exists M > 0,\ N > 0$ such that $\forall x \in X_1$

$$|T_1(x)|_{q_2} \leqslant M|x|_{q_1}$$

and $|T_2(x)|_{q_2} \leqslant N|x|_{q_1}$.

$$\begin{aligned}\text{Now } |(k_1T_1 + k_2T_2)(x)|_{q_2} &= |k_1(T_1(x)) + k_2(T_2(x))|_{q_2}\\ &= |T_1(k_1(x)) + T_2(k_2(x))|_{q_2}\\ &\leqslant K\{|T_1(k_1x)|_{q_2} + |T_2(k_2x)|_{q_2}\}.\end{aligned}$$

$$\begin{aligned}\text{Thus } |(k_1T_1 + k_2T_2)(x)|_{q_2} &\leqslant MK|k_1x|_{q_1} + NK|k_2x|_{q_1}\\ &= K(M|k_1| + N|k_2|)|x|_{q_1};\end{aligned}$$

$$\Rightarrow |(k_1T_1 + k_2T_2)(x)|_{q_2} \leqslant P|x|_{q_1} \quad \text{where } P = K(M|k_1| + M|k_2|).$$

Thus $(k_1T_1 + k_2T_2) \in B(X_1, X_2)$. □

Theorem 3.2. *Let* $(X_1, |\cdot|_{q_1})$ *and* $(X_2, |\cdot|_{q_2})$ *be two quasi-normed linear spaces. For* $T \in B(X_1, X_2)$ *we define*

$$|T|_q = \bigvee_{x(\neq\theta)\in X_1} \frac{|T(x)|_{q_2}}{|x|_{q_1}}.$$

Then $(B(X_1, X_2), |\cdot|_q)$ *is a quasi-normed linear space.*

Proof. Clearly $|T|_q \geqslant 0$ and conditions (QN-1) and (QN-2) are directly followed from definition.

For (QN-3), let $T_1, T_2 \in B(X_1, X_2)$, then

$$\begin{aligned}|T_1 + T_2|_q &= \bigvee_{x(\neq\theta)\in X_1} \frac{|(T_1 + T_2)(x)|_{q_2}}{|x|_{q_1}}\\ &= \bigvee_{x(\neq\theta)\in X_1} \frac{|T_1(x) + T_2(x)|_{q_2}}{|x|_{q_1}}\\ &\leqslant \bigvee_{x(\neq\theta)\in X_1} K\frac{|T_1(x)|_{q_2} + |T_2(x)|_{q_2}}{|x|_{q_1}}\\ &\leqslant \bigvee_{x(\neq\theta)\in X_1} K\frac{|T_1(x)|_{q_2}}{|x|_{q_1}} + \bigvee_{x(\neq\theta)\in X_1} K\frac{|T_2(x)|_{q_2}}{|x|_{q_1}}.\end{aligned}$$

So $|T_1 + T_2|_q \leqslant K\{|T_1|_q + |T_2|_q\}$.

Hence $(B(X_1, X_2), |\cdot|_q)$ is a quasi-normed linear space. □

Remark 3.1. In Theorem 3.2, we can also define

$$|T|_q = \bigvee_{x\in X_1} \{|T(x)|_{q_2} : |x|_{q_1} \leqslant 1\}.$$

Then $(B(X_1, X_2), |\cdot|_q)$ is a quasi-normed linear space and two quasi-norm in two cases are same.

Again

$$|T(x)|_{q_2} \leqslant |T|_q|x|_{q_1} \quad \forall x \in X_1.$$

Lemma 3.1. *Let* $(X, |\cdot|_q)$ *be a quasi-normed linear space and* $\{x_n\}$ *be a sequence in X such that*

$$\lim_{n\to\infty} x_n = x \text{ i.e. } \lim_{n\to\infty} |x_n - x|_q = 0.$$

Then $|x|_q = |\lim_{n\to\infty} x_n|_q \leqslant K \lim_{n\to\infty} |x_n|_q$.

Proof. By (QN-3),

$$\begin{aligned}&|x|_q \leqslant K\{|x - x_n|_q + |x_n|_q\};\\ &\Rightarrow |x|_q \leqslant \lim_{n\to\infty} K\{|x - x_n|_q + |x_n|_q\} = K\lim_{n\to\infty} |x_n|_q;\\ &\Rightarrow |x|_q = |\lim_{n\to\infty} x_n|_q \leqslant K\lim_{n\to\infty} |x_n|_q. \quad \square\end{aligned}$$

Theorem 3.3. *Let* $(X_1, |\cdot|_{q_1})$ *be a quasi-normed linear space and* $(X_2, |\cdot|_{q_2})$ *be a complete quasi-normed linear space. Then* $(B(X_1, X_2), |\cdot|_q)$ *is a complete quasi-normed linear space.*

Proof. By Theorem 3.2, $(B(X_1, X_2), |\cdot|_q)$ is a quasi-normed linear space. Next we shall prove it is complete.

Now we consider an arbitrary Cauchy sequence $\{T_n\}$ in $(B(X_1, X_2), |\cdot|_q)$ and show that $\{T_n\}$ converges to an operator in $(B(X_1, X_2), |\cdot|_q)$. Since $\{T_n\}$ is Cauchy sequence

$$\lim_{m,n\to\infty} |T_n - T_m|_q = 0.$$

So corresponding to any $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that

$$|T_n - T_m|_q < \epsilon \quad \forall m, n \geqslant N(\epsilon).$$

For all $x \in X_1$ and $m, n \geqslant N(\epsilon)$,

$$\begin{aligned}|T_n(x) - T_m(x)|_{q_2} = |(T_n - T_m)(x)|_{q_2} &\leqslant |T_n - T_m|_q|x|_{q_1}\\ &< \epsilon|x|_{q_1}.\end{aligned} \tag{3.1}$$

Now for any fixed $x \in X_1$ we see that $\{T_n(x)\}$ is a Cauchy sequence in $(X_2, |\cdot|_{q_2})$. Since $(X_2, |\cdot|_{q_2})$ is complete, $\{T_n(x)\}$ converges, say, $\lim_{n\to\infty} T_n(x) = y$. Clearly, the limit $y \in X_2$ depends on the choice of $x \in X_1$. This defines an operator $T : (X_1, |\cdot|_{q_1}) \to (X_2, |\cdot|_{q_2})$, where $y = T(x)$. We shall show that the operator T is linear.

Let $\alpha, \beta \in F$ then,

$$T(\alpha x + \beta z) = \lim_{n\to\infty} T_n(\alpha x + \beta z) = \lim_{n\to\infty} T_n(\alpha x) + T_n(\beta z).$$

Now

$$\begin{aligned}&\lim_{n\to\infty} |(T_n(\alpha x) + T_n(\beta z)) - (\alpha Tx + \beta Tz)|_{q_2}\\ &\quad\leqslant \lim_{n\to\infty} K\{|\alpha||T_n(x) - T(x)|_{q_2} + |\beta||T_n(z) - T(z)|_{q_2}\} = 0.\end{aligned}$$

Hence $T(\alpha x + \beta z) = \alpha T(x) + \beta T(z)$.

Next we have to show that T is bounded and $\lim_{n\to\infty} T_n = T$

From (2.1) we have

$$\begin{aligned}|T_n(x)-T(x)|_{q_2} &= |T_n(x)-\lim_{m\to\infty}T_m(x)|_{q_2}\\ &\leqslant K\lim_{m\to\infty}|T_n(x)-T_m(x)|_{q_2}, \quad \text{[by Lemma 3.1]}\\ &\leqslant K\lim_{m\to\infty}|T_n-T_m|_q|x|_{q_1}\\ &\leqslant K\epsilon|x|_{q_1} \quad \forall n\geqslant N(\epsilon), \quad \forall x\in X_1.\end{aligned}$$

Since $T_{N(\epsilon)}$ is bounded, there exists $M_{N(\epsilon)}>0$ such that $|T_{N(\epsilon)}(x)|_{q_2}\leqslant M_{N(\epsilon)}|x|_{q_1} \quad \forall x\in X_1$.

Thus

$$\begin{aligned}|T(x)|_{q_2} &= |T(x)-T_{N(\epsilon)}(x)+T_{N(\epsilon)}(x)|_{q_2}\\ &\leqslant K|T(x)-T_{N(\epsilon)}(x)|_{q_2}+K|T_{N(\epsilon)}(x)|_{q_2} \quad \forall x\in X_1\\ &< K^2\epsilon|x|_{q_1}+KM_{N(\epsilon)}|x|_{q_1} \quad \forall x\in X_1\\ &= (K^2\epsilon+KM_{N(\epsilon)})|x|_{q_1} \quad \forall x\in X_1.\end{aligned}$$

So T is bounded.

Now

$$\begin{aligned}|T_n-T|_q &= \bigvee_{x(\neq\theta)\in X_1}\frac{|T_n(x)-T(x)|_{q_2}}{|x|_{q_1}}\leqslant\epsilon \quad \forall n\geqslant N(\epsilon)\\ &\Rightarrow \lim_{n\to\infty}|T_n-T|_q=0\\ &\Rightarrow \lim_{n\to\infty}T_n=T. \quad \square\end{aligned}$$

Definition 3.1. Let $(X_1,|\cdot|_{q_1})$ and $(X_2,|\cdot|_{q_2})$ be two quasi-normed linear spaces. For $T\in B(X_1,X_2)$ we define

$$|T|_q=\bigvee_{x(\neq\theta)\in X_1}\frac{|T(x)|_{q_2}}{|x|_{q_1}}.$$

Then $(B(X_1,X_2),|\cdot|_q)$ is a quasi-normed linear space. The space $(B(X_1,X_2),|\cdot|_q)$ is called the Dual space of $(X_1,|\cdot|_{q_1})$ if $X_2=R$ and $|\cdot|_{q_2}=|\cdot|$. We denote the set of all bounded linear functional defined on $(X,|\cdot|_q)$ by $B(X,|\cdot|_q)$ which is the Dual space of $(X,|\cdot|_q)$.

Example 3.1. Let $X=R^2$ be a linear space. For $x=(x_1,x_2)\in X$ define

$$|x|_q=\left(\sqrt{|x_1|}+\sqrt{|x_2|}\right)^2.$$

Then $(X,|\cdot|_q)$ is a quasi-normed linear space. Let $f:X\to X$ be an operator defined by $f(x)=x\cdot a=x_1a_1+x_2a_2$. Then T is a bounded linear functional.

Proof

$$\text{Now } \frac{|f(x)|}{|x|_q}=\frac{|x_1a_1+x_2a_2|}{\left(\sqrt{|x_1|}+\sqrt{|x_2|}\right)^2}\leqslant\frac{|x_1||a_1|+|x_2||a_2|}{(|x_1|+|x_2|+2\sqrt{|x_1||x_2|})}$$
$$=\frac{|x_1||a_1|}{\left(|x_1|+|x_2|+2\sqrt{|x_1||x_2|}\right)}+\frac{|x_2||a_2|}{\left(|x_1|+|x_2|+2\sqrt{|x_1||x_2|}\right)}$$
$$\leqslant(|a_1|+|a_2|)\Rightarrow|f(x)|\leqslant(|a_1|+|a_2|)|x|_q \quad \forall x\in X. \quad \square$$

Remark 3.2. In Example 3.1, $(X,|\cdot|_q)$ is a quasi-normed linear space for quasi index $K=2$ but not a normed linear space (see [12]). Therefore there exists such type of function which are quasi-norm but not a norm.

Theorem 3.4. *Let $(X,|\cdot|_q)$ be a quasi-normed linear space. Then the Dual space $B(X,|\cdot|_q)$ of $(X,|\cdot|q)$ is a complete normed linear space.*

Proof. Proof follows from Theorem 3.3. $\square$

4. Open problems-questions

(Q1) A question naturally arises that, what will be the forms of fundamental theorems of functional analysis in this setting?
(Q2) Is this space Reflexive?
(Q3) Which type of Topology is associated with this space?
(Q4) How much it generalized the normed linear space?
(Q5) What will be the fuzzy version of this function i.e. what will be the form of quasi-fuzzy normed linear space in this contest?

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References

[1] L. Kikina, K. Kikina, A related fixed point theorem for m mapping on m complete quasi-metric spaces, Mathematica Tome 55 (78), No. 1 (2013) 51–61.
[2] G. Rano, T. Bag, Quasi-metric space and fixed point theorems, Int. J. Appl. Math. Comput. Sci. 3 (2) (2013) 27–31.
[3] P. Das, L.K. Dey, A fixed point theorem in generalized metric space, Soochow J. Math. 33 (2007) 33–39.
[4] L. Kikina, K. Kikina, Fixed point on two generalized metric spaces, Int. J. Math. Anal. 5 (30) (2011) 1459–1467.
[5] L. Kikina, K. Kikina, K. Gjino, A new fixed theorem on generalized quasi-metric spaces, ISRN Math. Anal. 2012 (2012) 1–9. Article ID: 457846.
[6] C.T. Aage, J.N. Salunke, Some results of fixed point theorem in dislocated quasi-metric spaces, Bull. Marathwada Math. Soc. 9 (2) (2008) 1–5.
[7] T. Bandyopadhyay, S.K. Samanta, P. Das, Fuzzy metric spaces redefined and a fixed point theorem, Bull. Cal. Math. Soc. 81 (1989) 247–252.
[8] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets Syst. 12 (1984) 215–229.
[9] P.V. Subrahmanyam, A common fixed point theorems in fuzzy metric spaces, Inform. Sci. 83 (3&4) (1995) 109–112.
[10] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica (1975) 326–334.
[11] B. Fisher, Fixed point in two metric spaces, Glasnik Mat 16 (36) (1981) 333–337.
[12] G. Rano, Finite dimensional quasi-normed linear space, The J. Fuzzy Math. 22 (2) (2014) 669–676.

[13] T. Bag, S.K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. (3) (2003) 687–705.
[14] R. Saadati, S.M. Vaezpour, Some results on fuzzy Banach spaces, Chaos Solitons Fract. 27 (2006) 331–344.
[15] K.W. Jun, H.M. Kim, Remarks on the stability of additive functional equation, Bull. Korean. Math. Soc 38 (2001) 679–687.
[16] K.W. Jun, H.M. Kim, The generalized Hyers–Ulam–Rassias stability of a cubic functional equation, J. Math. Anal. Appl. 274 (2002) 867–878.
[17] K. Ravi, J.M. Rassias, B.V. Senthil Kumar, A fixed point approach to the generalized Hyers–Ulam stability of reciprocal difference and adjoint functional equations, Thai J. Math. 8 (3) (2010) 469–481.
[18] K. Ravi, M. Arunkumar, J.M. Rassias, Ulam stability for the orthogonally general Euler–Lagrange type functional equations, Int. J. Math. Stat. 3 (A08) (2008) 36–46.
[19] G.Z. Eskandani, On Hyers–Ulam–Rassias stability of an additive functional equation in qiasi-Banach spaces, J. Math. Anal. Appl. 345 (2008) 405–409.
[20] T.Z. Xu, J.M. Rassias, W.X. Xu, A generalized mixed additive cubic functional equation, J. Comput. Anal. Appl. 13 (2011) 1273–1282.
[21] M. Eshaghi Gordji, S. Zolfaghari, S.K. Gharetaph, A. Ebadian, C. Park, Solution and stability of generalized mixed type additive and quadratic functional equation in non-Archimedean spaces, Ann. Univ. Ferrara 58 (2012) 49–64.