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ORIGINAL ARTICLE

Quintic hyperbolic nonpolynomial spline and finite difference method for nonlinear second order differential equations and its application

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Abstract An efficient numerical method based on quintic nonpolynomial spline basis and high order finite difference approximations has been presented. The scheme deals with the space containing hyperbolic and polynomial functions as spline basis. With the help of spline functions we derive consistency conditions and high order discretizations of the differential equation with the significant first order derivative. The error analysis of the new method is discussed briefly. The new method is analyzed for its efficiency using the physical problems. The order and accuracy of the proposed method have been analyzed in terms of maximum errors and root mean square errors.

MATHEMATICS SUBJECT CLASSIFICATION: 65L12; 65L10; 34K28

1. Introduction

We consider the nonlinear second order boundary value problems

$$\begin{cases} -\frac{d^2y}{dx^2} + \Phi\left(x, y, \frac{dy}{dx}\right) = 0 \\ y(a) = y_a, y(b) = y_b, a < x < b \end{cases} \quad (1)$$

where the function Φ is supposed to be sufficiently smooth in order to a unique solution exists and y_a, y_b are finite constants.

The mathematical formulations of a wide variety of engineering problems are modeled by nonlinear systems of differential equations depending upon second order derivative of unknowns [1–4]. But we have limited scope of solving the vast majority of differential equations in explicit, analytic form, hence the design of suitable numerical algorithms for accurately approximating solutions is essential. The ubiquity of deferential equations throughout mathematics and its

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applications has driven the tremendous research efforts devoted to numerical solution schemes. Nowadays, one has the luxury of choosing from a wide range of excellent software packages that provide reliable and accurate results for a broad range of systems, at least, for solutions over moderately long time periods. However, all of these packages and the underlying methods have their limitations. Explicit solutions, when known, can also be used as test cases for tracking the reliability and accuracy of a chosen numerical scheme.

We survey some of the computational techniques for solving second order boundary value problems. In [5], author considered spline scaling functions and wavelets for singularly perturbed problems and discussed their convergence theory. A numerical algorithm based on optimal monitor function for mesh selection was developed for second order differential equations in [6]. The quintic spline difference scheme for linear boundary value problem was developed by [7]. In [8], author had provided an exhaustive list of references based on spline solution of two point boundary value problems in his review article. In [9], exponential spline basis for the numerical solution of two point boundary value problems over a semi-infinite range was discussed in detail. A quartic B-spline collocation scheme for fifth order boundary value problems had been developed by [10]. A uniformly convergent monotone iterative method for nonlinear boundary value problems occurring in population dynamics and heat conduction phenomenon were discussed by [11,12]. Reference [13] had developed quintic nonpolynomial spline functions to obtain approximate solution of boundary value problems with singular perturbation. Quintic nonpolynomial spline function with polynomial and trigonometric parts to obtain numerical solution of second order differential equations subject to the neumann boundary conditions was discussed by [14]. In the recent past, researchers have developed a C^∞-differentiable trigonometric and exponential spline basis for the solution of second and higher order differential equations. The detail study of nonpolynomial spline basis for second and higher order differential equations were presented by [15–17].

In the present paper, an efficient numerical algorithm based on hyperbolic functions as nonpolynomial spline basis and finite difference approximations has been developed for solving second order boundary value problems. The essence of the method lies in the fact that it offers sixth order of accuracy for differential equations with significant first order derivative. The proposed method uses evaluations at two off steps nodes. The frequency of the trigonometric parts in the nonpolynomial spline basis optimizes the accuracy of the solution. The method shows superiority in terms of order and accuracy over existing methods because it provides continuous approximations for y and dy/dx. Also the C^∞ -differentiability of the trigonometric part of nonpolynomial spline basis compensates for the loss of smoothness inherited by the polynomial approximations functions [18]. For practical purposes, we have solved both nonlinear and linear problems. The resulting difference equations are solved using standard Newton's method in case of nonlinear problem while Gauss-elimination method is used for linear problems.

2. Nonpolynomial spline finite difference approximations

The numerical scheme has been developed on the domain of integration $\Omega = [a, b]$ with the partitions $\{a = x_0 < x_1 < \cdots < x_{n+1} = b\}$ using the uniform mesh step size $h = (b-a)/(n+1)$, where $n \in \mathbb{Z}_+$. The nonpolynomial spline function spaces are more flexible than polynomials [19,20]. Our approach includes the basis

$$\mathfrak{I} = \{1, x, x^2, x^3, \sinh(\mu_1 x), \cosh(\mu_2 x)\}$$
$$\text{or } \mathfrak{I}_* = \{1, x, x^2, x^3, x^4, x^5\} \text{ when } (\mu_1, \mu_2) \to (0, 0)$$

The basis $\mathfrak{I}$ in the limiting case $(\mu_1, \mu_2) \to 0$, is consistent with the polynomial spline basis $\mathfrak{I}_*$ and it can be easily investigated using the series expansions of hyperbolic functions.

Now, for each segment $[x_k, x_{k+1}]$, $k = 0(1)n$, let $S_k(x)$ be the interpolating nonpolynomial function which interpolate $y(x)$ at x_k defined as follows:

$$S_k(x) = a_k \sinh(\mu_1(x - x_k)) + b_k \cosh(\mu_2(x - x_k)) + c_k(x - x_k)^3 + d_k(x - x_k)^2 + e_k(x - x_k) + f_k \tag{2}$$

where μ_1 and μ_2 are the frequency of hyperbolic part of the spline basis and the coefficients a_k, b_k, c_k, d_k, e_k and f_k are obtained using the following relations

$$S_k(x_k) = y_k, \quad S''_k(x_k) = M_k, \quad S''_k(x_{k\pm 1/2}) = M_{k\pm 1/2}, \quad S''''_k(x_k) = F_k \tag{3}$$

The algebraic manipulations yields the following expressions

$$a_k = \frac{2h^4 F_k \cosh(\upsilon_2)(\cosh(\upsilon_2) - 1) + h^2\theta_2^2(2M_{k+1/2} - M_k - M_{k+1})}{2\theta_1^2\theta_2^2 \sinh(\upsilon_1)(1 - \cosh(\upsilon_1))}$$

$$b_k = \frac{h^4 F_k}{\theta_2^4}$$

$$c_k = \frac{(h^2 F_k(\cosh(\upsilon_2) - 1) + \theta_2^2(M_k - M_{k+1}))\sinh(\theta_1)}{6h\theta_2^2 \sinh(\upsilon_1)(1 - \cosh(\upsilon_1))} + \frac{h^2 F_k(1 - \cosh(\upsilon_2)) + \theta_2^2(M_{k+1} - M_k)}{6h\theta_2^2(1 - \cosh(\upsilon_1))}$$

$$d_k = \frac{\theta_2^2 M_k - h^2 F_k}{2\theta_2^2}$$

$$e_k = \frac{\left(\begin{array}{l} 6h^4F_k(\theta_2^2-\theta_1^2)\sinh(\theta_1)+2h^4F_k\theta_1^2(6-\theta_2^2)\sinh(\upsilon_1)\cosh(\upsilon_2)+2h^4F_k\theta_2^2(\theta_1^2-6) \\ \sinh(\theta_1)\cosh(\upsilon_2)+(((6(y_{k+1}-y_k)-h^2(2M_{k+1/2}+M_k))\theta_2^4+(\theta_2^2+6)h^4F_k)\theta_1^2 \\ +6h^2(2M_{k+1/2}-M_k-M_{k+1})\theta_2^4+6h^4\theta_2^2F_k)\sinh(\theta_1)+(((2h^2(M_{k+1}+2M_k) \\ +12(y_k-y_{k+1}))\theta_2^4-4h^4F_k(\theta_2^2+3))\theta_1^2)\sinh(\upsilon_1) \end{array}\right)}{12h\theta_1^2\theta_2^4(\cosh(\upsilon_1)-1)\sinh(\upsilon_1)}$$

$$f_k = \frac{\theta_2^4 y_k - h^4 F_k}{\theta_2^4}, \quad \theta_1 = h\mu_1, \quad \theta_2 = h\mu_2, \quad \upsilon_1 = \frac{\theta_1}{2}, \quad \upsilon_2 = \frac{\theta_2}{2}$$

Now, using the continuity conditions of first and third derivatives, that is

$$\begin{cases} S'_{k-1}(x_k) = S'_k(x_k) \\ S'''_{k-1}(x_k) = S'''_k(x_k) \end{cases} \tag{4}$$

Eliminating F_k from Eq. (4), we obtain the following nonpolynomial spline relation

$$y_{k-1} - 2y_k + y_{k+1} + \alpha(M_{k-1} + M_{k+1}) + \beta M_k + \gamma(M_{k-1/2} + M_{k+1/2}) = 0 \tag{5}$$

where

$$\alpha = -\frac{1}{6}(h^2((12\theta_2^2(\cosh(\upsilon_2)-1)\sinh(\theta_1)+(6\theta_1^2(2+\theta_2^2-2\times\cosh(\theta_2))\sinh(\upsilon_1)+\theta_1^3(6\cosh(\theta_2)-6-2\theta_2^2\cosh(\upsilon_2)-\theta_2^2)+12\theta_1\theta_2^2(1-\cosh(\upsilon_2))))/(\theta_1^2\theta_2^2(2(\cosh(\upsilon_2)-1)\times\sinh(\theta_1)+2(1-\cosh(\theta_2))\sinh(\upsilon_1)+\theta_1(1-2\cosh(\upsilon_2)+\cosh(\theta_2))))))$$

$$\beta = -\frac{1}{3}\left(h^2\left(\left(12\left(\theta_2^2-\theta_1^2\right)\sinh(\theta_1)-6\theta_1^2\theta_2^2\sinh(\upsilon_1)+\theta_1^3\left(6+\theta_2^2\right)+12\theta_1(\theta_1\sinh(\upsilon_1)-\theta_2^2)\right)\cosh(\theta_2)+\left(6\theta_2^2\left(\theta_1^2-2\right)\sinh(\theta_1)+4\theta_1\theta_2^2(3-\theta_1^2)\right)\cosh(\upsilon_2)+12\theta_1^2(\sinh(\theta_1)-\sinh(\upsilon_1))-6\theta_1^3\right)\right)/\left(\theta_1^2\theta_2^2((\theta_1-2\sinh(\upsilon_1))\cosh(\theta_2)+2(\sinh(\theta_1)-\theta_1)\cosh(\upsilon_2)+2\sinh(\upsilon_1)+\theta_1-2\sinh(\theta_1)\right))$$

$$\gamma = -\frac{1}{3}\left(h^2\left(\left(6\sinh(\theta_1)\left(\theta_1^2-\theta_2^2\right)+\theta_1^3(\theta_2^2-6)+6\theta_1\theta_2^2\right)\cosh(\theta_2)-3\theta_1^2\sinh(\theta_1)(2+\theta_2^2)+6\theta_2^2((1-\theta_1)\sinh(\theta_1)+2\theta_1^3(\theta_2^2+3))\right)\right)/\left(\theta_1^2\theta_2^2((\theta_1-2\sinh(\upsilon_1))\cosh(\theta_2)+2(\sinh(\theta_1)-\theta_1)\cosh(\upsilon_2)+2\sinh(\upsilon_1)+\theta_1-2\sinh(\theta_1)\right))$$

When $(\mu_1, \mu_2)\to(0, 0)$ or equivalently $(\theta_1, \theta_2)\to(0, 0)$ and $(\upsilon_1, \upsilon_2)\to(0, 0)$, we obtain $(\alpha,\beta,\gamma)\to-\frac{h^2}{60}[1,26,16]$ and the nonpolynomial spline relation defined by (5) reduces into usual quintic polynomial spline difference scheme.

Now, consider the following approximations

$$\tilde{y}'_k = \frac{1}{2h}(y_{k+1}-y_{k-1}) \tag{6}$$

$$\tilde{y}'_{k\pm1} = \frac{1}{2h}(\pm3y_{k\pm1}\mp4y_k\pm y_{k\mp1}) \tag{7}$$

$$\widetilde{F}_{k+\tau} = \Phi\left(x_{k+\tau}, y_{k+\tau}, \tilde{y}'_{k+\tau}\right), \tau=0,\pm1 \tag{8}$$

$$\tilde{y}_{k\pm1/2} = \frac{1}{32}(15y_{k\pm1}+18y_k-y_{k\mp1})-\frac{h^2}{64}(3\widetilde{F}_{k\pm1}+4\widetilde{F}_k-\widetilde{F}_{k\mp1}) \tag{9}$$

$$\tilde{y}'_{k\pm1/2} = \frac{1}{4h}(\pm5y_{k\pm1}\mp6y_k\pm y_{k\mp1})\mp\frac{h}{48}(3\widetilde{F}_{k\pm1}+8\widetilde{F}_k+\widetilde{F}_{k\mp1}) \tag{10}$$

$$\widehat{M}_{k\pm1/2} = \Phi\left(x_{k\pm1/2}, \tilde{y}_{k\pm1/2}, \tilde{y}'_{k\pm1/2}\right) \tag{11}$$

$$\hat{y}'_{k\pm1} = \tilde{y}'_k \pm\frac{h}{3}(2\widetilde{F}_k+\widetilde{F}_{k\mp1}) \tag{12}$$

$$\widehat{M}_{k\pm1} = \Phi\left(x_{k\pm1}, y_{k\pm1}, \hat{y}'_{k\pm1}\right) \tag{13}$$

$$\hat{\hat{y}}'_k = \tilde{y}'_k + h(\sigma_1(\widetilde{M}_{k+1}-\widetilde{M}_{k-1})+\sigma_2(\widehat{M}_{k+1}+\widehat{M}_{k-1})+\sigma_3(\widehat{M}_{k+1/2}+\widehat{M}_{k-1/2})) \tag{14}$$

$$\widetilde{M}_k = \Phi\left(x_k, y_k, \hat{\hat{y}}'_k\right) \tag{15}$$

where σ_1, σ_2 and σ_3 are to be determined so as to satisfy the approximations

$$M_k - \tilde{M}_k = O(h^8) \quad \text{and } M_k - \widehat{M}_{k\pm\wp} = O(h^8), \quad \wp = 1, 1/2$$

Now, incorporating the above approximations into the Eq. (5), we obtain the following nonpolynomial spline finite difference scheme for $k = 1(1)n$

$$y_{k-1}-2y_k+y_{k+1}+\alpha(\widehat{M}_{k-1}+\widehat{M}_{k+1})+\beta\widetilde{M}_k+\gamma(\widehat{M}_{k-1/2}+\widehat{M}_{k+1/2}) = O(h^8) \tag{16}$$

where $y_0 = y_a$ and $y_{n+1} = y_b$. If we choose $[\alpha,\beta,\gamma] = -\frac{h^2}{60}[1,26,16]$, the method (6) reduces to the method of [22,23]. The method (16) has local truncation error of sixth order. Since the difference scheme computes y_{k-1}, y_k and y_{k+1} and the resulting scheme has tri-diagonal iteration matrix, the method is compact. Neglecting $O(h^8)$ and higher order terms, the method is applicable to the numerical solution of linear and nonlinear second order differential equation with appropriate Dirichlet's boundary conditions. For the nonlinear problems, standard Newton's method can be used with sufficiently close initial guess and in case of linear problems Gauss-elimination solver is used for computations.

3. Formulation of the difference scheme

In this section, we explain the formulation of the hyperbolic spline finite difference method (16) and obtain its local truncation errors.

The series expansion yields the approximations (6) and (7) as

$$\tilde{y}'_k = y_k^{(1)}+\frac{h^2}{6}y_k^{(3)}+\frac{h^4}{120}y_k^{(5)}+O(h^6) \tag{17}$$

$$\tilde{y}'_{k\pm1} = y_{k\pm1}^{(1)}-\frac{h^2}{3}y_k^{(3)}\mp\frac{h^3}{12}y_k^{(4)}-\frac{h^4}{30}y_k^{(5)}\mp\frac{h^5}{180}y_k^{(6)}+O(h^6) \tag{18}$$

Define $P_k = \frac{\partial\Phi}{\partial y}\Big|_{x_k}$, $Q_k = \frac{\partial\Phi}{\partial y^{(1)}}\Big|_{x_k}$, $R_k = \frac{\partial^2\Phi}{\partial(y^{(1)})^2}\Big|_{x_k}$, etc.

With the help of Eq. (8), we get

$$\widetilde{M}_k = M_k+\frac{h^2}{6}Q_k y_k^{(3)}+\frac{h^4}{360}\omega_1+O(h^6) \tag{19}$$

$$\widetilde{M}_{k\pm1} = M_{k\pm1}-\frac{h^2}{3}Q_k y_k^{(3)}\mp\frac{h^3}{12}\omega_2-\frac{h^4}{180}\omega_3\mp\frac{h^5}{360}\omega_4+O(h^6) \tag{20}$$

where

$$\omega_1 = 5R_k\left(y_k^{(3)}\right)^2+3Q_k y_k^{(5)}\omega_2 = 4Q_k^{(1)}y_k^{(3)}+Q_k y_k^{(4)}\omega_3 = 3\left(2Q_k y_k^{(5)}+5Q_k^{(1)}y_k^{(4)}\right)+10\left(3Q_k^{(2)}-R_k y_k^{(3)}\right)y_k^{(3)}\omega_4 = \left(2Q_k+15Q_k^{(2)}\right)y_k^{(4)}+12Q_k^{(1)}y_k^{(5)}+10\left(2Q_k^{(3)}-R_k y_k^{(4)}-2R_k^{(1)}y_k^{(3)}\right)y_k^{(3)}$$

With the help of series expansion, we obtain

$$y_{k\pm1/2} = y_k \pm \frac{h}{2}y_k^{(1)} + \frac{h^2}{8}y_k^{(2)} \pm \frac{h^3}{48}y_k^{(3)} + \frac{h^4}{348}y_k^{(4)} + O(h^5) \tag{21}$$

$$y^{(1)}_{k\pm1/2} = y_k^{(1)} \pm \frac{h}{2}y_k^{(2)} + \frac{h^2}{8}y_k^{(3)} \pm \frac{h^3}{48}y_k^{(4)} + \frac{h^4}{348}y_k^{(5)} + O(h^5) \tag{22}$$

$$y^{(1)}_{k\pm1} = y_k^{(1)} \pm hy_k^{(2)} + \frac{h^2}{2}y_k^{(3)} \pm \frac{h^3}{6}y_k^{(4)} + \frac{h^4}{24}y_k^{(5)} + O(h^5) \tag{23}$$

Further let,

$$\begin{pmatrix} \tilde{y}_{k+1/2} \\ \tilde{y}_{k-1/2} \\ \tilde{y}'_{k+1/2} \\ \tilde{y}'_{k-1/2} \\ \widehat{y}'_{k+1} \\ \widehat{y}'_{k-1} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} & \sigma_{46} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} & \sigma_{56} \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma_{66} \end{pmatrix} \begin{pmatrix} y_k \\ y_{k+1} \\ y_{k-1} \\ \widetilde{M}_k \\ \widetilde{M}_{k+1} \\ \widetilde{M}_{k-1} \end{pmatrix} \tag{24}$$

where σ_{lm}, l, m = 1(1)6 to be determined so as to obtain the following approximations

$$\tilde{y}_{k\pm1/2} - y_{k\pm1/2} = O(h^5) \tag{25}$$

$$\tilde{y}'_{k\pm1/2} - y^{(1)}_{k\pm1/2} = O(h^5) \tag{26}$$

$$\widehat{M}'_{k\pm1} - y^{(1)}_{k\pm1} = O(h^4) \tag{27}$$

Using the approximations (19)–(23) in Eq. (24) and equating the like powers of h, we get the values of undermined coefficients α_{lm} that satisfies the relations (9), (10) and (12), further it leads to the following approximations

$$\tilde{y}_{k\pm1/2} = y_{k\pm1/2} \mp \frac{h^5}{768}\left(5y_k^{(5)} - 4\omega_2\right) - \frac{h^6}{46080}\left(33y_k^{(6)} + 8(\omega_1 - \omega_3)\right) + O(h^7) \tag{28}$$

$$\tilde{y}'_{k\pm1/2} = y^{(1)}_{k\pm1/2} - \frac{h^4}{5760}\left(7y_k^{(5)} - 20\omega_2\right) \mp \frac{h^5}{34560}(57y_k^{(6)} + 16(\omega_1 - \omega_3)) - \frac{h^6}{46080}y_k^{(7)} + O(h^7) \tag{29}$$

$$\widehat{M}'_{k\pm1} = y^{(1)}_{k\pm1} + \frac{h^4}{180}\left(4y_k^{(5)} - 5\omega_2\right) \pm \frac{h^5}{540}\left(3y_k^{(6)} + \omega_1 - \omega_3\right) - \frac{h^6}{720}y_k^{(7)} + O(h^7) \tag{30}$$

With the help of approximations (28)–(30), we may write the equations (11) and (13) as follows:

$$\widehat{M}_{k\pm1/2} = M_{k\pm1/2} - \frac{h^4 Q_k}{5760}\left(7y_k^{(5)} - 20\omega_2\right) \mp \frac{h^5\omega_5}{34560} + O(h^6) \tag{31}$$

$$\widehat{M}_{k\pm1} = M_{k\pm1} + \frac{h^4 Q_k}{180}\left(4y_k^{(5)} - 5\omega_2\right) \pm \frac{h^5\omega_6}{540} + O(h^6) \tag{32}$$

where

$$\omega_5 = 45\left(5y_k^{(5)} - 4\omega_2\right)P_k + 6\left(7y_k^{(5)} - 20\omega_2\right)Q_k^{(1)} + \left(57y_k^{(6)} + 16(\omega_1 - \omega_3)\right)Q_k\omega_6 = 3\left(4y_k^{(5)} - 5\omega_2\right)Q_k^{(1)} + \left(3y_k^{(6)} + \omega_1 - \omega_3\right)Q_k$$

Next, we obtain $O(h^4)$ approximations for $\widehat{\widehat{y}}'_k$, i.e. $\widehat{\widehat{y}}'_k - y_k^{(1)} = O(h^4)$. Using the approximations (20), (31) and (32) in (14), we get

$$\tilde{\tilde{y}}'_k = \tilde{y}'_k + h\sigma_1\left(M_{k+1} - M_{k-1} - \frac{h^3\omega_2}{6} - \frac{h^5\omega_4}{180}\right) + h\sigma_2\left(M_{k+1} - M_{k-1} + \frac{h^5\omega_6}{270}\right) + h\sigma_3\left(M_{k+1/2} - M_{k-1/2} - \frac{h^5\omega_5}{17280}\right) + O(h^7) \tag{33}$$

With the help of relations (3), we can easily obtain

$$M_{k+1} - M_{k-1} = y^{(2)}_{k+1} - y^{(2)}_{k-1} = 2h\left(y_k^{(3)} + \frac{h^2}{6}y_k^{(5)} + \frac{h^4}{120}y_k^{(7)}\right) + O(h^7) \tag{34}$$

$$M_{k+1/2} - M_{k-1/2} = y^{(2)}_{k+1/2} - y^{(2)}_{k-1/2} = 2h\left(\frac{1}{2}y_k^{(3)} + \frac{h^2}{48}y_k^{(5)} + \frac{h^4}{3840}y_k^{(7)}\right) + O(h^7) \tag{35}$$

Applying the approximation (17), (34) and (35), we can rewrite the Eq. (33) as follows:

$$\tilde{\tilde{y}}'_k = y_k^{(1)} + \frac{h^2}{6}y_k^{(3)} + \frac{h^4}{120}y_k^{(5)} + 2\sigma_1 h^2\left(y_k^{(3)} + \frac{h^2}{6}y_k^{(5)}\right) + 2\sigma_2 h^2\left(y_k^{(3)} + \frac{h^2}{6}y_k^{(5)}\right) + \sigma_3 h^2\left(y_k^{(3)} + \frac{h^2}{24}y_k^{(5)}\right) + O(h^6) \tag{36}$$

Form the Eq. (34), if

$$2(\sigma_1 + \sigma_2) + \sigma_3 + \frac{1}{6} = 0 \tag{37}$$

Then, we obtain $\tilde{\tilde{y}}'_k - y_k^{(1)} = O(h^4)$ and

$$\tilde{\tilde{y}}'_k = y_k^{(1)} + \frac{h^4}{120}\left((1 + 40(\sigma_1 + \sigma_2) + 5\sigma_3)y_k^{(5)} - 20\sigma_1\omega_2\right) + O(h^6) \tag{38}$$

Now, with the help of Eq. (15), we get

$$\widetilde{\widetilde{M}}_k = M_k + \frac{h^4}{120}((1 + 40(\sigma_1 + \sigma_2) + 5\sigma_3)y_k^{(5)} - 20\sigma_1\omega_2) + O(h^6) \tag{39}$$

Using the finite difference approximations (31), (32) and (39) in the hyperbolic spline relation (5), we obtain

$$y_{k-1} - 2y_k + y_{k+1} + \alpha(\widehat{M}_{k-1} + \widehat{M}_{k+1}) + \beta\widetilde{\widetilde{M}}_k + \gamma(\widehat{M}_{k-1/2} + \widehat{M}_{k+1/2}) = \frac{h^4}{2880}(128\alpha + 24(5\sigma_3 + 40\sigma_1 + 40\sigma_2 + 1)\beta - 7\gamma)Q_k y_k^{(5)} - \frac{h^4}{144}(8\alpha + 24\beta\sigma_1 - \gamma)Q_k\left(4Q_k^{(1)}y_k^{(3)} + Q_k y_k^{(4)}\right) + O(h^8) \tag{40}$$

The difference method (40) is to be of $O(h^8)$, if the coefficient of h^6 vanishes. Since α, β and γ are $O(h^2)$, we must have

$$8\alpha + 24\beta\sigma_1 - \gamma = 0 \tag{41}$$

$$128\alpha + 24(5\sigma_3 + 40(\sigma_1 + \sigma_2) + 1)\beta - 7\gamma = 0 \tag{42}$$

Solving the Eqs. (37), (41) and (42) for σ_1, σ_2 and σ_3, we obtain

$$\begin{aligned}\sigma_1 &= -(8\alpha-\gamma)/(24\beta)\\ \sigma_2 &= (112\alpha-4\beta-23\gamma)/(720\beta)\\ \sigma_3 &= (128\alpha-56\beta-7\gamma)/(360\beta)\end{aligned} \tag{43}$$

Thus, the local truncation error becomes $O(h^8)$ and hence the method is sixth order accurate.

4. Error analysis

In this section, we investigate the error of the hyperbolic spline finite difference scheme. Consider the second order differential equation

$$-\frac{d^2y}{dx^2}+a(x)\frac{dy}{dx}+b(x)y+g(x)=0, \quad 0\leqslant x\leqslant 1 \tag{44}$$

Applying the quintic nonpolynomial spline finite difference technique (16) to the differential Eq. (44), we obtain the following difference equation:

$$p_k y_{k-1}+q_k y_k+r_k y_{k+1}=R_k+T_k, \quad k=1(1)n \tag{45}$$

where

$$\begin{aligned}p_k=&\frac{-1}{1248h}\begin{pmatrix}(120+h(8a_k+9a_{k-1}-6hb_{k-1}-a_{k+1}))(2\gamma ha_k+13\beta)a_{k-1/2}\\ +(24+h(8a_k-3a_{k+1}-2hb_{k-1}+3a_{k-1}))(2\gamma ha_k-13\beta)a_{k+1/2}\end{pmatrix}\\ &+\frac{1}{1664}\begin{pmatrix}(60+h(a_{k+1}+9a_{k-1}-6hb_{k-1}+4a_k))(2\gamma ha_k+13\beta)b_{k-1/2}\\ -(-4+h(4a_k-3a_{k+1}-3a_{k-1}+2hb_{k-1}))(2\gamma ha_k-13\beta)b_{k+1/2}\end{pmatrix}\\ &+\frac{1}{312h}\begin{pmatrix}312h-156\alpha(a_{k+1}+a_{k-1})+104\alpha ha_k(a_{k-1}-a_{k+1})+156\alpha ha_{k-1}^2\\ -104\alpha h^2a_{k-1}b_{k-1}+52\alpha h(a_{k+1}^2+6b_{k-1})+\gamma ha_k(3a_{k-1}+5a_{k+1})\\ +2\gamma h^2a_k(b_{k-1}+a_k(a_{k+1}+a_{k-1}))+\gamma h^2a_k(3a_{k-1}^2-a_{k+1}^2)-2\gamma a_k(h^3a_{k-1}b_{k-1}+78)\end{pmatrix}\end{aligned}$$

$$\begin{aligned}q_k=&\frac{-1}{312h}\begin{pmatrix}(h(3a_{k+1}-4hb_k-a_{k-1})-36)(2\gamma ha_k-13\beta)a_{k+1/2}\\ +(h(a_{k+1}-3a_{k-1}-4hb_k)-36)(2\gamma ha_k+13\beta)a_{k-1/2}\end{pmatrix}\\ &-\frac{1}{416}\begin{pmatrix}(h(3a_{k+1}-2hb_k+a_{k-1})+18)(2\gamma ha_k-13\beta)b_{k+1/2}\\ +(h(3a_{k-1}+2hb_k+a_{k+1})-18)(2\gamma ha_k+13\beta)b_{k-1/2}\end{pmatrix}\\ &+\frac{1}{78}\begin{pmatrix}52h\alpha(a_{k+1}-a_{k-1})b_k-52\alpha(a_{k+1}^2+a_{k-1}^2)+78(\gamma b_k-2)\\ -\gamma(h^2b_k+2)(a_{k+1}+a_{k-1})a_k+\gamma ha_k(a_{k+1}^2-a_{k-1}^2)\end{pmatrix}\end{aligned}$$

$$\begin{aligned}r_k=&\frac{1}{1248h}\begin{pmatrix}(h(9a_{k+1}+6hb_{k+1}+8a_k-a_{k-1})-120)(2\gamma ha_k-13\beta)a_{k+1/2}\\ +(h(3a_{k+1}+2hb_{k+1}+8a_k-3a_{k-1})-24)(2\gamma ha_k+13\beta)a_{k-1/2}\end{pmatrix}\\ &+\frac{1}{1664}\begin{pmatrix}(h(9a_{k+1}+6hb_{k+1}+4a_k+a_{k-1})-60)(2\gamma ha_k-13\beta)b_{k+1/2}\\ -(h(4a_k-3a_{k-1}-3a_{k+1}-2hb_{k+1})+4)(2\gamma ha_k+13\beta)b_{k-1/2}\end{pmatrix}\\ &-\frac{1}{312h}\begin{pmatrix}-156\alpha(a_{k+1}+a_{k-1})+104\alpha ha_k(a_{k-1}-a_{k+1})-52\alpha h\left(a_{k-1}^2+3a_{k+1}^2\right)\\ -104\alpha h^2a_{k+1}b_{k+1}-312h(1+\alpha b_{k+1})+2(h^3a_{k+1}b_{k+1}-78)\gamma a_k\\ -\gamma ha_k(3a_{k+1}+5a_{k-1})+\gamma h^2a_k(2b_{k+1}-a_{k-1}^2)+\gamma h^2a_k(2a_ka_{k+1}\\ +3a_{k+1}^2+2a_ka_{k-1})\end{pmatrix}\end{aligned}$$

$$\begin{aligned}R_k=&\frac{-h}{624}((8g_k+g_{k-1}+3g_{k+1})(2\gamma ha_k-13\beta)a_{k+1/2}\\ &+\left(8g_k+g_{k+1}+3g_{k-1})(2\gamma ha_k+13\beta)a_{k-1/2}\right)\\ &-\frac{h^2}{832}((4g_k+3g_{k+1}-g_{k-1})(2\gamma ha_k-13\beta)b_{k+1/2}\\ &-\left(4g_k-g_{k+1}+3g_{k-1})(2\gamma ha_k+13\beta)b_{k-1/2}\right)\\ &+(2\gamma ha_k/13-\beta)g_{k+1/2}-(2\gamma ha_k/13+\beta)g_{k-1/2}\\ &+\frac{1}{156}\begin{pmatrix}2\gamma(h^2a_ka_{k+1}-78)g_k+\gamma ha_k(g_{k+1}-g_{k-1})-156\alpha(g_{k+1}+g_{k-1})\\ +104\alpha h(a_{k-1}-a_{k+1})g_k\\ +52\alpha h(a_{k-1}g_{k-1}-a_{k+1}g_{k+1})+\gamma h^2a_k(a_{k+1}g_{k+1}+2a_{k-1}g_k+a_{k-1}g_{k-1})\end{pmatrix}\end{aligned}$$

and $T_k = O(h^8)$.

Incorporating the boundary values, the system of algebraic equations given by (45) in matrix notations can be written as

$$\boldsymbol{P}\boldsymbol{y}+\boldsymbol{J}+\boldsymbol{T}=\boldsymbol{0} \tag{46}$$

where

$$\boldsymbol{P}=\begin{bmatrix}q_1 & r_1 & & & \bigcirc\\ p_2 & q_2 & r_2 & & \\ & \ddots & \ddots & \ddots & \\ & & p_{n-1} & q_{n-1} & r_{n-1}\\ \bigcirc & & & p_n & q_n\end{bmatrix}=\left(P_{ij}\right)_{n\times n},\ \boldsymbol{J}=\begin{bmatrix}R_1-y_0p_1\\ R_2\\ \vdots\\ R_{n-1}\\ R_n-y_{n+1}r_n\end{bmatrix},\ \boldsymbol{T}=\begin{bmatrix}T_1\\ T_2\\ \vdots\\ \vdots\\ T_n\end{bmatrix}$$

Let $\boldsymbol{Y} = [y(x_1),\ y(x_2),\ \ldots,\ y(x_n)]^T \cong \boldsymbol{y} = [y_1,\ y_2,\ \ldots,\ y_n]^T$ satisfies

$$\boldsymbol{P}\boldsymbol{Y}+\boldsymbol{J}=\boldsymbol{0} \tag{47}$$

Let $\varepsilon_k = |y_k - y(x_k)|$ be the discretization errors at the grid point x_k and $\boldsymbol{\varepsilon} = [\varepsilon_1,\ \varepsilon_2,\ \ldots,\ \varepsilon_n]^T$ be the error vector. With the help of Eqs. (46) and (47), we obtain the error equation

$$\boldsymbol{P}\boldsymbol{\varepsilon}+\boldsymbol{T}=\boldsymbol{0} \tag{48}$$

Let

$$\begin{aligned}\overset{\frown}{A}&=\max_k\{|a_k|,|a_{k\pm1}|,|a_{k\pm1/2}|\}, & \overset{\smile}{A}&=\min_k\{|a_k|,|a_{k\pm1}|,|a_{k\pm1/2}|\}\\ \overset{\frown}{B}&=\max_k\{|b_k|,|b_{k\pm1}|,|b_{k\pm1/2}|\}, & \overset{\smile}{B}&=\min_k\{|b_k|,|b_{k\pm1}|,|b_{k\pm1/2}|\}\\ \overset{\frown}{G}&=\max_k\{|g_k|,|g_{k\pm1}|,|g_{k\pm1/2}|\}, & \overset{\smile}{G}&=\min_k\{|g_k|,|g_{k\pm1}|,|g_{k\pm1/2}|\}\end{aligned}$$

Then, for $k = 1(1)n$

$$\begin{aligned}|P_{k,k+1}|\leqslant{}&1+\frac{\gamma h}{52}\overset{\frown}{A}{}^3+\left(\frac{h^2}{39}\gamma\overset{\frown}{B}+\frac{2\alpha}{3}-\frac{8\gamma}{39}-\frac{\beta}{12}\right)\overset{\frown}{A}{}^2-\frac{\beta h^2}{32}\overset{\frown}{B}{}^2\\ &+\left(\frac{7\beta}{16}+\alpha\right)\overset{\frown}{B}+\left(\frac{\gamma h^3}{104}\overset{\frown}{B}{}^2-\frac{h}{96}(13\beta+8\gamma-32\alpha)\overset{\frown}{B}+\frac{1}{2h}(2\alpha+2\beta+\gamma)\right)\overset{\frown}{A}\end{aligned}$$

For $k = 2(1)n$

$$\begin{aligned}|P_{k,k-1}|\leqslant{}&1-\frac{\gamma h}{52}\overset{\frown}{A}{}^3+\left(\frac{\gamma h^2}{39}\overset{\frown}{B}+\frac{2\alpha}{3}-\frac{8\gamma}{39}-\frac{\beta}{12}\right)\overset{\frown}{A}{}^2-\frac{\beta h^2}{32}\overset{\frown}{B}{}^2\\ &+\left(\frac{7\beta}{16}+\alpha\right)\overset{\frown}{B}+\left(-\frac{\gamma h^3}{104}\overset{\frown}{B}{}^2+\frac{h}{96}(13\beta+8\gamma-32\alpha)\overset{\frown}{B}-\frac{1}{2h}(2\alpha+2\beta+\gamma)\right)\overset{\frown}{A}\end{aligned}$$

Since α, β and γ are all $O(h^2)$, hence, for sufficiently small h and θ_i, υ_i, $i = 1, 2$, we obtain

$$|P_{k,k+1}|\leqslant 1, k=1(1)n-1 \quad \text{and } |P_{k,k-1}|\leqslant 1, k=2(1)n$$

Thus, $\boldsymbol{P}$ is irreducible and monotone [21]. Consequently, we find $\boldsymbol{P}^{-1}$ exists and $\boldsymbol{P}^{-1}\geqslant 0$.

Now, let

$$S_k=\begin{cases}q_k+r_k, & k=1\\ p_k+q_k+r_k, & k=2(1)n-1\\ p_k+q_k, k=n & \end{cases}$$

From the error Eq. (48), we obtain the error bound

$$\|\boldsymbol{\varepsilon}\|\leqslant\|\boldsymbol{P}^{-1}\|.\|\boldsymbol{T}\| \tag{49}$$

If P_{ij}^{-1} be the (i, j)th element of $\boldsymbol{P}^{-1}$ and we define the matrix-vector norms as

$$\|\boldsymbol{P}^{-1}\|=\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\left|P_{ij}^{-1}\right| \quad \text{and } \|\boldsymbol{T}\|=\max_{1\leqslant i\leqslant n}|T_i| \tag{50}$$

Also, from the theory of matrix, we have

$$\sum_{j=1}^{n}P_{ij}^{-1}S_j=1, i=1(1)n \tag{51}$$

With the help of Taylor's expansion, for $i = 1(1)n$, we obtain

$$P_{i1}^{-1} \leqslant \left|\frac{1}{S_1}\right| \leqslant \frac{2h\breve{A}}{(2\alpha+2\beta+\gamma)} + O(h^2) \tag{52}$$

$$\sum_{j=2}^{n-1} P_{ij}^{-1} \leqslant \left|\frac{1}{\min\limits_{2\leqslant j\leqslant n-1} S_j}\right| \leqslant \frac{2h\breve{B}}{2\alpha+2\beta+\gamma} + O(h^2) \tag{53}$$

$$P_{in}^{-1} \leqslant \left|\frac{1}{S_n}\right| \leqslant \frac{2h\breve{A}}{2\alpha+2\beta+\gamma} + O(h^2) \tag{54}$$

With the help of Eqs. 48 and (52)–(54), we obtain

$$\|\boldsymbol{\varepsilon}\| \leqslant \left(\frac{\breve{B}}{2\alpha+2\beta+\gamma} + O(h^2)\right) O(h^8) \tag{55}$$

Also, we have $2\alpha + 2\beta + \gamma = -h^2$. Hence, it follows that $\|\boldsymbol{\varepsilon}\| \leqslant O(h^6)$. The error analysis of nonlinear problems can be investigated in a similar manner and we conclude above results in the following theorem:

Theorem 4.1. *The hyperbolic spline finite difference method defined by (16) for solving the second order boundary value problems, with sufficiently small h and (μ_1, μ_2) gives a sixth order convergent solution.*

5. Computational illustrations

In this section, we have implemented our method for solving some of the problems arising in population dynamics, first order equation in chemical reactor, heat transfer and convection-diffusion problems. We compute the maximum absolute errors: $L_\infty^n(\varepsilon)$ and root mean square errors: $L_2^n(\varepsilon)$ using the following formula

$$L_\infty^n(\varepsilon) = \max_{1\leqslant k\leqslant n}|y_k - y(x_k)|, \quad L_2^n(\varepsilon) = \frac{1}{n}\left(\sum_{k=1}^{n}|y_k - y(x_k)|^2\right)^{1/2}$$

The numerical accuracy are obtained using following three approaches

(a) Sixth order finite difference method.
(b) Hyperbolic spline finite difference method ($\mu_1 = \mu_2$).
(c) Hyperbolic spline finite difference method ($\mu_1 \neq \mu_2$).

The symbols $L_\infty^{n*}(\varepsilon)$ and $L_2^{n*}(\varepsilon)$ represents the estimates of $L_\infty^n(\varepsilon)$ and $L_2^n(\varepsilon)$ in case of hyperbolic spline finite difference method ($\mu_1 = \mu_2$) while $L_\infty^{n**}(\varepsilon)$ and $L_2^{n**}(\varepsilon)$ represents the estimates of $L_\infty^n(\varepsilon)$ and $L_2^n(\varepsilon)$ in case of sixth order polynomial spline finite difference method.

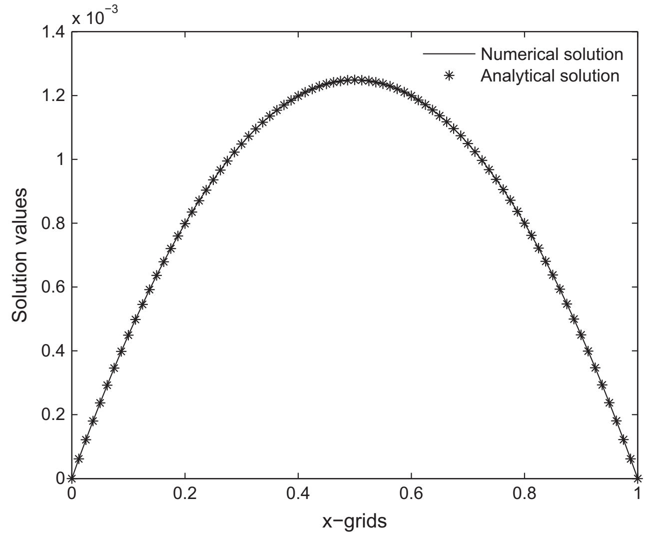

Figure 1 Numerical solution vs. analytical solution for Problem 5.1.

Problem 5.1. Consider the convection-dominated equation (see [24])

$$\begin{cases} -\frac{d^2y}{dx^2} + \lambda\left(\frac{dy}{dx} + y - 1\right) = 0 \\ y(0) = y(1) = 0, \quad 0 < x < 1 \end{cases}$$

The analytical solution is give by

$$y(x) = 1 + \frac{\left(\left(e^{\frac{1-\sqrt{1+4/\lambda}}{2/\lambda}} - 1\right)e^{\frac{1+\sqrt{1+4/\lambda}}{2/\lambda}x} + \left(1 - e^{\frac{1+\sqrt{1+4/\lambda}}{2/\lambda}}\right)e^{\frac{1-\sqrt{1+4/\lambda}}{2/\lambda}x}\right)}{\left(e^{\frac{1+\sqrt{1+4/\lambda}}{2/\lambda}} - e^{\frac{1-\sqrt{1+4/\lambda}}{2/\lambda}}\right)}$$

The numerical results comprising maximum absolute errors and root mean square errors are reported in Table 1 for various mesh arrangements corresponding to $\lambda = 0.01$ and optimum frequency parameter (μ_1, μ_2). The numerical solution obtained using hyperbolic spline finite difference scheme closely approximate the analytical solution. Fig. 1 illustrate the comparison of numerical solution and analytical solution values at $n = 80$. The proposed numerical solution gives almost overlapping behavior with the corresponding exact solution values.

Problem 5.2. Consider the heat transfer in a thin wire with the given temperatures $A \geqslant 0$ and $B \geqslant 0$ at two ends. The temperature distribution $y(x)$ in the thin wire based on Boltzmann fourth power law is given by

Table 1 The maximum absolute error and root mean square error for Problem 5.1.

n	μ_1	μ_2	$L_2^n(\varepsilon)$	$\mu_1 = \mu_2$	$L_2^{n*}(\varepsilon)$	$L_2^{n**}(\varepsilon)$
10	0.010	0.110	1.02e−03	0.0400	1.68e−02	1.88e−02
20	0.010	2.760	2.96e−05	0.0500	4.77e−04	1.76e−03
30	2.290	0.030	3.69e−06	0.1430	2.71e−05	2.86e−04
40	0.860	0.098	5.11e−08	0.2039	1.08e−06	6.61e−05

Table 2 The maximum absolute error and root mean square error for Problem 5.2.

n	μ_1	μ_2	$L^n_\infty(\varepsilon)$	$L^n_2(\varepsilon)$	$L^{n**}_\infty(\varepsilon)$	$L^{n**}_2(\varepsilon)$
10	3.680	0.010	1.23e−06	7.80e−07	2.17e−06	1.55e−06
20	0.503	0.303	1.65e−08	9.08e−09	5.41e−08	3.65e−08
40	3.281	1.282	2.84e−10	1.54e−10	1.02e−09	6.78e−10
80	6.046	6.000	4.98e−12	2.53e−12	1.74e−11	1.15e−11

Table 3 The maximum absolute error and root mean square error for Problem 5.3.

n	μ_1	μ_2	$L^n_\infty(\varepsilon)$	$L^n_2(\varepsilon)$	$L^{n**}_\infty(\varepsilon)$	$L^{n**}_2(\varepsilon)$
10	0.200	0.100	1.60e−07	1.25e−07	*overflow*	*overflow*
20	3.200	0.100	7.18e−09	5.45e−09	*overflow*	*overflow*
40	4.600	0.800	9.45e−11	7.08e−11	*overflow*	*overflow*
80	8.000	2.200	5.71e−12	4.25e−12	*overflow*	*overflow*

Table 4 The maximum absolute error and root mean square error for Problem 5.4.

n	μ_1	μ_2	$L^n_\infty(\varepsilon)$	$L^n_2(\varepsilon)$	$L^{n**}_\infty(\varepsilon)$	$L^{n**}_2(\varepsilon)$
10	0.2	0.1	8.43e−07	6.52e−07	*overflow*	*overflow*
20	3.2	0.1	2.62e−08	1.98e−08	*overflow*	*overflow*
40	4.6	0.8	2.14e−10	1.59e−10	*overflow*	*overflow*
80	8.0	2.2	3.12e−12	2.31e−12	*overflow*	*overflow*

$$\frac{d}{dx}\left(\frac{1}{1+y^2}\frac{dy}{dx}\right) + \lambda g(x) = \lambda y^4, \quad y(0) = A, \quad y(1) = B$$

where $\lambda > 0$ is a constant and $g(x)$ denotes the surrounding temperature (see [1]). The proposed method is tested for the solution of above problem using exact solution $y(x) = \sin(\pi x)$. The function $g(x)$ may be obtained from analytical solution as a test procedure. The maximum absolute errors are obtained in Table 2 from the presented method corresponding to $\lambda = 1$ and optimum value of frequency (μ_1, μ_2).

Problem 5.3. Consider the logistic equation in population dynamics

$$\begin{cases} -\frac{d}{dx}\left(\tan^{-1}(y)\frac{dy}{dx}\right) = \lambda y(1-y) + g(x) \\ y(0) = y(1) = 0, \quad 0 < x < 1, \end{cases}$$

where λ is a positive constant and $g(x) \geqslant 0$ is an internal source (see [4,11]). The analytical solution of the problem is given by $y(x) = x - x^2$. The problem is solved using proposed method using standard Newton's solver with $\lambda = 1$. Table 3 presents the numerical accuracy of computed solution with error tolerance of 10^{-15}. The method takes very little iteration to converge the desired accuracy.

Problem 5.4. Consider the nonlinear problem arising in chemical reactor with first order equation (see [2,12])

$$\begin{cases} -\frac{d}{dx}\left(\sqrt{1+y^2}\frac{dy}{dx}\right) = \lambda(1-y)e^{-v/(1+y)} + g(x) \\ y(0) = 1, \quad y(1) = 1/2, \quad 0 < x < 1 \end{cases}$$

The analytical solution of the problem is $y(x) = 1 - \frac{x^2}{2}$. The computational results are shown in Table 4 for $\lambda = 1$, $v = 0.5$ and various values of n.

The tabulated results show that two parameter hyperbolic spline finite difference method shows superiority over single parameter method and classical finite difference technique. In case of nonlinear problems, the finite difference method gives divergent solution. The proposed method can be extended to nonlinear elliptic problems.

6. Conclusion

The proposed method is sixth order accurate using two parameter hyperbolic nonpolynomial spline basis for the numerical treatment of two point boundary value problems with significant first order derivative. The significance of two parameters has been shown while solving convection dominated equations. Proposed method provides the convergent solution to the problems related with logistic equation in population dynamics and chemical reactor theory, compared with classical finite difference method. The graphical illustration of numerical results shows that the proposed method maintains a very high accuracy for dealing with the solution.

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