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A comparative study of numerical methods for solving the generalized Ito system

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Abstract   This paper is devoted to the numerical comparison of methods applied to solve the generalized Ito system. Four numerical methods are compared, namely, the Laplace decomposition method (LDM), the variation iteration method (VIM), the homotopy perturbation method (HPM) and the Laplace decomposition method with the Pade approximant (LD–PA) with the exact solution.

1. Introduction

In recent years 1999, the variational iteration method (VIM) was proposed by He in [1–6]. This method is now widely used by many researchers to study linear and nonlinear partial differential equations. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications, linear or nonlinear, homogeneous or inhomogeneous, equations and systems of equations as well. Many authors [7–11] that this method is more powerful than existing

techniques such as the Adomian method [12,16], perturbation method, etc. showed it. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. Another important advantage is that the VIM method is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications. Many authors for different cases have obtained some exact and numerical solutions of the generalized Ito system (see [17–20]).

Consider the generalized Ito system [21]:

$$u_t = v_x, \quad v_t = -2v_{xxx} - 6(uv)_x + aww_x + bpw_x + cwp_x$$
$$+ dpp_x + fw_x + gp_x,$$
$$w_t = w_{xxx} + 3uw_x, \quad p_t = p_{xxx} + 3up_x. \tag{1}$$

The general exact solution of above system is:

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$$u(x,t) = \frac{2-\beta}{3} - 2\tanh(x-t\beta)^2, v(x,t)$$

$$= \frac{\beta^2-2}{3} + 4\beta\tanh(x-t\beta)^2,$$

$$w(x,t) = \frac{-bf+ag-b^2c_0+adc_0}{a(b-c)}$$

$$+ \frac{2\left(-\frac{b^2c\sqrt{-8a\beta-5a\beta^2}}{a\sqrt{-bc+ad}} + \frac{bd\sqrt{-8a\beta-5a\beta^2}}{\sqrt{-bc+ad}}\right)\tanh(x-t\beta)}{-bc+ad},$$

$$P(x,t) = c_0 - \frac{2\sqrt{-8a\beta-5a\beta^2}\tanh(x-t\beta)}{\sqrt{-bc+ad}}, \tag{2}$$

For simplicity, we take: $a = -37$, $b = 2$, $c = \frac{1}{2}$, $d = -1$, $f = 2$, $g = 2$, $c_0 = -1$, $\beta \to \frac{1}{4}$. We have:

$$u(x,t) = \frac{7}{12} - 2\tanh\left(x-\frac{t}{4}\right)^2, \quad v(x,t) = -\frac{7}{48} + \frac{1}{2}\tanh\left(x-\frac{t}{4}\right)^2,$$

$$w(x,t) = -\frac{36}{37} + \frac{1}{6}\tanh\left(x-\frac{t}{4}\right), \quad p(x,t) = 4 + \frac{37}{12}\tanh\left(x-\frac{t}{4}\right), \tag{3}$$

with initial conditions:

$$u(x,0) = \frac{7}{12} - 2\tanh(x)^2, v(x,0) = -\frac{7}{48} + \frac{1}{2}\tanh(x)^2,$$

$$w(x,0) = -\frac{36}{37} + \frac{1}{6}\tanh(x), p(x,0) = 4 + \frac{37}{12}\tanh(x). \tag{4}$$

The aim of this paper is to use the Laplace decomposition method (LDM), the variation iteration method (VIM), Homotopy perturbation method (HPM) and the Pade approximant (LD–PA) to find the numerical solution of Eq. (1), compare our obtained results with the exact solution, and compute the error.

2. Methods and its applications

2.1. The variational iteration method [3–8]

For the purpose of illustration of the methodology to the proposed method, using variational iteration method [22–26], we write a system in an operator form as:

$$L_t u + R_1(u,v,w,p) + N_1(u,v,w,p) = g_1,$$

$$L_t v + R_2(u,v,w,p) + N_2(u,v,w,p) = g_2,$$

$$L_t w + R_3(u,v,w,p) + N_3(u,v,w,p) = g_3,$$

$$L_t p + R_4(u,v,w,p) + N_4(u,v,w,p) = g_4, \tag{5}$$

with initial data

$$u(x,0) = f_1(x), \quad v(x,0) = f_2(x),$$

$$w(x,0) = f_3(x), \quad p(x,0) = f_4(x), \tag{6}$$

where L_t is considered a first-order partial differential operator, R_j, $1 \leqslant j \leqslant 4$, and N_j, $1 \leqslant j \leqslant 4$, are linear and nonlinear operators respectively, and g_1, g_2 and g_3 are source terms. In what follows we give the main steps of He's variational iteration method in handling scientific and engineering problems. The system (5) can be written as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1\{Lu_n(\tau) + R_1(\breve{u},\breve{v},\breve{w},\breve{p})$$

$$+ N_1(\breve{u},\breve{v},\breve{w},\breve{p}) - g_1(\tau)\}d\tau,$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2\{Lv_n(\tau) + R_2(\breve{u},\breve{v},\breve{w},\breve{p})$$

$$+ N_2(\breve{u},\breve{v},\breve{w},\breve{p}) - g_2(\tau)\}d\tau,$$

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3\{Lw_n(\tau) + R_3(\breve{u},\breve{v},\breve{w},\breve{p})$$

$$+ N_3(\breve{u},\breve{v},\breve{w},\breve{p}) - g_3(\tau)\}d\tau,$$

$$p_{n+1}(x,t) = p_n(x,t) + \int_0^t \lambda_4\{Lp_n(\tau) + R_4(\breve{u},\breve{v},\breve{w},\breve{p})$$

$$+ N_4(\breve{u},\breve{v},\breve{w},\breve{p}) - g_4(\tau)\}d\tau, \tag{7}$$

where λ_j, $1 \leqslant j \leqslant 4$, are general Lagrange multipliers [7], which can be identified optimally via the variational theory, and u_n, v_n, w_n and p_n are restricted variations which means $\delta u_n = 0$, $\delta v_n = 0$, $\delta w_n = 0$ and $\delta p_n = 0$. It is required first to determine the Lagrange multipliers λ_j that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)$, $v_{n+1}(x,t)$, $w_{n+1}(x,t)$, $p_{n+1}(x,t)$, $n \geqslant 0$, of the solutions $u(x,t)$, $v(x,t)$, $w(x,t)$ and $p(x,t)$ will follow immediately upon using the Lagrange multipliers obtained and by using selected functions u_0, v_0, and w_0. The initial values are usually used for the selected zeroth approximations. With the Lagrange multipliers λj determined, then several approximations $u_j(x,t)$, $v_j(x,t)$, $w_j(x,t)$, $p_j(x,t)$ $j \geqslant 0$, can be determined. Consequently, the solutions are given by

$$u(x,t) = \lim_{n\to\infty\partial} u_n(x,t), \quad v(x,t) = \lim_{n\to\infty\partial} v_n(x,t),$$

$$w(x,t) = \lim_{n\to\infty\partial} p_n(x,t), \quad p(x,t) = \lim_{n\to\infty\partial} p_n(x,t). \tag{8}$$

2.2. The Laplace decomposition method (LDM) [27]

In this section, Laplace decomposition method [27–29] is applied to the system of partial differential Eq. (1). The method consists of first applying the Laplace transformation to both sides of (1)

$$\pounds[L_t u] = \pounds[g_1] + \pounds[R_1(u,v,w,p) + N_1(u,v,w,p)],$$

$$\pounds[L_t v] = \pounds[g_2] + \pounds[R_2(u,v,w,p) + N_2(u,v,w,p)],$$

$$\pounds[L_t w] = \pounds[g_3] + \pounds[R_3(u,v,w,p) + N_3(u,v,w,p)],$$

$$\pounds[L_t p] = \pounds[g_4] + \pounds[R_4(u,v,w,p) + N_4(u,v,w,p)]. \tag{9}$$

Using the formulas of the Laplace transform, we get

$$s\pounds[u] - u(0) = \pounds[g_1] + \pounds[R_1(u,v,w,p) + N_1(u,v,w,p)],$$

$$s\pounds[v] - v(0) + \pounds[R_2(u,v,w,p) + N_2(u,v,w,p)],$$

$$s\pounds[w] - w(0) = \pounds[g_3] + \pounds[R_3(u,v,w,p) + N_3(u,v,w,p)],$$

$$s\pounds[p] - p(0) = \pounds[g_4] + \pounds[R_4(u,v,w,p) + N_4(u,v,w,p)]. \tag{10}$$

In the Laplace decomposition method, we assume the solution as an infinite series, given as follows:

$$u = \sum_n^\infty u_n, \quad v = \sum_n^\infty v_n, \quad w = \sum_n^\infty w_n, \quad p = \sum_n^\infty p_n, \tag{11}$$

where the terms u_n are to be recursively computed. In addition, the linear and nonlinear terms R_1, R_2, R_3, R_4 and N_1, N_2, N_3, N_4 are decompose as an infinite series of Adomian polynomials (see [12–16]):

From a numerical point of view, the approximation

$$u(x) = \lim_{n \to \infty} [\phi_{1n}], \quad v(x) = \lim_{n \to \infty} [\phi_{2n}], \quad w(x)$$
$$= \lim_{n \to \infty} [\phi_{3n}], \quad p(x) = \lim_{n \to \infty} [\phi_{4n}], \tag{12}$$

where

$$\phi_{1n} = \sum_{k=0}^{n-1} u_k(x), \quad \phi_{2n} = \sum_{k=0}^{n-1} v_k(x), \quad \phi_{3n}$$
$$= \sum_{k=0}^{n-1} w_k(x), \quad \phi_{4n} = \sum_{k=0}^{n-1} p_k(x), \tag{13}$$

can be used in the Laplace decomposition scheme for computing the approximate solution. It is also clear that a better approximation can be obtained by evaluating more components of the series solution (11) of $u(x, t)$, $v(x, t)$, $w(x, t)$ and $p(x, t)$.

2.3. The Pade approximant [27]

Here we will investigate the construction of the Pade approximates [27,30–32] for the functions studied. The main advantage of Pade approximation over the Taylor series approximation is that the Taylor series approximation can exhibit oscillati, which may produce an approximation error bound. Moreover, Taylor series approximations can never blow-up in a fin region. To overcome these demerits we use the Pade approximations. The Pade approximation of a function is given by ratio of two polynomials. The coefficients of the polynomial in both the numerator and the denominator are determined using the coefficients in the Taylor series expansion of the function. The Pade approximation of a function, symbolized by $[m/n]$, is a rational function defined by

$$\left[\frac{m}{n}\right] = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}, \tag{14}$$

where we considered $b_0 = 1$, and the numerator and denominator have no common factors. In the LD–PA method, we use the method of Pade approximation as an after-treatment method to the solution obtained by the Laplace decomposition method. This after-treatment method improves the accuracy of the proposed method.

2.4. Homotopy perturbation method [33–36]

One considers the following nonlinear differential equation to represent the procedure of this method,

$$A(U) - f(r) = 0, \quad r \in \Omega, \tag{15}$$

with the boundary conditions:

$$B\left(U, \frac{\partial U}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{16}$$

where A and B are general differential operator and boundary operator, respectively. Γ is the boundary of the domain Ω, and $f(r)$ is a given analytical function. After dividing the general operator into a linear part (L) and a nonlinear part (N), one can rewrite the Eq. (15) as

$$L(U) + N(U) - f(r) = 0. \tag{17}$$

By constructing the homotopy technique to Eq. (4), one can write a homotopy in the form

$$H(V, p) = (1 - p)[L(V) - L(U_0)] + p[A(V) - f(r)]$$
$$= 0, p \in [0, 1], \quad r \in \Omega, \tag{18}$$

where $p \in [0, 1]$ is an embedding parameter and U_0 is an initial approximation of the Eq. (15) that satisfies Eq. (16). In HPM, one can use the embedding parameter as a small parameter. Therefore, the solution of Eq. (17) can be written a power series of p in the form

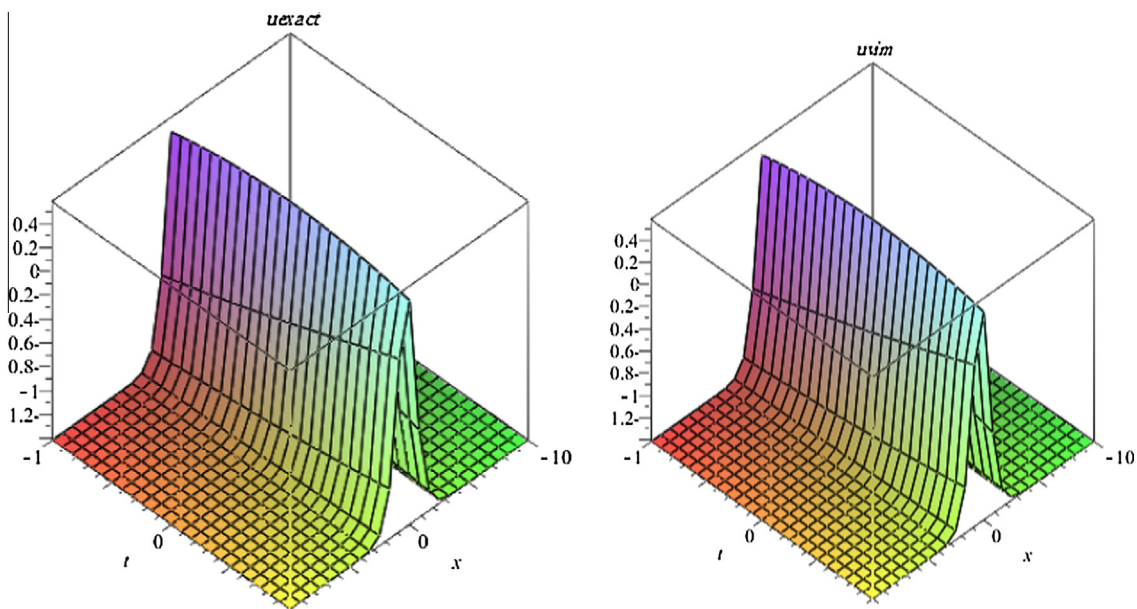$$V = V_0 + p V_1 + p^2 V_2 + \cdots. \tag{19}$$

Fig. 1a Exact solution of $u(x, t)$ and numerical solution of $u(x, t)$, $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.
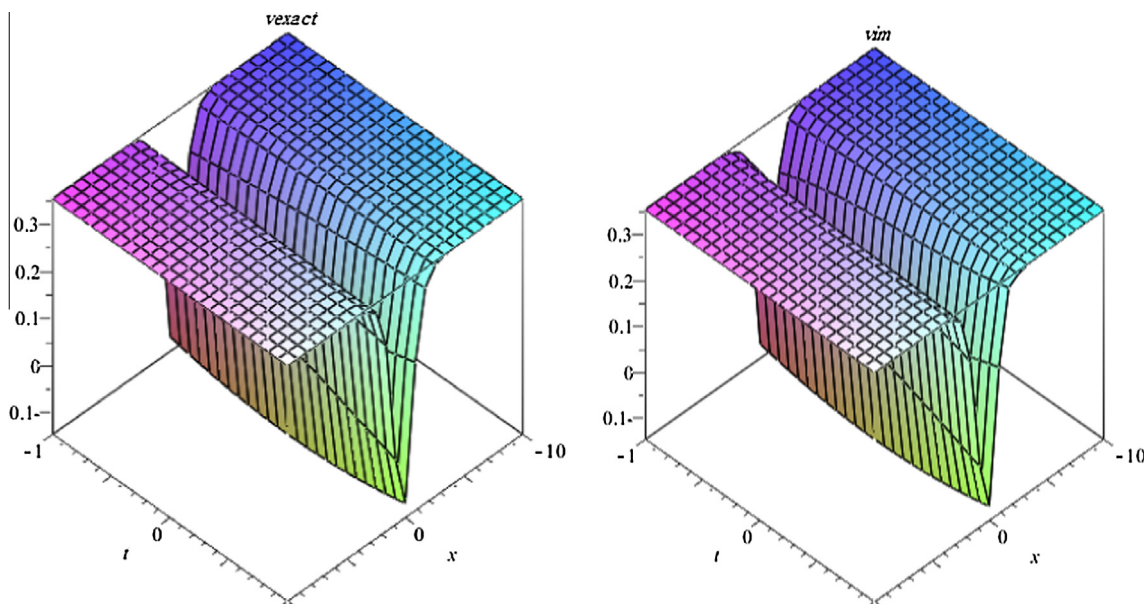
Fig. 1b Exact solution of $v(x, t)$ and numerical solution of $v(x, t)$, $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.
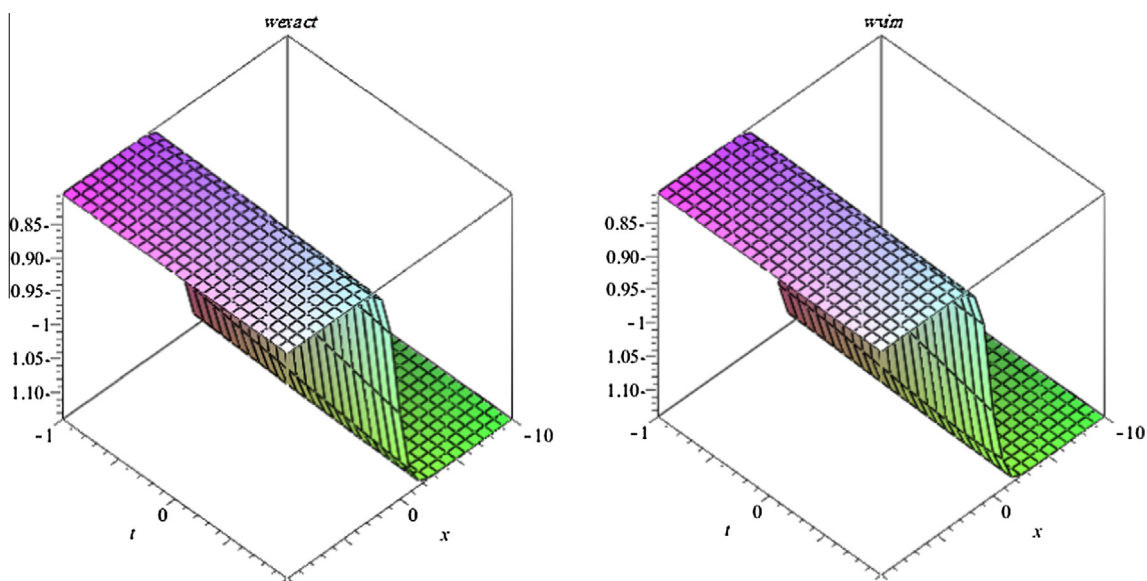

Fig. 1c Exact solution of $w(x, t)$ and numerical solution of $w(x, t)$, $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.

By setting $p = 1$, one can obtain an approximate solution of Eq. (2) as,

$$U = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \cdots, \tag{20}$$

presented in Eq. (20).The combination of a small parameter (perturbation parameter) with a homotopy is called HPM, as

3. Application

In this section, we demonstrate the analysis of all the numerical methods by applying methods to the system of partial differential Eq. (1). A comparison of all methods is also given in the form of graphs and tables, presented here.

3.1. The variational iteration method [37]

To solve the system of Eq. (1), by means of variational iteration method, we construct a correctional functional which reads

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1 \{u_{nt} - v_{nx}\} d\tau,$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \lambda_2 \left\{ \begin{array}{l} v_{nt} + 2v_{nxxx} + 6u_n v_{nx} \\ +6v_n u_{nx} + 37w_n w_{nx} - 2p_n w_{nx} \\ -\frac{1}{2}w_n p_{nx} + p_n p_{nx} - 2w_{nx} - 2p_{nx} \end{array} \right\} d\tau,$$

Fig. 1d Exact solution of $p(x, t)$ and numerical solution of $p(x, t)$, $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.

Fig. 2a Numerical solution of $u(x, t)$ and $v(x, t)$ with ten terms by (LDM), $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3 \{w_{nt} - w_{nxxx} - 3u_n w_{nx}\} d\tau,$$

$$p_{n+1}(x,t) = p_n(x,t) + \int_0^t \lambda_4 \{p_{nt} - p_{nxx} - 3u_n p_{nx}\} d\tau, \tag{21}$$

where λ_1, λ_2, λ_3, and λ_4 are general Lagrangian multipliers are to be determined [7]. With the aid of the above correction functional stationary, we obtain

$$\lambda_1'(\tau) = 0, \quad \lambda_2'(\tau) = 0, \quad \lambda_3'(\tau) = 0, \quad \lambda_4'(\tau) = 0,$$

$$1 + \lambda_1(\tau)|_{\tau=t} = 0, \quad 1 + \lambda_2(\tau)|_{\tau=t} = 0, \quad 1 + \lambda_3(\tau)|_{\tau=t} = 0,$$

$$1 + \lambda_4(\tau)|_{\tau=t} = 0. \tag{22}$$

Eq. (21) are called Lagrange–Euler equations, one natural boundary conditions. The Lagrange multipliers, therefore, can be identified as $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, and the following variational iteration formula is given by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_{nt} - v_{nx}\} d\tau,$$

$$v_{n+1}(x,t) = v_n(x,t)$$
$$- \int_0^t \left\{ \begin{array}{l} v_{nt} + 2v_{nxxx} + 6u_n v_{nx} \\ +6v_n u_{nx} + 37 w_n w_{nx} - 2p_n w_{nx} \\ -\frac{1}{2} w_n p_{nx} + p_n p_{nx} - 2w_{nx} - 2p_{nx} \end{array} \right\} d\tau,$$
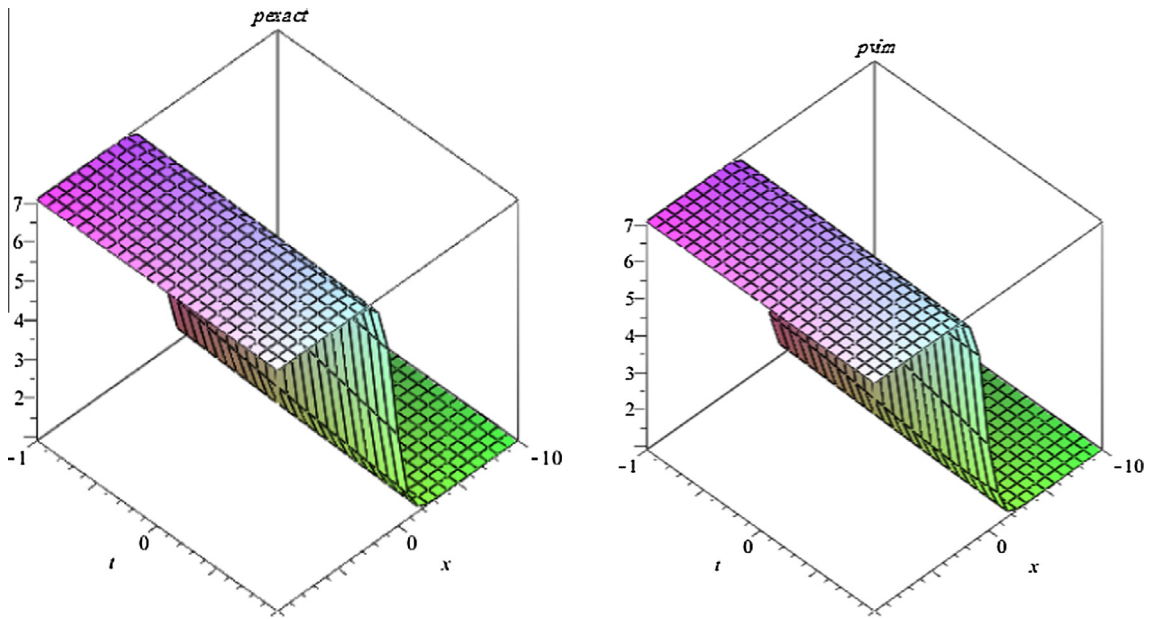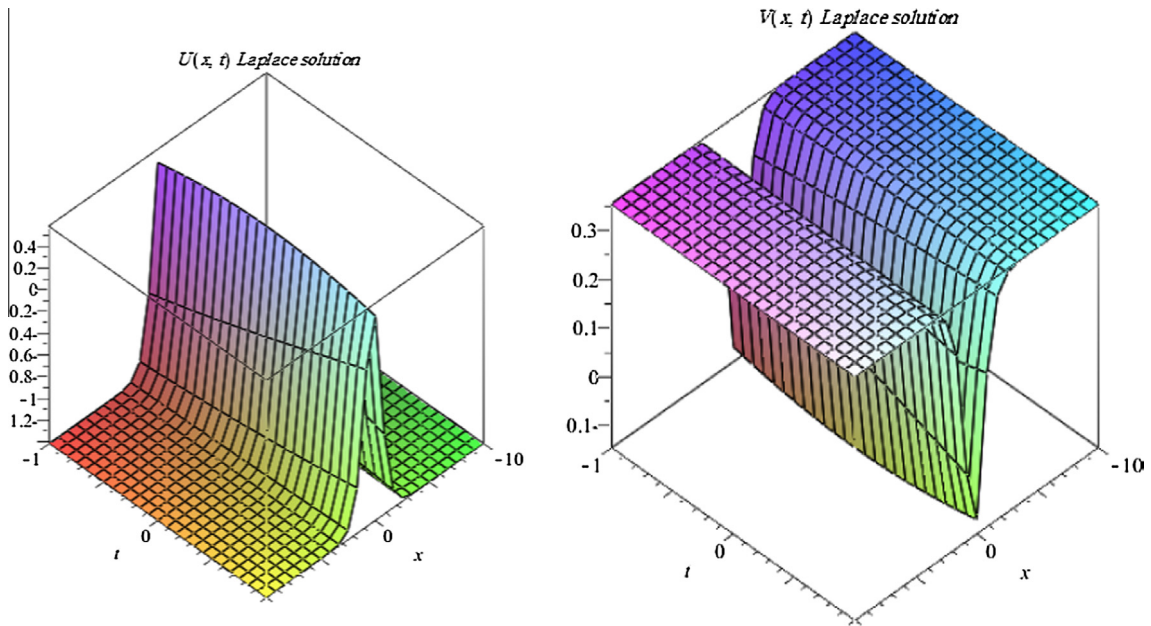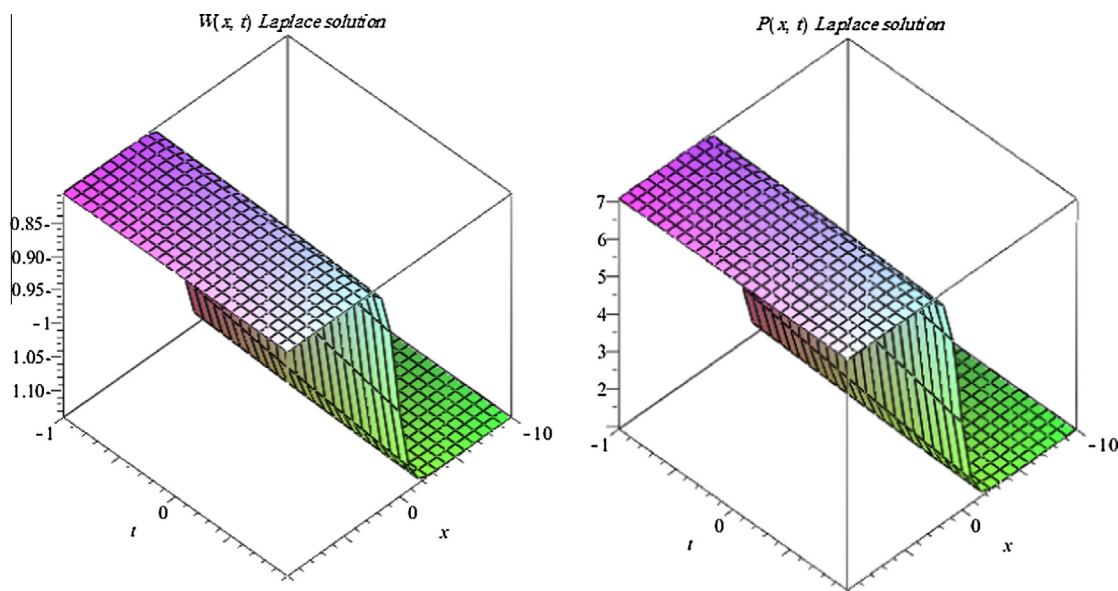
Fig. 2b   Numerical solution of $w(x, t)$ and $p(x, t)$ with ten terms by (LDM), $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.

$$w_{n+1}(x, t) = w_n(x, t) - \int_0^t \{w_{nt} - w_{nxxx} - 3u_n w_{nx}\} d\tau,$$

$$p_{n+1}(x, t) = p_n(x, t) - \int_0^t \{p_{nt} - p_{nxx} - 3u_n p_{nx}\} d\tau. \tag{23}$$

We start with initial approximations $u_n(x, 0)$, $v_0(x, 0)$, $w_n(x, 0)$ and $p_n(x, 0)$ given by Eq. (4) with the above iteration formula (22), we get

different values of times is shown in Figs. 1a, 1b, 1c and 1d. The comparison shows that the two solutions obtained are in excellent agreement.

3.2. The Laplace decomposition method

To solve the system of Eq. (1) by means of Laplace decomposition method, we construct a correctional functional which reads

$$u_1 := \frac{7}{12} - 2\tanh(x)^2 + \tanh(x)t - \tanh(x)^3 t,$$

$$v_1 := -\frac{1}{48} \frac{-17\cosh(x)^3 + 12\sinh(x)t + 24\cosh(x)}{\cosh(x)^3},$$

$$w_1 := \frac{1}{888} \frac{148\sinh(x)\cosh(x) - 864\cosh(x)^2 - 37t}{\cosh(x)^2},$$

$$p_1 := \frac{1}{48} \frac{148\sinh(x)\cosh(x) + 192\cosh(x)^2 - 37t}{\cosh(x)^2},$$

$$u_2 := \frac{1}{24} \frac{24\cosh(x)t\sinh(x) - 34\cosh(x)^4 + 6\cosh(x)^2 t^2 + 48\cosh(x)^2 - 9t^2}{\cosh(x)^4},$$

$$v_2(x, t) = -\frac{1}{96} \frac{155\cosh(x)^2 t^3 \sinh(x) + 24\cosh(x)^4 t\sinh(x) - 288\sinh(x)t^3 + 6\cosh(x)^5 t^2 - 34\cosh(x)^7 + 48\cosh(x)^5 - 9t^2\cosh(x)^3}{\cosh(x)^7},$$

$$w_2(x, t) = \frac{1}{3552} \frac{592\sinh(x)\cosh(x)^5 - 37\cosh(x)^3 t^2 \sinh(x) - 296t^3 + 296\cosh(x)^2 t^3 - 3456\cosh(x)^6 - 148t\cosh(x)^4}{\cosh(x)^6},$$

$$p_2(x, t) == \frac{1}{192} \frac{592\sinh(x)\cosh(x)^5 - 37\cosh(x)^3 t^2 \sinh(x) - 296t^3 + 296\cosh(x)^2 t^3 + 768\cosh(x)^6 - 148t\cosh(x)^4}{\cosh(x)^6}.$$

$$\tag{24}$$

The behavior of the approximate solutions obtained by variational iteration method with the exact solutions (Eq. (3)) for

$$£[u] = \frac{1}{s}u(0) + \frac{1}{s}£[v_x],$$

$$\pounds[v] = \frac{1}{s}v(0) + \frac{1}{s}\pounds\left[-2v_{xxx} - 6(uv)_x - 37ww_x + 2pw_x\right.$$
$$\left. + \frac{1}{2}wp_x - pp_x + 2w_x + 2p_x\right],$$

$$\pounds[w] = \frac{1}{s}w(0) + \frac{1}{s}\pounds[w_{xxx} + 3uw_x],$$

$$\pounds[p] = \frac{1}{s}p(0) + \frac{1}{s}\pounds[p_{xxx} + 3up_x]. \tag{25}$$

We can define the Adomian polynomial as follows:

$$A_n = \sum_{i=0}^{n} u_i w_{(n-i)x}, \quad B_n = \sum_{i=0}^{n} u_i p_{(n-i)x}, \quad C_n = \sum_{i=0}^{n} u_i v_{(n-i)x},$$

$$E_n = \sum_{i=0}^{n} v_i u_{(n-i)x}, \quad F_n = \sum_{i=0}^{n} w_i w_{(n-i)x},$$

$$G_n = \sum_{i=0}^{n} p_i w_{(n-i)x}, \quad H_n = \sum_{i=0}^{n} w_i p_{(n-i)x}, \quad M_n = \sum_{i=0}^{n} p_i p_{(n-i)x}, \tag{26}$$

we define an iterative scheme

$$\pounds[u_{n+1}] = \frac{1}{s}\pounds[v_{nx}], \quad \pounds[w_{n+1}] = \frac{1}{s}\pounds[w_{nxxx} + 3A_n],$$

$$\pounds[p_{n+1}] = \frac{1}{s}\pounds[p_{nxxx} + 3B_n],$$

$$\pounds[v_{n+1}] = \frac{1}{s}\pounds\left[-2v_{nxxx} - 6C_n - 6E_n - 37F_n + 2G_n\right.$$
$$\left. + \frac{1}{2}H_n - M_n + 2w_{nx} + 2p_{nx}\right], \quad n \geq 1. \tag{27}$$

Applying the inverse Laplace transform, finally we get

$$u_1 = \frac{t\sinh(x)}{\cosh(x)^3}, \quad v_1(x,t) = -\frac{1}{4}\frac{t\sinh(x)}{\cosh(x)^3},$$

$$w_1(x,t) = -\frac{1}{24}\frac{t}{\cosh(x)^2}, \quad p_1(x,t) = -\frac{37}{48}\frac{t}{\cosh(x)^2},$$

$$u_2(x,t) = \frac{1}{8}\frac{t^2(2\cosh(x)^2 - 3)}{\cosh(x)^4},$$

$$v_2(x,t) = -\frac{1}{32}\frac{t^2(2\cosh(x)^2 - 3)}{\cosh(x)^4},$$

$$w_2(x,t) = -\frac{1}{96}\frac{t^2\sinh(x)}{\cosh(x)^3},$$

$$p_2(x,t) = -\frac{37}{192}\frac{t^2\sinh(x)}{\cosh(x)^3},$$

$$u_3(x,t) = \frac{1}{24}\frac{t^3\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5},$$

$$v_3(x,t) = -\frac{1}{96}\frac{t^3\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5},$$

$$w_3(x,t) = -\frac{1}{1152}\frac{t^3(2\cosh(x)^2 - 3)}{\cosh(x)^4},$$

$$p_3(x,t) = -\frac{37}{2304}\frac{t^3(2\cosh(x)^2 - 3)}{\cosh(x)^4}, \tag{28}$$

and so on,

Similarly, for the t6th term the solution takes the following form:

$$u(x,t) \cong \sum_{i=0}^{16} u_i(x,t), \quad p(x,t) \cong \sum_{i=0}^{16} u_i(x,t), \quad w(x,t)$$
$$\cong \sum_{i=0}^{16} u_i(x,t), \quad p(x,t) \cong \sum_{i=0}^{16} u_i(x,t), \tag{29}$$

or

$$u(x,t) = \frac{7}{12} - 2\tanh(x)^2 + \frac{t\sinh(x)}{\cosh(x)^3} + \frac{1}{8}\frac{t^2(2\cosh(x)^2 - 3)}{\cosh(x)^4}$$

$$+ \frac{1}{24}\frac{t^3\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5}$$

$$+ \frac{1}{384}\frac{t^4(2\cosh(x)^4 - 15\cosh(x)^2 + 15)}{\cosh(x)^6}$$

$$+ \frac{1}{3840}\frac{t^5\sinh(x)(2\cosh(x)^4 - 30\cosh(x)^2 + 45)}{\cosh(x)^7} + \cdots,$$

$$v(x,t) = -\frac{7}{48} + \frac{1}{2}\tanh(x)^2 - \frac{1}{4}\frac{t\sinh(x)}{\cosh(x)^3}$$

$$- \frac{1}{32}\frac{t^2(2\cosh(x)^2 - 3)}{\cosh(x)^4} - \frac{1}{96}\frac{t^3\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5}$$

$$- \frac{1}{1536}\frac{t^4(2\cosh(x)^4 - 15\cosh(x)^2 + 15)}{\cosh(x)^6}$$

$$- \frac{1}{15360}\frac{t^5\sinh(x)(2\cosh(x)^4 - 30\cosh(x)^2 + 45)}{\cosh(x)^7} + \cdots,$$

$$w(x,t) = -\frac{36}{37} + \frac{1}{6}\tanh(x) - \frac{1}{24}\frac{t}{\cosh(x)^2} - \frac{1}{96}\frac{t^2\sinh(x)}{\cosh(x)^3}$$

$$- \frac{1}{1152}\frac{t^3(2\cosh(x)^2 - 3)}{\cosh(x)^4}$$

$$- \frac{1}{4608}\frac{t^4(\cosh(x)^2 - 3)\sinh(x)}{\cosh(x)^5}$$

$$- \frac{1}{92160}\frac{t^5(2\cosh(x)^4 - 15\cosh(x)^2 + 15)}{\cosh(x)^6} + \cdots,$$

$$p(x,t) = 4 + \frac{37}{12}\tanh(x) - \frac{37}{48}\frac{t}{\cosh(x)^2}$$

$$- \frac{37}{192}\frac{t^2\sinh(x)}{\cosh(x)^3} - \frac{37}{2304}\frac{t^3(2\cosh(x)^2 - 3)}{\cosh(x)^4}$$

$$- \frac{37}{9216}\frac{t^4(\cosh(x)^2 - 3)\sinh(x)}{\cosh(x)^5}$$

$$- \frac{37}{184320}\frac{t^5(2\cosh(x)^4 - 15\cosh(x)^2 + 15)}{\cosh(x)^6} + \cdots. \tag{30}$$

And Figs. 2a and 2b shows the numerical solution of system (1) with ten terms by (LDM)

3.3. The Pade approximation

We use Maple to calculate the [6/4] the Pade approximant of the infinite series solution (28), which gives the following rational fraction approximation to the solution:

$$u(x,t) = \frac{1}{7431782400} - 779371851720142356480000 \cosh{(x)}^{20} + 115359246933804711936000 \cosh{(x)}^{24} + \cdots,$$

$$v(x,t) = \frac{1}{29727129600} \, 779371851720142356480000 \cosh{(x)}^{20} - 115359246933804711936000 \cosh{(x)}^{24} + \cdots,$$

$$w(x,t) = \frac{1}{3299711385600} \, 90034069038453227520000 \cosh{(x)}^{7} + 17370703537310269440000 \cosh{(x)}^{5} + \cdots,$$

$$p(x,t) = \frac{1}{178362777600} \, 5223968936873164800 \cosh{(x)}^{12} \sinh(x) - 5235475476381696000 \cosh{(x)}^{10} \sinh{(x)}^{3}$$
$$+ 695311416151769088000 \sinh(x) \cosh{(x)}^{8} + \cdots. \tag{31}$$
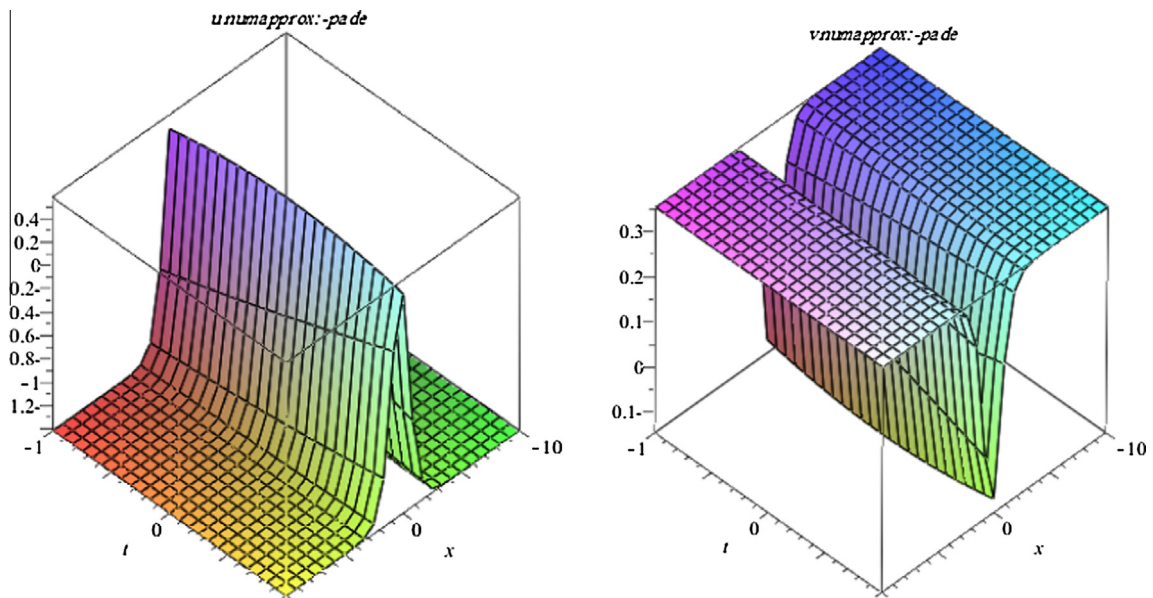
Fig. 3a Numerical solution of $u(x, t)$ and $v(x, t)$ with ten terms by (LD–PA method), $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.
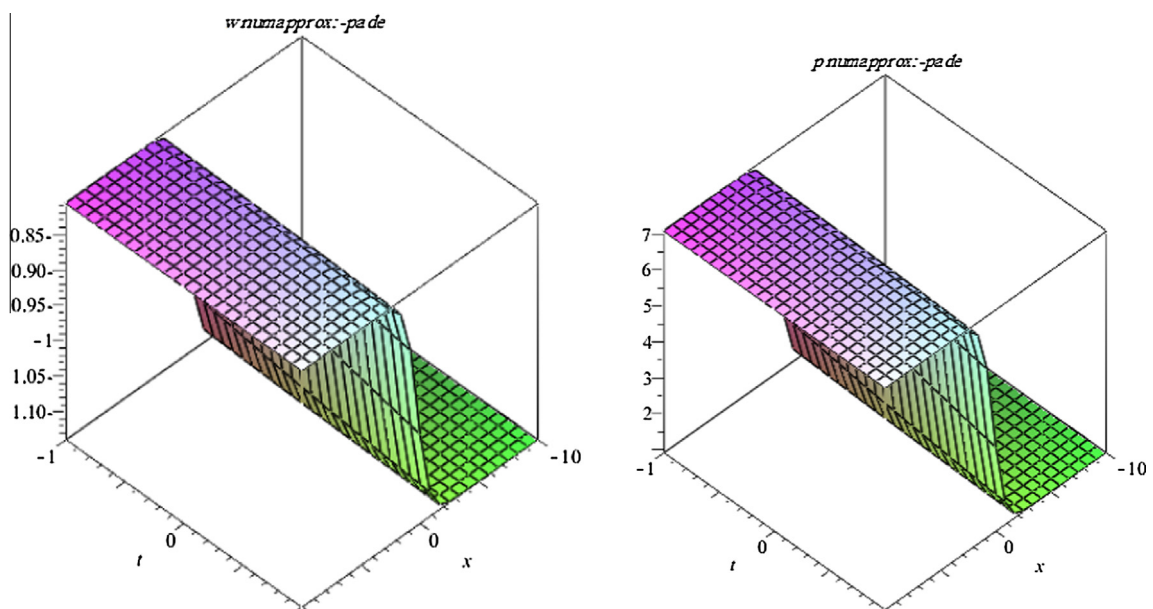
Fig. 3b Numerical solution of $u(x, t)$ and $v(x, t)$ with ten terms by (LD–PA method), $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.
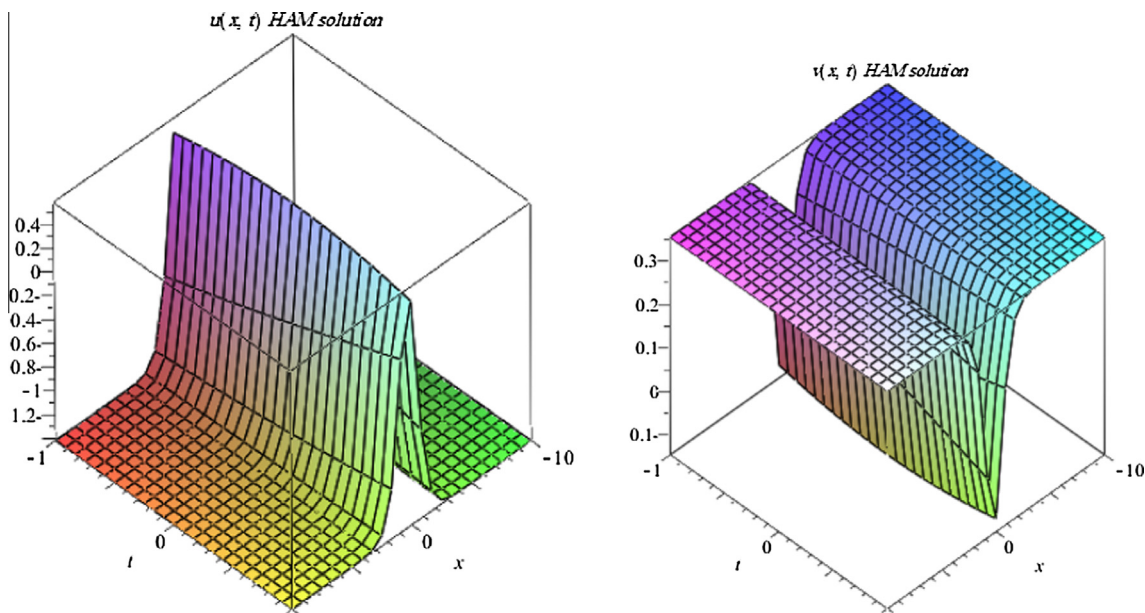
Fig. 4a Numerical solution of $u(x, t)$ and $v(x, t)$ with ten terms by (HPM) method, $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.
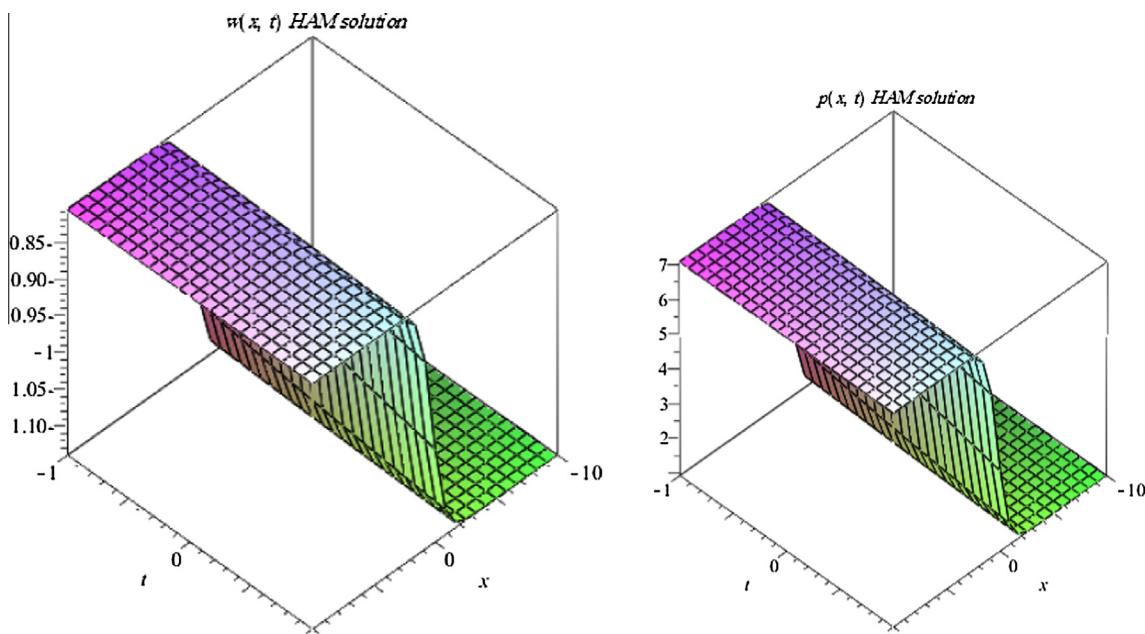
Fig. 4b Numerical solution of $w(x, t)$ and $p(x, t)$ with ten terms by (HPM) method, $-10 \leqslant x \leqslant 10$, $-1 \leqslant t \leqslant 1$.

And the numerical solution (24) of system (1) obtained by (LD–PA) method showed in Figs. 3a and 3b, and comparison of the results obtained by the Laplace decomposition method (LDM), the variation iteration method (VIM) and the Pade approximant (LD–PA) with exact solution of system (1) presented in Figs. 4a and 4b.

3.4. Homotopy perturbation method [33]

To solve system (1) by means of HPM, we choose that:

$$u(x, t) = U_0(x, t) + pU_1(x, t) + p^2 \cdot U_2(x, t)$$
$$+ p^3 \cdot U_3(x, t) + p^4 \cdot U_4(x, t) + \cdots,$$
$$v(x, t) = V_0(x, t) + p \cdot V_1(x, t) + p^2 \cdot V_2(x, t)$$
$$+ p^3 \cdot V_3(x, t) + p^4 \cdot V_4(x, t) + \cdots,$$
$$w(x, t) = W_0(x, t) + p \cdot W_1(x, t) + p^2 \cdot W_2(x, t)$$
$$+ p^3 \cdot W_3(x, t) + p^4 \cdot W_4(x, t) + \cdots,$$
$$h(x, t) = H_0(x, t) + p \cdot H_1(x, t) + p^2 \cdot H_2(x, t)$$
$$+ p^3 \cdot H_3(x, t) + p^4 \cdot H_4(x, t) + \cdots. \quad (32)$$

We choose the initial approximations (4) and using (18) we have:

$$p^0 : \frac{\partial}{\partial t} U_0(x,t) - \left(\frac{\partial}{\partial t} r_0(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} V_0(x,t) - \left(\frac{\partial}{\partial t} s_0(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} W_0(x,t) - \left(\frac{\partial}{\partial t} m_0(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} H_0(x,t) - \left(\frac{\partial}{\partial t} n_0(x,t)\right) = 0, \tag{33}$$

$$p^1 : \frac{\partial}{\partial t} U_1(x,t) + \frac{\partial}{\partial t} r_0(x,t) - \left(\frac{\partial}{\partial x} V_0(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} V_1(x,t) + \frac{\partial}{\partial t} s_0(x,t) + 2\left(\frac{\partial^3}{\partial x^3} V_0(x,t)\right) - \left(\frac{\partial}{\partial x} W_0(x,t)\right)$$
$$- \left(\frac{\partial}{\partial x} H_0(x,t)\right) + 6\left(\frac{\partial}{\partial x} U_0(x,t)\right) V_0(x,t)$$
$$+ 6U_0(x,t)\left(\frac{\partial}{\partial x} V_0(x,t)\right) - 6W_0(x,t)\left(\frac{\partial}{\partial x} W_0(x,t)\right)$$
$$- 7H_0(x,t)\left(\frac{\partial}{\partial x} W_0(x,t)\right) - W_0(x,t)\left(\frac{\partial}{\partial x} H_0(x,t)\right)$$
$$- H_0(x,t)\left(\frac{\partial}{\partial x} H_0(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} W_1(x,t) + \frac{\partial}{\partial t} m_0(x,t) - \left(\frac{\partial^3}{\partial x^3} W_0(x,t)\right)$$
$$- 3U_0(x,t)\left(\frac{\partial}{\partial x} W_0(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} H_1(x,t) + \frac{\partial}{\partial t} n_0(x,t) - \left(\frac{\partial^3}{\partial x^3} H_0(x,t)\right)$$
$$- 3U_0(x,t)\left(\frac{\partial}{\partial x} H_0(x,t)\right) = 0,$$

$$\ldots, \tag{34}$$

Solving system (33)–(35) we have:

$$U_1 = \frac{t\sinh(x)}{\cosh(x)^3}, \quad V_1 = -\frac{1}{4}\frac{t\sinh(x)}{\cosh(x)^3},$$

$$W_1 = -\frac{1}{24}\frac{t}{\cosh(x)^2}, \quad P_1 = -\frac{37}{48}\frac{t}{\cosh(x)^2},$$

$$U_2 = \frac{1}{8}\frac{t^2(2\cosh(x)^2 - 3)}{\cosh(x)^4},$$

$$V_2 = -\frac{1}{32}\frac{t^2(2\cosh(x)^2 - 3)}{\cosh(x)^4},$$

$$W_2 = -\frac{1}{96}\frac{t^2\sinh(x)}{\cosh(x)^3}, \quad P_2 = -\frac{37}{192}\frac{t^2\sinh(x)}{\cosh(x)^3},$$

$$U_3 = \frac{1}{24}\frac{t^3\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5},$$

$$V_3 = -\frac{1}{96}\frac{t^3\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5},$$

$$W_3 = -\frac{1}{1152}\frac{t^3(2\cosh(x)^2 - 3)}{\cosh(x)^4},$$

$$H_3 = -\frac{37}{2304}\frac{t^3(2\cosh(x)^2 - 3)}{\cosh(x)^4},$$

$$U_4 = \frac{1}{384}\frac{t^4(2\cosh(x)^4 - 15\cosh(x)^2 + 15)}{\cosh(x)^6},$$

$$V_4 = -\frac{1}{1536}\frac{t^4(2\cosh(x)^4 - 15\cosh(x)^2 + 15)}{\cosh(x)^6},$$

$$W_4 = -\frac{1}{4608}\frac{t^4\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5},$$

$$H_4 = -\frac{37}{9216}\frac{t^4\sinh(x)(\cosh(x)^2 - 3)}{\cosh(x)^5}. \tag{36}$$

For calculating 13th term and using (20), we obtain the following solution:

$$p^n : \frac{\partial}{\partial t} U_n(x,t) - \left(\frac{\partial}{\partial x} V_{n-1}(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} V_n(x,t) + 2\left(\frac{\partial^3}{\partial x^3} V_{n-1}(x,t)\right) + 6\sum_{i=1}^{n-1} U_i(x,t)\left(\frac{\partial}{\partial x} V_{n-i-1}(x,t)\right) + 6\sum_{i=1}^{n-1}\left(\frac{\partial}{\partial x} U_i(x,t)\right) V_{n-i-1}(x,t)$$
$$- 6\sum_{i=1}^{n-1} W_i(x,t)\left(\frac{\partial}{\partial x} W_{n-i-1}(x,t)\right) - 7\sum_{i=1}^{n-1} H_i(x,t)\left(\frac{\partial}{\partial x} W_{n-i-1}(x,t)\right) - \sum_{i=1}^{n-1} W_i(x,t)\left(\frac{\partial}{\partial x} H_{n-i-1}(x,t)\right)$$
$$- \sum_{i=1}^{n-1} H_i(x,t)\left(\frac{\partial}{\partial x} H_{n-i-1}(x,t)\right) - \left(\frac{\partial}{\partial x} W_{n-1}(x,t)\right) - \left(\frac{\partial}{\partial x} H_{n-1}(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} W_n(x,t) - \left(\frac{\partial^3}{\partial x^3} W_{n-1}(x,t)\right) - 3\sum_{i=1}^{n-1} U_i(x,t)\left(\frac{\partial}{\partial x} W_{n-i-1}(x,t)\right) = 0,$$

$$\frac{\partial}{\partial t} H_n(x,t) - \left(\frac{\partial^3}{\partial x^3} H_{n-1}(x,t)\right) - 3\sum_{i=1}^{n-1} U_i(x,t)\left(\frac{\partial}{\partial x} H_{n-i-1}(x,t)\right) = 0. \tag{35}$$

$$u(x, t) = -\frac{1}{102023508787200} \frac{1}{\cosh{(x)}^{15}} \left(144533304115200 \cosh{(x)}^{15} + 139485265920t^6 \cosh{(x)}^{11} + \cdots\right),$$

$$v(x, t) = \frac{1}{408094035148800} \frac{1}{\cosh{(x)}^{15}} \left(144533304115200 \cosh{(x)}^{15} + 139485265920t^6 \cosh{(x)}^{11} + \cdots\right),$$

$$w(x, t) = -\frac{1}{905968758030336000} \frac{1}{\cosh{(x)}^{14}} \left(88148311592140800 \cosh{(x)}^{14} + \cdots\right),$$

$$p(x, t) = \frac{1}{4897128421785600} \frac{1}{\cosh{(x)}^{14}} \left(19588513687142400 \cosh{(x)}^{14} + \cdots\right). \tag{37}$$

Table 1 Absolute error of $u(x, t)$.

t	$|uex - u_{LDm}|$	$|uex - u_{pade}|$	$|uex - u_{HPM}|$	$|uex - u_{VIM}|$
0.2	3.1×10^{-71}	8.7×10^{-59}	1.0×10^{-63}	3.0×10^{-39}
0.4	4.1×10^{-66}	1.8×10^{-55}	1.6×10^{-59}	2.4×10^{-38}
0.6	4.1×10^{-63}	1.6×10^{-53}	4.8×10^{-57}	8.4×10^{-38}
0.8	5.4×10^{-61}	4.1×10^{-52}	2.7×10^{-55}	2.0×10^{-37}
1.0	2.4×10^{-59}	5.0×10^{-51}	6.2×10^{-54}	4.0×10^{-37}
1.2	5.3×10^{-85}	3.9×10^{-50}	8.0×10^{-53}	7.0×10^{-37}
1.4	7.3×10^{-57}	2.2×10^{-49}	7.0×10^{-52}	1.1×10^{-36}
1.6	7.1×10^{-56}	1.0×10^{-48}	4.5×10^{-51}	1.7×10^{-36}
1.8	5.3×10^{-55}	3.8×10^{-48}	2.3×10^{-50}	2.4×10^{-36}
2.0	3.1×10^{-54}	1.3×10^{-47}	1.0×10^{-49}	3.4×10^{-36}

Table 2 Absolute error of $v(x, t)$.

t	$|vex - v_{LDm}|$	$|vex - v_{pade}|$	$|vex - u_{HPM}|$	$|vex - v_{VIM}|$
0.2	7.7×10^{-72}	2.2×10^{-59}	2.5×10^{-64}	7.6×10^{-40}
0.4	1.0×10^{-66}	4.6×10^{-56}	4.1×10^{-60}	6.1×10^{-39}
0.6	1.0×10^{-63}	4.2×10^{-54}	1.2×10^{-57}	2.1×10^{-38}
0.8	1.3×10^{-61}	1.0×10^{-52}	6.8×10^{-56}	5.0×10^{-38}
1.0	6.0×10^{-60}	1.3×10^{-51}	1.5×10^{-54}	1.0×10^{-37}
1.2	1.3×10^{-58}	9.8×10^{-51}	2.0×10^{-53}	1.7×10^{-37}
1.4	1.8×10^{-57}	5.5×10^{-50}	1.7×10^{-52}	2.8×10^{-37}
1.6	1.7×10^{-56}	2.5×10^{-49}	1.1×10^{-51}	4.2×10^{-37}
1.8	1.3×10^{-55}	9.6×10^{-49}	5.9×10^{-51}	6.1×10^{-37}
2.0	7.9×10^{-55}	3.2×10^{-48}	2.6×10^{-50}	8.5×10^{-37}

Tables 1–4 displays the absolute error obtained by (LDM), (LD-PD), (HAP) and (VIM) respectively.

4. Concluding remarks

In this work, we have applied the Laplace decomposition method (LDM), the variation iteration method (VIM), the Pade approximant (LD–PA) and Homotopy perturbation method (HPM) to the Generalized Ito System. The numerical study presented in Section 3 showed that all the methods give a highly accurate results for a given system of partial differential equations. The LDM, HAM and the VIM are simple and easy to use, required less computational complexity. Since the methods of the VIM and the LDM are based on an approximation of the solution function with polynomial expansion, this kind of approximation can exhibit oscillations which may produce an approximation error bound Moreover, the approximate solution obtained by the LDM and the VIM can never blow-up in a finite region. To overcome these demerits, we use the Pade approximations. This fact is also verified by the system of our study. (see Figs. 5a and 5b.)

Table 3 Absolute error of $w(x, t)$.

t	$|wex - w_{LDm}|$	$|wex - w_{pade}|$	$|wex - w_{HPM}|$	$|wex - w_{VIM}|$
0.2	1.3×10^{-72}	3.6×10^{-60}	4.1×10^{-65}	1.2×10^{-40}
0.4	1.7×10^{-67}	7.8×10^{-57}	9.6×10^{-61}	1.0×10^{-39}
0.6	1.7×10^{-64}	7.0×10^{-55}	2.0×10^{-58}	3.5×10^{-39}
0.8	2.2×10^{-62}	1.7×10^{-53}	1.1×10^{-56}	8.4×10^{-39}
1.0	9.9×10^{-61}	2.1×10^{-52}	2.6×10^{-55}	1.6×10^{-38}
1.2	2.2×10^{-59}	1.6×10^{-51}	3.3×10^{-54}	2.9×10^{-38}
1.4	3.1×10^{-58}	9.2×10^{-51}	2.9×10^{-53}	4.7×10^{-38}
1.6	2.9×10^{-57}	4.2×10^{-50}	1.9×10^{-52}	7.1×10^{-38}
1.8	2.2×10^{-56}	1.6×10^{-49}	9.9×10^{-52}	1.0×10^{-37}
2.0	1.3×10^{-55}	5.3×10^{-49}	4.3×10^{-51}	1.4×10^{-37}

Table 4   Absolute error of $p(x, t)$.

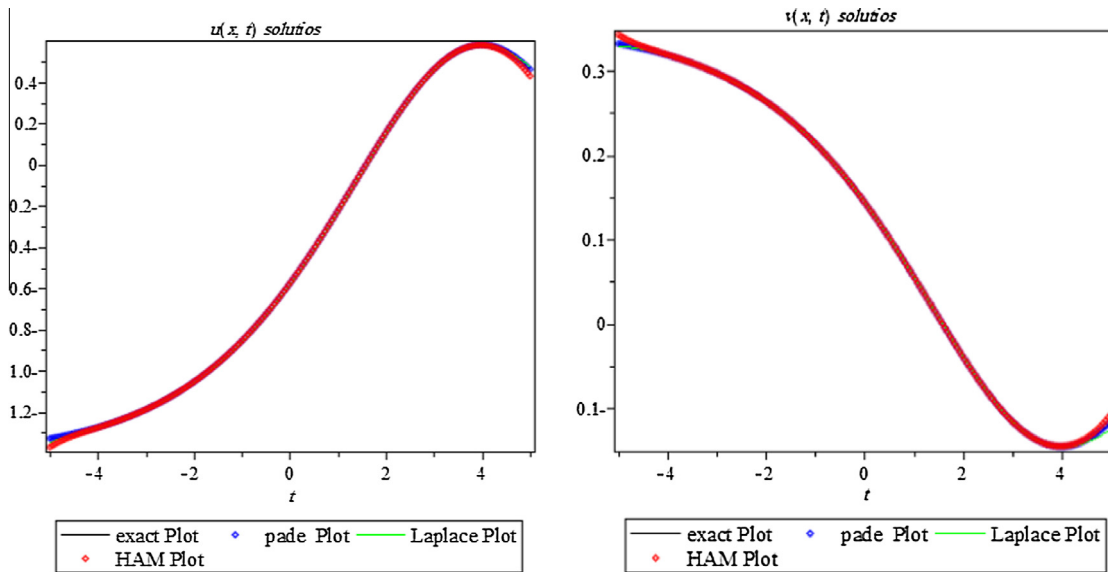| t	$|pex - p_{LDm}|$	$|pex - u_{pade}|$	$|pex - u_{HPM}|$	$|pex - p_{VIM}|$
0.2	2.4×10^{-71}	6.7×10^{-59}	7.7×10^{-64}	2.3×10^{-39}
0.4	3.1×10^{-66}	1.4×10^{-55}	1.2×10^{-59}	1.9×10^{-38}
0.6	3.1×10^{-63}	1.3×10^{-53}	3.7×10^{-57}	6.5×10^{-38}
0.8	4.1×10^{-61}	3.2×10^{-52}	2.1×10^{-55}	1.5×10^{-37}
1.0	1.8×10^{-59}	3.9×10^{-51}	4.8×10^{-54}	3.1×10^{-37}
1.2	4.1×10^{-58}	3.0×10^{-50}	6.2×10^{-53}	5.4×10^{-37}
1.4	5.6×10^{-57}	1.7×10^{-49}	5.4×10^{-52}	8.7×10^{-37}
1.6	5.6×10^{-56}	7.7×10^{-49}	3.5×10^{-51}	1.3×10^{-36}
1.8	4.1×10^{-55}	2.9×10^{-48}	1.8×10^{-50}	1.8×10^{-36}
2.0	2.4×10^{-54}	9.8×10^{-48}	8.1×10^{-50}	2.6×10^{-36}

Fig. 5a   Comparison exact and approximate solutions by LDM, (LD-PD) and HPM at $x = 1$ of $u(x, t)$ and $v(x, t)$.
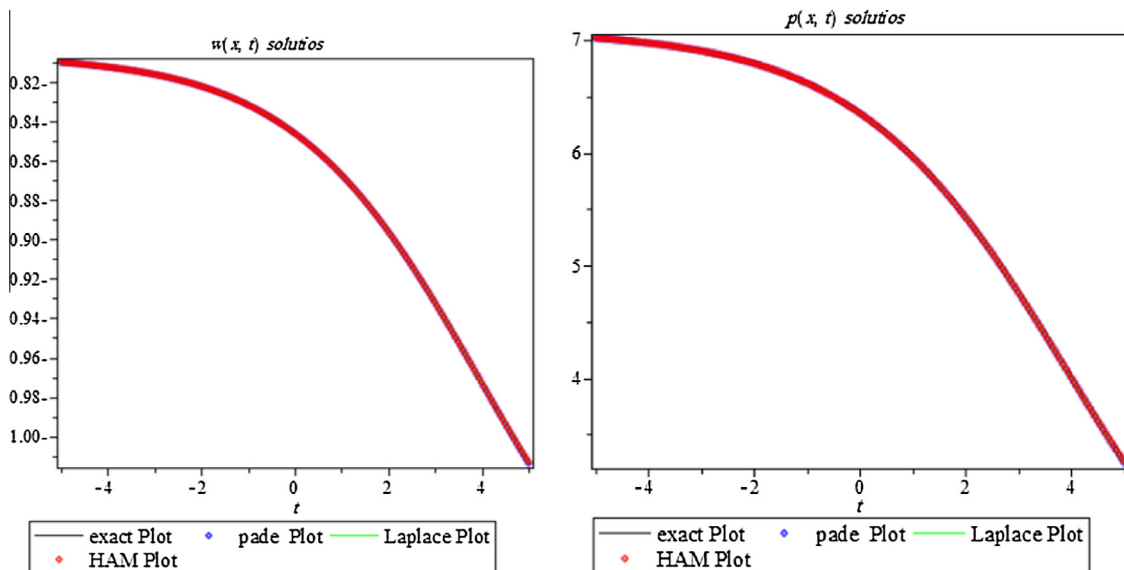

Fig. 5b   Comparison exact and approximate solutions by LDM, (LD-PD) and HPM at $x = 1$ of $w(x, t)$ and $p(x, t)$.

References

[1] D.D. Ganji, M. Nourollahi, M. Rostamian, A comparison of variational iteration method with Adomian's decomposition method in some highly nonlinear equations, International Journal of Science & Technology 2 (2) (2007) 179–188.

[2] M.N. abadi, F.M. Ghaini, M. Arab, Application of He's variational iteration method for Lie'nard equations, World Applied Sciences Journal 7 (9) (2009) 1077–1079.

[3] S.A. Yousefi, A. Lotfi, M. Dehghanb, He's variational iteration method for solving nonlinear mixed Volterra–Fredholm integral equations, Computers and Mathematics with Applications 58 (2009) 2172–2176.

[4] S. Abbasbandy, E. Shivanian, application of the variational iteration method for system of nonlinear Volterra's integro-differential equations, Mathematical and Computational Applications 14 (2) (2009) 147–158.

[5] M. Matinfar, M. Ghanbari, Solution of systems of integral-differential equations by variational iteration method, Journal of Mathematics and Computer Science 1 (2) (2010) 46–57.

[6] S.Q. Wang, J. He, Variational iteration method for solving integro-differential equations, Physics Letters A 367 (2007) 188–191.

[7] Z.M. Odibat, A study on the convergence of variational iteration method, Mathematical and Computer Modelling 51 (2010) 1181–1192.

[8] H. Jafari, A. Alipoor, A New Method for Calculating General Lagrange Multiplier in the Variational Iteration Method 27 (4) (2011) 996–1001.

[9] M. Labidi, A. Biswas, Application of He's principles to Rosenau-Kawahara equation, Mathematics in Engineering, Science and Aerospace 2 (2) (2011) 183–197.

[10] G. Ebadi, E.V. krishnan, M. Labidi, E. Zerrad, A. Biswas, Analytical and numerical solutions for Davey-Stewartson equation with power law nonlinearity, Waves in Random and Complex Media 21 (4) (2011) 559–590.

[11] M. Labidi, G. Ebadi, E. zerrad, A. Biswas, Analytical and numerical solutions of the Schrodinger-KdV equation, Pramana 78 (1) (2012) 59–90.

[12] J. Biazar, Solution of systems of integral–differential equations by Adomian decomposition method, Applied Mathematics and Computation 168 (2005) 1232–1238.

[13] M.I. Syam, Adomian decomposition method for approximating the solution of the Korteweg–deVries equation, Applied Mathematics and Computation 162 (2005) 1465–1473.

[14] A.M.A. El-Sayed, M. Gaber, The Adomian decomposition method for solving partial differential equations of fractal order in finite domains, Physics Letters A 359 (2006) 175–182.

[15] H. Jafari, V.D. Gejji, Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition, Applied Mathematics and Computation 180 (2006) 488–497.

[16] Qi Wang, Numerical solutions for fractional KdV–Burgers equation by Adomian decomposition method, Applied Mathematics and Computation 182 (2006) 1048–1055.

[17] G. ebadi, A.H. kara, M.D. Petkovic, A. Yildirim, A. Biswas, Solitons and conserved quantities of the Ito equation, Proceedings of the Romanian Academy Series, A 13 (3) (2012) 215–224.

[18] H.A. Zedan, E. Al-Aidrous, Numerical solutions for a generalized Ito system by using Adomian decomposition method, International Journal of Mathematics and Computation 4 (S09) (2009) 9–19.

[19] E.S.A. Alaidarous, F-expansion method for the nonlinear generalized Ito system, International Journal of Basic & Applied Sciences 10 (2) (2010) 90–117.

[20] H.A. Zedan, Symmetry analysis of an integrable Ito coupled system, Computers and Mathematics with Applications 60 (12) (2010) 3088–3097.

[21] A. Karasu, A. Karasu, S.Y. Sakovich, Integrability of a generalized Ito system: the Painlev'e test, Journal of the Physical Society of Japan 70 (5) (2001) 1165–1166.

[22] B. Raftari, A. Yildirim, Analytical solution of second-order hyperbolic telegraph equation by variational iteration and homotopy perturbation methods, Results in Mathematics 61 (1–2) (2012) 13–28.

[23] M. Labidi, H. Triki, E.V. Krishnan, A. Biswas, Soliton solutions of the long-short wave equation with power law nonlinearity, Journal of Applied Nonlinear Dynamics 1 (2) (2012) 125–140.

[24] A.J.M. Jawad, M.D. Petkovic, A. Biswas, Soliton solutions of a few nonlinear wave equations, Applied Mathematics and Computation 216 (2010) 2649–2658.

[25] A.J.M. Jawad, M.D. Petkovic, A. Biswas, Applications of He's principles to partial differential equations, Applied Mathematics and Computation 217 (2011) 7039–7047.

[26] S. Duangpithak, Variational iteration method for special nonlinear partial differential equations, International Journal on Mathematical Analysis 6 (22) (2012) 1071–1077.

[27] D. Bahuguna, A. Ujlayan, D.N. Pandey, A comparative study of numerical methods for solving an integro-differential equation, Computers and Mathematics with Applications 57 (2009) 1485–1493.

[28] Y. Khan, N. Faraz, Application of modified Laplace decomposition method for solving boundary layer equation, Journal of King Saud University (Science) 23 (2011) 115–119.

[29] E. Yusufoglu (Agadjanov), Numerical solution of Duffing equation by the Laplace decomposition algorithm, Applied Mathematics and Computation 177 (2006) 572–580.

[30] M. Torabia, H. Yaghoobi, K. Boubaker, Accurate solution for motion of a spherical solid particle in plane Couette Newtonian fluid mechanical flow using HPM–Padé approximant and the Boubaker polynomials expansion scheme BPES, International Journal of Heat and Mass Transfer 58 (2013) 224–228.

[31] H. Yaghoobi, M. Torabi, Novel solution for acceleration motion of a vertically falling non-spherical particle by VIM–Padé approximant, Powder Technology 215–216 (2012) 206–209.

[32] T.A. Abassya, M.A. El-Tawil, H. El-Zoheiry, Exact solutions of some nonlinear partial differential equations using the variational iteration method linked with Laplace transforms and the Pade technique, Computers and Mathematics with Applications 54 (2007) 940–954.

[33] J. Biazara, H. Aminikhahb, Study of convergence of homotopy perturbation method for systems of partial differential equations, Computers and Mathematics with Applications 58 (2009) 2221–2230.

[34] P.K. Gupta, M. Singh, Homotopy perturbation method for fractional Fornberg–Whitham equation, Computers and Mathematics with Applications 61 (2011) 250–254.

[35] Q. Wang, Homotopy perturbation method for fractional KdV equation, Applied Mathematics and Computation 190 (2007) 1795–1802.

[36] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Physics Letters A 365 (2007) 345–350.

[37] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burger's and coupled Burger's equations, Journal of Computational and Applied Mathematics 181 (2005) 245–251.