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ORIGINAL ARTICLE

On an explicit formula for inverse of triangular matrices

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Abstract In the present article, we define difference operators $B_L(a[m])$ and $B_U(a[m])$ which represent a lower triangular and upper triangular infinite matrices, respectively. In fact, the operators $B_L(a[m])$ and $B_U(a[m])$ are defined by $(B_L(a[m])x)_k = \sum_{i=0}^{m} a_{k-i}(i)x_{k-i}$ and $(B_U(a[m])x)_k = \sum_{i=0}^{m} a_{k+i}(i)x_{k+i}$ for all $k, m \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, where $a[m] = \{a(0), a(1), \dots a(m)\}$, the set of convergent sequences $a(i) = (a_k(i))_{k\in\mathbb{N}_0} (0 \leqslant i \leqslant m)$ of real numbers. Indeed, under different limiting conditions, both the operators unify most of the difference operators defined by various triangles such as $\Delta, \Delta^{(1)}, \Delta^m, \Delta^{(m)} (m \in \mathbb{N}_0), \Delta^\alpha, \Delta^{(\alpha)} (\alpha \in \mathbb{R}), B(r,s), B(r,s,t), B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$, and many others. Also, we derive an alternative method for finding the inverse of infinite matrices $B_L(a[m])$ and $B_U(a[m])$ and as an application of it we implement this idea to obtain the inverse of triangular matrices with finite support.

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1. Introduction, preliminaries and definitions

Difference operators are one of the important subclasses of Toeplitz operators where most of them are reduced to triangles under different limiting conditions. Triangular matrices have several applications in scientific computations and engineering, and the most useful contributions are solving the system of linear equations and finding spectral properties of bounded linear operators. Several methods have been employed to find the inverse of a triangle such as back ward substitution and elimination methods. The main idea of this note is to study certain triangles and derive an alternative method for their inverses.

Let w be the space all real valued sequences and for $m \in \mathbb{N}_0, a[m]$ be the set of convergent sequences $a(i) = (a_k(i))_{k\in\mathbb{N}_0} (0 \leqslant i \leqslant m)$ of real numbers. Let $x = (x_k)$ be any sequence in w, then we define the generalized difference operators $B_L(a[m])$ and $B_U(a[m])$ as :

$$(B_L(a[m])x)_k = a_k(0)x_k + a_{k-1}(1)x_{k-1} + a_{k-2}(2)x_{k-2} + \cdots + a_{k-m}(m)x_{k-m} \quad \text{and}$$
$$(B_U(a[m])x)_k = a_k(0)x_k + a_{k+1}(1)x_{k+1} + a_{k+2}(2)x_{k+2} + \cdots + a_{k+m}(m)x_{k+m} (k \in \mathbb{N}_0).$$

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It is being understood conventionally that any term with negative subscript is equal to zero. The operators $B_L(a[m])$ and $B_U(a[m])$ can be expressed as a lower triangular matrix (L_{nk}) and an upper triangular matrix (U_{nk}), respectively, where

$$(L_{nk}) = \begin{pmatrix} a_0(0) & 0 & 0 & \dots & 0 & 0 & \dots \\ a_0(1) & a_1(0) & 0 & \dots & 0 & 0 & \dots \\ a_0(2) & a_1(1) & a_2(0) & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_0(m) & a_1(m-1) & a_2(m-2) & \dots & a_m(0) & 0 & \dots \\ 0 & a_1(m) & a_2(m-1) & \dots & a_m(1) & a_{m+1}(0) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$(U_{nk}) = \begin{pmatrix} a_0(0) & a_0(1) & a_0(2) & \dots & a_0(m) & 0 & \dots \\ 0 & a_1(0) & a_1(1) & \dots & a_1(m-1) & a_1(m) & \dots \\ 0 & 0 & a_2(0) & \dots & a_2(m-2) & a_2(m-1) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_m(0) & a_m(1) & \dots \\ 0 & 0 & 0 & \dots & 0 & a_{m+1}(0) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Different kinds of triangles via difference operators have been studied by various authors. For instance, triangles such as double banded Δ, triple banded $B(r,s,t)$, fourth banded $B(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$, and $(m+1)$ banded Δ^m matrices have been introduced by Kızmaz [1], Furkan et al. [2], Dutta and Baliarsingh [3] and Et and Çolak [4], respectively. Altay and Başar [5] and Dutta and Baliarsingh [6] have studied the spectral properties of difference operators $B(r,s)$ and Δ^2, respectively. In fact, the detailed study of these operators involving topological properties, duals, matrix transformations and spectral properties is only possible by determining their inverse operators. Recently, Baliarsingh [7] and Dutta and Baliarsingh [8] have introduced fractional order difference matrix Δ^α and $(m+1)$sequential band matrix $B(a[m])$ and derived their corresponding inverse operators. However, the explicit formula for inverse of the lower triangle $B(a[m])$ has been employed in [8]. In fact, in that article this result has been proved by using counter examples, but in this investigation, we demonstrate these results in a more general way and extend those to upper triangular matrices.

Now, we define certain triangles generated by various means of the sequence $x=(x_k)$. Let $\mathcal{U}$ be the set of all sequences $u=(u_k)$ of real numbers such that $u_k \neq 0$ for all $k \in \mathbb{N}_0$. Let $r=(r_k), s=(s_k)$, and $t=(t_k)$ be three sequences in $\mathcal{U}$ and

$$T_n := \sum_{k=0}^{n} t_k (n \in \mathbb{N}_0).$$

Then the Cesàro mean of order one and Riesz mean with respect to the sequence $t=(t_k)$ are defined by the matrices $\mathcal{C}_1=(c_{nk})$ and $R^t=(r^t_{nk})$, respectively (see [9,10]), where

$$c_{nk} := \begin{cases} \frac{1}{n+1}, & (0 \leqslant k \leqslant n) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0)$$

$$r^t_{nk} := \begin{cases} \frac{t_n}{T_n}, & (0 \leqslant k \leqslant n) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0).$$

The generalized mean of the sequence $x=(x_k)$ can be computed by using $A(r,s,t)-$transform of x (see [11]), where $A(r,s,t)$ represents an infinite matrix a_{nk} and

$$a_{nk} := \begin{cases} \frac{s_{n-k}t_k}{r_n}, & (0 \leqslant k \leqslant n) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0).$$

2. Main results

In this section, we study certain results concerning the linearity, boundedness, and inverse properties of the infinite difference matrices $B_L(a[m])$ and $B_U(a[m])$. However, the results are valid for a matrix of infinite order, but it is convenient to implement those for the matrices of finite order.

Theorem 1. *The operators $B_L(a[m])$ and $B_U(a[m])$ defined from w to w are bounded linear operators.*

Proof. Linearity of the operators $B_L(a[m])$ and $B_U(a[m])$ are obvious and for boundedness,

$$\|B_L(a[m])\| = \|B_U(a[m])\| = \sup_{(k \in \mathbb{N}_0, 0 \leqslant i \leqslant m)} (m+1)|a_k(i)|. \qquad \square$$

Theorem 2. [8], Theorem 2*If $a_k(0) \neq 0$ for all $k \in \mathbb{N}_0$, then an explicit formula for inverse of the difference operator $B_L(a[m])$ is given by*

$$L^{-1}_{nk} = \begin{cases} \frac{1}{a_n(0)}, & (k=n) \\ \frac{(-1)^{n-k}}{\prod_{j=k}^{n} a_j(0)} D^{(k)}_{n-k}(a[m]), & (0 \leqslant k \leqslant n-1), \quad (n,k \in \mathbb{N}_0). \\ 0, & (k>n) \end{cases}$$

where

$$D^{(k)}_n(a[m]) = \begin{vmatrix} a_k(1) & a_{k+1}(0) & 0 & \dots & 0 & 0 & \dots & 0 \\ a_k(2) & a_{k+1}(1) & a_{k+2}(0) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_k(m) & a_{k+1}(m-1) & a_{k+2}(m-2) & \dots & a_{m-1}(1) & a_m(0) & \dots & 0 \\ 0 & a_{k+1}(m) & a_{k+2}(m) & \dots & a_{m-1}(2) & a_m(1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-m}(m) & \dots & \dots & \dots & a_{n+k-1}(1) \end{vmatrix}, \quad (n \geqslant 1).$$

Proof. We apply induction method for proving this theorem. Now, consider the system of linear equations $B_L(a[m])x = y$ and its solution in terms of y_k for $k \in \mathbb{N}_0$ can be obtained as follows:

$$x_0 = \frac{y_0}{a_0(0)}$$
$$x_1 = \frac{y_1}{a_1(0)} - \frac{a_0(1)y_0}{a_0(0)a_1(0)}$$
$$x_2 = \frac{y_2}{a_2(0)} - \frac{a_1(1)y_1}{a_1(0)a_2(0)} + \left(\frac{a_1(1)a_0(1)}{a_0(0)a_1(0)a_2(0)} - \frac{a_0(2)}{a_0(0)a_2(0)}\right)y_0$$
$$x_3 = \frac{y_3}{a_3(0)} - \frac{a_2(1)y_2}{a_2(0)a_3(0)} + \left(\frac{a_2(1)a_1(1)}{a_1(0)a_2(0)a_3(0)} - \frac{a_1(2)}{a_1(0)a_3(0)}\right)y_1 + \left(\frac{a_2(1)[a_0(2)a_1(0) - a_1(1)a_1(0)]}{a_0(0)a_1(0)a_2(0)a_3(0)} + \frac{a_0(1)a_1(2)}{a_0(0)a_1(0)a_3(0)} - \frac{a_0(3)}{a_0(0)a_3(0)}\right)y_0$$
$$\vdots$$

For simplicity we write D_n^k for $D_n^k(a[m])$ and using this notation in the above equations we obtain that

$$x_0 = \frac{y_0}{a_0(0)}, \quad x_1 = \frac{y_1}{a_1(0)} - \frac{D_1^{(0)}y_0}{\prod_{j=0}^{1}a_j(0)},$$
$$x_2 = \frac{y_2}{a_2(0)} - \frac{D_1^{(1)}y_1}{\prod_{j=1}^{2}a_j(0)} + \frac{D_2^{(0)}y_0}{\prod_{j=0}^{2}a_j(0)}$$
$$x_3 = \frac{y_3}{a_3(0)} - \frac{D_1^{(2)}y_2}{\prod_{j=2}^{3}a_j(0)} + \frac{D_2^{(1)}y_1}{\prod_{j=1}^{3}a_j(0)} - \frac{D_3^{(0)}y_0}{\prod_{j=0}^{3}a_j(0)}$$

Therefore, for $n = 1, 2, 3$, Theorem 2 holds, and by assuming so for all $k \leqslant n$, and we have

$$x_n = \frac{y_n}{a_n(0)} - \frac{D_1^{(n-1)}y_(n-1)}{\prod_{j=n-1}^{n}a_j(0)} + \frac{D_2^{(n-2)}y_{n-2}}{\prod_{j=n-2}^{n}a_j(0)} - \frac{D_3^{(n-3)}y_{n-3}}{\prod_{j=n-3}^{n}a_j(0)} + \ldots + (-1)^n\frac{D_n^{(0)}y_0}{\prod_{j=0}^{n}a_j(0)}$$

Now, for $k = n+1$, we may deduce

$$\begin{aligned}
x_{n+1} &= \frac{1}{a_{n+1}(0)}\left[y_{n+1} - a_n(1)\left(\frac{y_n}{a_n(0)} - \frac{D_1^{(n-1)}y_(n-1)}{\prod_{j=n-1}^{n}a_j(0)} + \ldots\right.\right.\\
&\quad \left. +(-1)^n\frac{D_n^{(0)}y_0}{\prod_{j=0}^{n}a_j(0)}\right) - a_{n-1}(2)\left(\frac{y_{n-1}}{a_{n-1}(0)} - \frac{D_1^{(n-2)}y_(n-2)}{\prod_{j=n-2}^{n-1}a_j(0)}\right.\\
&\quad \left.\left. + \ldots + (-1)^{n-1}\frac{D_{n-1}^{(0)}y_0}{\prod_{j=0}^{n-1}a_j(0)}\right) - \ldots - a_0(n+1)\frac{y_0}{a_0(0)}\right]\\
&= \frac{y_{n+1}}{a_{n+1}(0)} - \frac{a_n(1)y_n}{a_n(0)a_{n+1}(0)} + \left(\frac{a_n(1)D_1^{(n-1)} - a_{n-1}(2)a_n(0)}{\prod_{j=n-1}^{n+1}a_j(0)}\right)y_{n-1}\\
&\quad - \left(\frac{a_n(1)D_2^{(n-2)} - a_{n-1}(2)a_n(0)D_1^{n-2} + a_{n-2}(3)a_{n-1}(0)a_n(0)}{\prod_{j=n-2}^{n+1}a_j(0)}\right)y_{n-2}\\
&\quad + \ldots + (-1)^{n+1}\frac{D_{n+1}^{(0)}}{\prod_{j=0}^{n+1}a_j(0)} = \frac{y_{n+1}}{a_{n+1}(0)} - \frac{D_1^{(n)}y_(n)}{\prod_{j=n}^{n+1}a_j(0)}\\
&\quad + \frac{D_2^{(n-1)}y_{n-1}}{\prod_{j=n-1}^{n+1}a_j(0)} - \frac{D_3^{(n-2)}y_{n-2}}{\prod_{j=n-2}^{n+1}a_j(0)} + \ldots + (-1)^{n+1}\frac{D_{n+1}^{(0)}y_0}{\prod_{j=0}^{n+1}a_j(0)}
\end{aligned}$$

This completes the proof. $\square$

Theorem 3. *If $a_k(0) \neq 0$ for all $k \in \mathbb{N}_0$, then an explicit formula for inverse of the difference operator $B_U(a[m])$ is given by*

$$U_{nk}^{-1} = \begin{cases} \frac{1}{a_n(0)}, & (k = n) \\ \frac{(-1)^{k-n}}{\prod_{j=n}^{k}a_j(0)}D_{k-n}^{(n)}(a[m]), & (0 \leqslant k \leqslant n-1), \quad (n, k \in \mathbb{N}_0). \\ 0, & (k > n) \end{cases}$$

where

$$D_n^{(n)}(a[m]) = \begin{vmatrix} a_0(1) & a_0(2) & a_0(3) & \ldots & a_0(m-1) & \ldots & a_0(n) \\ a_1(0) & a_1(1) & a_1(2) & \ldots & a_1(m-2) & \ldots & a_0(n-1) \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & a_m(0) & a_m(1) & a_m(2) & \ldots & a_m(n-m) \\ 0 & \ldots & 0a_{k+2}(m) & a_{m+1}(0) & a_{m+1}(1) & \ldots & a_m(n-m-1) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_{n-1}(0) & a_{n-1}(1) \end{vmatrix},$$
$(n \geqslant 1)$.

Proof. Proof. Let us consider the matrix $B_U(a[m])$ of finite order $n+1$ for all $n = 0, 1, 2, \ldots$. On solving the system of equations $B_U(a[m])x = y$, it is observed that

$$x_n = \frac{y_n}{a_n(0)},$$
$$x_{n-1} = \frac{y_{n-1}}{a_{n-1}(0)} - \frac{y_n}{a_{n-1}(0)a_n(0)} = \frac{y_{n-1}}{a_{n-1}(0)} - \frac{D_1^{(n)}y_n}{\prod_{j=n-1}^{n}a_j(0)},$$
$$\begin{aligned}
x_{n-2} &= \frac{y_{n-2}}{a_{n-2}(0)} - \frac{a_{n-2}(1)y_{n-1}}{a_{n-2}(0)a_{n-1}(0)a_n(0)} + \left(\frac{a_{n-1}(1)a_{n-2}(1)}{a_{n-2}(0)a_{n-1}(0)a_n(0)} - \frac{a_{n-2}(2)}{a_{n-2}(0)a_n(0)}\right)y_n\\
&= \frac{y_{n-2}}{a_{n-2}(0)} - \frac{D_1^{(n)}y_{n-1}}{\prod_{j=n-2}^{n-1}a_j(0)} + \frac{D_2^{(n)}y_n}{\prod_{j=n-2}^{n}a_j(0)},
\end{aligned}$$
$$x_{n-3} = \frac{y_{n-3}}{a_{n-3}(0)} - \frac{D_1^{(n)}y_{n-2}}{\prod_{j=n-3}^{n-2}a_j(0)} + \frac{D_2^{(n)}y_{n-1}}{\prod_{j=n-3}^{n-1}a_j(0)} + \frac{D_3^{(n)}y_n}{\prod_{j=n-3}^{n}a_j(0)}$$
$$\vdots$$

For deducing x_0, we apply induction method as discussed in proof of Theorem 2. $\square$

Corollary 1. *The inverse of generalized fractional difference operator $\Delta^\alpha, \alpha \in \mathbb{R}$ is given by*

$$\Delta_{nk}^{-\alpha} = \begin{cases} 1, & (k = n) \\ \frac{\prod_{i=0}^{n-k-1}(\alpha+i)}{(n-k)!}, & (0 \leqslant k \leqslant n-1), \quad (n, k \in \mathbb{N}_0). \\ 0, & (k > n) \end{cases}$$

Proof. The generalized fractional difference operator Δ^α represents a lower triangular Toeplitz matrix (see [7] for detail), i.e.

$$\Delta^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ -\alpha & 1 & 0 & 0 & 0 & \ldots \\ \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & 0 & 0 & \ldots \\ -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & 0 & \ldots \\ \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!} & -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As a direct consequence of Theorem 2, the inverse of Δ^{α} can be computed as

$$\Delta^{-\alpha}_{nk}=\begin{cases}1, & (k=n)\\ (-1)^{n-k}D^{(k)}_{n-k}(\Delta^{\alpha}), & (0\leqslant k\leqslant n-1)\,,\quad (n,k\in\mathbb{N}_0),\\ 0, & (k>n)\end{cases}$$

where

$$D^{(k)}_{n}(\Delta^{\alpha})=D^{(0)}_{n}(\Delta^{\alpha})$$
$$=\begin{vmatrix}-\alpha & 1 & 0 & 0 & \dots & 0\\ \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & 0 & \dots & 0\\ -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & \dots & 0\\ \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!} & -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ (-1)^n\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} & (-1)^{n-1}\frac{\alpha(\alpha-1)\dots(\alpha-n+2)}{(n-1)!} & (-1)^{n-2}\frac{\alpha(\alpha-1)\dots(\alpha-n+3)}{(n-2)!} & \dots & \dots & -\alpha\end{vmatrix}.$$

However, using suitable row or column operations, the determinant $D^{(0)}_{n}(\Delta^{\alpha})$ can be reduced to

$$\alpha\times\begin{vmatrix}-1 & 1 & 0 & 0 & \dots & 0\\ \frac{(\alpha-1)}{2!} & -\alpha & 1 & 0 & \dots & 0\\ -\frac{(\alpha-1)(\alpha-2)}{3!} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & 1 & \dots & 0\\ \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{4!} & -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ (-1)^n\frac{(\alpha-1)\dots(\alpha-n+1)}{n!} & (-1)^{n-1}\frac{\alpha(\alpha-1)\dots(\alpha-n+2)}{(n-1)!} & (-1)^{n-2}\frac{\alpha(\alpha-1)\dots(\alpha-n+3)}{(n-2)!} & \dots & \dots & -\alpha\end{vmatrix}$$
$$=\frac{\alpha(\alpha+1)}{2}$$
$$\times\begin{vmatrix}-1 & 0 & 0 & 0 & \dots & 0\\ \frac{(\alpha-1)}{2!} & -1 & 1 & 0 & \dots & 0\\ -\frac{(\alpha-1)(\alpha-2)}{3!} & \frac{2(\alpha-1)}{3} & -\alpha & 1 & \dots & 0\\ \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{4!} & -\frac{(\alpha-1)(\alpha-2)}{4} & \frac{\alpha(\alpha-1)}{2!} & -\alpha & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ (-1)^n\frac{(\alpha-1)\dots(\alpha-n+1)}{n!} & (-1)^{n-1}\frac{(\alpha-1)\dots(\alpha-n+2)2(n-1)}{n!} & (-1)^{n-2}\frac{\alpha(\alpha-1)\dots(\alpha-n+3)}{(n-2)!} & \dots & \dots & -\alpha\end{vmatrix}$$
$$=\frac{\alpha(\alpha+1)}{2}\times\frac{(\alpha+2)}{3}$$
$$\times\begin{vmatrix}-1 & 0 & 0 & 0 & \dots & 0\\ \frac{(\alpha-1)}{2!} & -1 & 0 & 0 & \dots & 0\\ -\frac{(\alpha-1)(\alpha-2)}{3!} & \frac{2(\alpha-1)}{3} & -1 & 1 & \dots & 0\\ \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{4!} & -\frac{(\alpha-1)(\alpha-2)}{4} & \frac{3(\alpha-1)}{4} & -\alpha & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ (-1)^n\frac{(\alpha-1)\dots(\alpha-n+1)}{n!} & (-1)^{n-1}\frac{(\alpha-1)\dots(\alpha-n+2)2(n-1)}{n!} & (-1)^{n-2}\frac{(\alpha-1)\dots(\alpha-n+3)3(n-1)(n-2)}{n!} & \dots & \dots & -\alpha\end{vmatrix}$$
$$=\frac{\alpha(\alpha+1)}{2}\times\frac{(\alpha+2)}{3}\times\frac{(\alpha+3)}{4}\times\dots\times\frac{(\alpha+n-1)}{n}\times(-1)^n$$
$$=(-1)^n\frac{\alpha(\alpha+1)(\alpha+3)(\alpha+4)\dots(\alpha+n-1)}{n!}.$$

Therefore, it is being concluded that

$$\Delta^{-\alpha}_{nk}=\begin{cases}1, & (k=n)\\ \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-k-1)}{(n-k)!}, & (0\leqslant k\leqslant n-1)\,,\quad (n,k\in\mathbb{N}_0),\\ 0, & (k>n)\end{cases}$$

This completes the proof. □

Corollary 2. *The inverse of the Riesz mean operator R^t is given by*

$$(R^t)^{-1}_{nk}=\begin{cases}\frac{T_n}{t_n}, & (k=n)\\ -\frac{T_{n-k-1}}{t_{n-k}}, & (k=n-1)\,,(n,k\in\mathbb{N}_0),\\ 0, & (\text{otherwise})\end{cases}$$

Proof. The proof of this Corollary is analogous to that of Corollary 1. □

Corollary 3. *The inverse of the generalized mean operator $A(r,s,t)$ is given by*

$$a_{nk}^{-1} = \begin{cases} \frac{1}{s_0}, & (k=n) \\ -(n)^{n-k}\frac{D_{n-k}}{t_{n-k}}t_k, & (0 \leqslant k \leqslant n-1), \quad (n,k \in \mathbb{N}_0), \\ 0, & (\text{otherwise}) \end{cases}$$

where

$$D_n = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & \dots & 0 \\ s_2 & s_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \ddots & s_0 \\ s_n & s_{n-1} & s_{n-3} & \dots & s_1 \end{vmatrix}.$$

Proof. Proof of this corollary follows from the fact that for all $n, k \in \mathbb{N}_0$

$$D_n^{(k)} = \begin{vmatrix} \frac{s_1 t_k}{r_{k+1}} & \frac{s_0 t_{k+1}}{r_{k+1}} & 0 & \dots & 0 \\ \frac{s_2 t_k}{r_{k+2}} & \frac{s_1 t_{k+2}}{r_{k+1}} & \frac{s_0 t_{k+2}}{r_{k+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{s_{n-1} t_k}{r_{n-1}} & \frac{s_{n-2} t_{k+1}}{r_{n-1}} & \frac{s_{n-3} t_{k+2}}{r_{n-1}} & \ddots & \frac{s_0 t_{n-1}}{r_{n-1}} \\ \frac{s_n t_k}{r_n} & \frac{s_{n-1} t_{k+1}}{r_n} & \frac{s_{n-2} t_{k+2}}{r_{n-2}} & \dots & \frac{s_1 t_n}{r_n} \end{vmatrix}$$
$$= \frac{t_k t_{k+1} \dots t_{n-1}}{r_k r_{k+1} \dots r_n} \begin{vmatrix} s_1 & s_0 & 0 & \dots & 0 \\ s_2 & s_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \ddots & s_0 \\ s_n & s_{n-1} & s_{n-3} & \dots & s_1 \end{vmatrix}. \quad \square$$

Remark 1. Algorithm for inverse of a lower triangular matrix

- Input $n, (a_{ij})_{n\times n} = (r^1; r^2; \dots r^n)^T$, where $r^k = (r_1^k, r_2^k \dots r_n^k)$, kth row vector of $(a_{ij})_{n\times n}$ with $r_j^k = a_{kj}, (0 \leqslant j < k), a_{jj} \neq 0$ and 0 otherwise.
- Compute : $(d_{ij})_{(n-k)\times(n-k)} = (r^{k+1}; r^{k+2}; \dots r^n)^T$.
- Compute: $b_{nk} = (-1)^{n-k}\frac{\det((d_{ij})_{(n-k)\times(n-k)})}{\prod_{j=k}^{n} a_{jj}}$ for $0 \leqslant k < n$ and set $b_{nk} = \frac{1}{a_{nn}}$ for $k = n$ and 0 for $k > n$.

Example 1. let A be a lower triangular matrix of order 10 and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 8 & 9 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 0 \\ 6 & 7 & 8 & 9 & 10 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 0 & 0 \\ 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 0 \\ 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

Being a comprehensive calculation for finding the inverse of A, we omit the detail. However, using Theorem 2 and Remark 1, we mention different values of $D_n^{(k)}$ for all $n, k \in \mathbb{N}_0$ via this table assuming that $D_n^{(k)} = 0$ for all $k > n$.

$k \downarrow / n \rightarrow$	0	1	2	3	4	5	6	7	8	9
0	1	2	−2	12	168	1540	10472	22152	39888	−545568
1	3	5	−3	−42	−385	−2618	−23538	−81972	208392	0
2	6	9	6	55	374	−66	−2004	−16056	0	0
3	10	4	−5	−34	6	2364	21096	0	0	0
4	5	10	64	24	−144	−2016	0	0	0	0
5	1	7	3	−18	−252	0	0	0	0	0
6	8	5	2	28	0	0	0	0	0	0
7	6	4	−4	0	0	0	0	0	0	0
8	5	9	0	0	0	0	0	0	0	0
9	10	0	0	0	0	0	0	0	0	0

Indeed, in componentwise the elements of A^{-1} are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.6667 & 0.3333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1111 & -0.2778 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0667 & -0.0167 & -0.15 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1867 & 0.0467 & 0.02 & -0.08 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ -1.7111 & -0.4278 & -0.1833 & -0.1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1.4544 & 0.3636 & 0.1558 & 0.085 & 1.6 & -0.875 & 0.125 & 0 & 0 & 0 \\ -0.5128 & -0.5449 & 0.00046 & 0.0025 & -0.1 & 0.0625 & -0.1042 & 0.1667 & 0 & 0 \\ 0.1847 & 0.3795 & -0.0278 & -0.1970 & -0.12 & 0.075 & 0.0083 & -0.1333 & 0.2 & 0 \\ 0.2526 & 0.0965 & 0.0223 & 0.1758 & 0.168 & -0.1056 & -0.0117 & -0.0133 & -0.18 & 0.1 \end{pmatrix}$$

Example 2. Let us consider an upper triangular matrix B of order 5 and

$$B=\begin{pmatrix}1&2&3&4&5\\0&6&7&8&9\\0&0&10&11&12\\0&0&0&13&14\\0&0&0&0&15\end{pmatrix}.$$

Using Theorem 3, $B^{-1}=(u_{nk})$, the inverse of B can be directly computed as follows:

$$u_{00}=1,\quad u_{01}=\frac{(-1)D_1^{(0)}}{6}=-\frac{2}{6}=-0.3333$$

$$u_{02}=\frac{(-1)^2D_2^{(0)}}{60}=\frac{-4}{60}=-0.0667\quad u_{03}=\frac{(-1)^3D_3^{(0)}}{780}=-\frac{36}{780}=-0.0462$$

$$u_{04}=\frac{(-1)^4D_4^{(0)}}{11700}=\frac{-432}{11700}=-0.0369\quad u_{11}=\frac{1}{6}=0.1667$$

$$u_{12}=\frac{(-1)D_1^{(1)}}{60}=-\frac{7}{60}=-0.1167\quad u_{13}=\frac{(-1)^2D_2^{(1)}}{780}=\frac{-3}{780}=-0.0038$$

$$u_{14}=\frac{(-1)^3D_2^{(1)}}{11700}=-\frac{36}{11700}=-0.0031\quad u_{22}=\frac{1}{10}=0.1$$

$$u_{23}=\frac{(-1)D_1^{(2)}}{130}=-\frac{11}{130}=-0.0846\quad u_{24}=\frac{(-1)^2D_2^{(2)}}{1950}=\frac{-2}{1950}=-0.001$$

$$u_{33}=\frac{1}{13}=0.0769\quad u_{34}=\frac{(-1)D_1^{(3)}}{195}=-\frac{14}{195}=-0.718$$

$$u_{44}=\frac{1}{15}=0.0667 \text{ and } u_{nk}=0 \text{ forall } k<n.$$

3. Conclusion

The main key factor of an algorithm is its computational time. Lesser computational time corresponds to the higher efficiency of the algorithm. While finding the inverse of an $n\times n$ matrix, first, the matrix is converted to a triangle by Gauss elimination, then each element of the inverse matrix is computed recursively. Due to this recursive calculations sometimes more computational time is needed to find exactly one particular element of the inverse matrix. The main advantage of this study is to reduce the computational time for the inverse of the matrix remarkably by taking the properties of the determinant $D_n^{(k)}$ into account. For instance, the inverse of the $n\times n$ Cesàro matrix $\mathcal{C}_1$ of order 1 is given by $\mathcal{C}_1^{-1}=c_{nk}^{-1}$ and

$$c_{nk}^{-1}=\begin{cases}n, & (k=n)\\ -(n-1), & (k=n-1)\,,\quad (n,k\in\mathbb{N}_0),\\ 0, & (\text{otherwise})\end{cases}$$

This follows from the fact that $D_1^{(0)}=-(n-1)$ and $D_n^{(k)}=0$ for all $n,k>1$. Therefore, we may construct a fast algorithm which includes the properties of determinant of $D_n^{(k)}$ for evaluation of inverse of any arbitrary matrix.

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