Journal of the Egyptian Mathematical Society (2015) 23, 286–291

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Solving systems of high-order linear differential–difference equations via Euler matrix method

Farshid Mirzaee *, Saeed Bimesl

Department of Mathematics, Faculty of Science, Malayer University, Malayer, Iran

Received 21 July 2013; revised 4 December 2013; accepted 4 May 2014
Available online 5 June 2014

KEYWORDS

Differential–difference equation;
Collocation points;
Polynomial solutions

Abstract This paper contributes a new matrix method for solving systems of high-order linear differential–difference equations with variable coefficients under given initial conditions. On the basis of the presented approach, the matrix forms of the Euler polynomials and their derivatives are constructed, and then by substituting the collocation points into the matrix forms, the fundamental matrix equation is formed. This matrix equation corresponds to a system of linear algebraic equations. By solving this system, the unknown Euler coefficients are determined. Some illustrative examples with comparisons are given. The results demonstrate reliability and efficiency of the proposed method.

2010 AMS (MOS) SUBJECT CLASSIFICATION: 35B30; 37M05; 65M22; 65N22

1. Introduction

In the recent years, the systems of differential–difference equations [1], treated as models of some physical phenomena, have been received considerable attention. They are usually difficult to solve analytically; so a numerical method is needed. Recently, much attention has been given in the literature to the development, analysis, and implementation of methods for differential and differential–difference equations (see [2–14] for instance). In this research we try to introduce a solution of a system of high-order linear differential–difference equations in the form

$$\sum_{n=0}^{m}\sum_{j=1}^{k}F_{i,j}^{n}(x)y_j^{(n)}(\mu x+\lambda)=g_i(x),\quad a\leqslant x\leqslant b,\quad i=1,2,\ldots,k, \tag{1}$$

subject to the initial conditions

$$\sum_{j=0}^{m-1}(a_{i,j}^{n}y_n^{(j)}(a)+b_{i,j}^{n}y_n^{(j)}(b)+c_{i,j}^{n}y_n^{(j)}(c))=\mu_{n,i},$$
$$a\leqslant c\leqslant b,\quad i=0,1,\ldots,m-1,\quad n=1,\ldots,k, \tag{2}$$

where $a_{i,j}^n$, $b_{i,j}^n$, $c_{i,j}^n$, μ, λ and $\mu_{n,i}$ are real or complex constants, meanwhile $F_{i,j}^n(x)$, $g_i(x)$ are continuous functions defined on the interval $a\leqslant x\leqslant b$.

* Corresponding author. Tel./fax: +98 851 2355466.
E-mail addresses: f.mirzaee@malayeru.ac.ir, f.mirzaee@iust.ac.ir (F. Mirzaee), saeed.bimesl@stu.malayeru.ac.ir (S. Bimesl).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2014 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society.
http://dx.doi.org/10.1016/j.joems.2014.05.003

2. Basic matrix relations for solution

The classical Euler polynomials $E_n(x)$ is usually defined as [15]

$$\sum_{k=0}^{n}\binom{n}{k}E_k(x)+E_n(x)=2x^n,\quad n\in N. \tag{3}$$

Let $X^T(x)$ be the $(N+1)\times 1$ matrix and P_{N+1} be the $(N+1)\times(N+1)$ lower triangular matrix defined by

$$X^T(x)=\left[1,x,\ldots,x^N\right]^T,\quad [P_{N+1}]_{ij}=\binom{i-1}{j-1},\quad i\geqslant j.$$

If n varies from 0 to N, the property (3) can be represented as matrix systems of equations

$$\frac{1}{2}(P_{N+1}+I_{N+1})E^T(x)=X^T(x).$$

Thus, the Euler vector can be given directly from

$$E^T(x)=D^{-1}X^T(x)\iff E(x)=X(x)(D^{-1})^T. \tag{4}$$

A relation between Euler polynomials and their derivatives is as follows ($E'_n(x)=nE_{n-1}(x), n=1,2,\ldots$)

$$\underbrace{[E_0(x),E_1(x),\ldots,E_N(x)]'}_{E'(x)}=\underbrace{[E_0(x),E_1(x),\ldots,E_N(x)]}_{E(x)}\underbrace{\begin{bmatrix}0&1&0&\ldots&0\\0&0&2&\ldots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&N\\0&0&0&\ldots&0\end{bmatrix}}_{M}. \tag{5}$$

We recall that, M is the Euler operational matrix of differentiation. Trivially $E^{(n)}(x)=E(x)M^n$ for all positive integers n, where our purpose from $E^{(n)}(x)$ is the n*th* derivative of $E(x)$. We can write $y_i(x)$ in the matrix form as follows;

$$y_i(x)=\sum_{n=0}^{N}a_{i,n}E_n(x)=E(x)A_i,\quad i=1,2,\ldots,k,$$
$$a\leqslant x\leqslant b, \tag{6}$$

where the Euler coefficient vector A_i and the Euler vector $E(x)$ are given by

$$A_i=[a_{i,0},a_{i,1},\ldots,a_{i,N}]^T,\quad E(x)=[E_0(x),E_1(x),\ldots,E_N(x)],$$

then the n*th* derivative of $y_i(x)$ can be expressed in the matrix form by

$$y_i^{(n)}(x)=E^{(n)}(x)A_i,\quad i=1,2,\ldots,k,\quad n=0,1,\ldots m. \tag{7}$$

Making use of (4), (5) and (7) yields

$$y_i^{(n)}(x)=E(x)M^nA_i=X(x)(D^{-1})^TM^nA_i,$$
$$i=1,2,\ldots,k,\quad n=0,1,\ldots m. \tag{8}$$

By putting $x\to\mu x+\lambda$ in the relation (8), we obtain the matrix form

$$y_i^{(n)}(\mu x+\lambda)=E(\mu x+\lambda)M^nA_i=X(\mu x+\lambda)(D^{-1})^TM^nA_i. \tag{9}$$

To obtain the matrix $X(\mu x+\lambda)$ in terms of the matrix $X(x)$, we can use the following relation:

$$X(\mu x+\lambda)=X(x)B(\mu,\lambda), \tag{10}$$

where

$$X(\mu x+\lambda)=\left[1,(\mu x+\lambda),(\mu x+\lambda)^2,\ldots,(\mu x+\lambda)^N\right],$$

for $\mu\neq 0$ and $\lambda\neq 0$,

$$B(\mu,\lambda)=\begin{bmatrix}\binom{0}{0}\mu^0\lambda^0&\binom{1}{0}\mu^0\lambda^1&\binom{2}{0}\mu^0\lambda^2&\ldots&\binom{N}{0}\mu^0\lambda^N\\0&\binom{1}{1}\mu^1\lambda^0&\binom{2}{1}\mu^1\lambda^1&\ldots&\binom{N}{1}\mu^1\lambda^{N-1}\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&\binom{N}{N}\mu^N\lambda^0\end{bmatrix}$$ and for

$$\mu\neq 0 \text{ and } \lambda=0, B(\mu,\lambda)=\begin{bmatrix}\mu^0&0&0&\ldots&0\\0&\mu^1&0&\ldots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&\mu^N\end{bmatrix}.$$

By substituting the relation (10) into (9), we get

$$y_i^{(n)}(\mu x+\lambda)=X(x)B(\mu,\lambda)(D^{-1})^TM^nA_i,$$
$$i=1,2,\ldots,k,\quad n=0,1,\ldots m. \tag{11}$$

Hence, the matrices $y^{(n)}(\mu x+\lambda)$, $n=0,1,\ldots m$ can be expressed as follows

$$y^{(n)}(\mu x+\lambda)=\overline{X}(x)\overline{B}(\mu,\lambda)\overline{D}\overline{M}^nA, \tag{12}$$

where

$$y^{(n)}(\mu x+\lambda)=\begin{bmatrix}y_1^{(n)}(\mu x+\lambda)\\y_2^{(n)}(\mu x+\lambda)\\\vdots\\y_k^{(n)}(\mu x+\lambda)\end{bmatrix},$$

$$\overline{X}(x)=\begin{bmatrix}X(x)&0&\cdots&0\\0&X(x)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&X(x)\end{bmatrix},$$

$$\overline{B}(\mu,\lambda)=\begin{bmatrix}B(\mu,\lambda)&0&\cdots&0\\0&B(\mu,\lambda)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&B(\mu,\lambda)\end{bmatrix},$$

$$\overline{D}=\begin{bmatrix}(D^{-1})^T&0&\cdots&0\\0&(D^{-1})^T&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&(D^{-1})^T\end{bmatrix},$$

$$\overline{M}^n=\begin{bmatrix}M^n&0&\cdots&0\\0&M^n&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&M^n\end{bmatrix},\quad A=\begin{bmatrix}A_1\\A_2\\\vdots\\A_k\end{bmatrix}.$$

2.1. Matrix representation for the conditions

We can write Eq. (2) for $i=1,2,\ldots,k$ in the matrix form as $\sum_{j=0}^{m-1}[a_j^iy_i^{(j)}(a)+b_j^iy_i^{(j)}(b)+c_j^iy_i^{(j)}(c)]=[\mu_i]$, where

$$\mu_i = \begin{bmatrix} \mu_{i,0} \\ \mu_{i,1} \\ \vdots \\ \mu_{i,m-1} \end{bmatrix}_{m\times 1}, \quad a_j^i = \begin{bmatrix} a_{0,j}^i \\ a_{1,j}^i \\ \vdots \\ a_{m-1,j}^i \end{bmatrix}_{m\times 1}, \quad b_j^i = \begin{bmatrix} b_{0,j}^i \\ b_{1,j}^i \\ \vdots \\ b_{m-1,j}^i \end{bmatrix}_{m\times 1},$$

$$c_j^i = \begin{bmatrix} c_{0,j}^i \\ c_{1,j}^i \\ \vdots \\ c_{m-1,j}^i \end{bmatrix}_{m\times 1},$$

or briefly

$$\sum_{j=0}^{m-1}[a_j y^{(j)}(a) + b_j y^{(j)}(b) + c_j y^{(j)}(c)] = \bar{\mu}, \tag{13}$$

where

$$\bar{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}, \quad a_j = \begin{bmatrix} a_j^1 & 0 & \cdots & 0 \\ 0 & a_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_j^k \end{bmatrix}_{k\times k},$$

$$b_j = \begin{bmatrix} b_j^1 & 0 & \cdots & 0 \\ 0 & b_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_j^k \end{bmatrix}_{k\times k}, \quad c_j = \begin{bmatrix} c_j^1 & 0 & \cdots & 0 \\ 0 & c_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_j^k \end{bmatrix}_{k\times k}.$$

Substituting the matrices $y^{(j)}(a)$, $y^{(j)}(b)$ and $y^{(j)}(c)$, which depend on the matrix of Euler coefficients A into Eq. (13) and simplifying the result we obtain $\sum_{j=0}^{m-1}[a_j\overline{X}(a) + b_j\overline{X}(b) + c_j\overline{X}(c)]\bar{D}\bar{M}^j A = \bar{\mu}$. Let us define U as

$U = \sum_{j=0}^{m-1}[a_j\overline{X}(a) + b_j\overline{X}(b) + c_j\overline{X}(c)]\bar{D}\bar{M}^j$, thus the fundamental matrix form for the conditions becomes

$$UA = \bar{\mu}, \quad or \ [U;\bar{\mu}]. \tag{14}$$

3. Description of the numerical method

First, we can write the system (1) in the matrix form

$$\sum_{n=0}^{m} F_n(x) y^{(n)}(\mu x + \lambda) = g(x), \tag{15}$$

where

$$F_n(x) = \begin{bmatrix} F_{1,1}^n(x) & F_{1,2}^n(x) & \ldots & F_{1,k}^n(x) \\ F_{2,1}^n(x) & F_{2,2}^n(x) & \ldots & F_{2,k}^n(x) \\ \vdots & \vdots & \ddots & \vdots \\ F_{k,1}^n(x) & F_{k,2}^n(x) & \cdots & F_{k,k}^n(x) \end{bmatrix},$$

$$y^{(n)}(\mu x + \lambda) = \begin{bmatrix} y_1^{(n)}(\mu x + \lambda) \\ y_2^{(n)}(\mu x + \lambda) \\ \vdots \\ y_k^{(n)}(\mu x + \lambda) \end{bmatrix}, \quad g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}.$$

By placing in Eq. (15) the collocation points defined by

$$x_s = a + \frac{b-a}{N}s, \quad s = 0, 1, 2, \ldots, N, \tag{16}$$

we obtain

$$\sum_{n=0}^{m} F_n Y^{(n)} = G; \quad F_n = \begin{bmatrix} F_n(x_0) & 0 & \ldots & 0 \\ 0 & F_n(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & F_n(x_N) \end{bmatrix},$$

$$Y^{(n)} = \begin{bmatrix} y^{(n)}(\mu x_0 + \lambda) \\ y^{(n)}(\mu x_1 + \lambda) \\ \vdots \\ y^{(n)}(\mu x_N + \lambda) \end{bmatrix}, \quad G = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Using relation (12) and the collocation points (16), we obtain

$$y^{(n)}(\mu x_s + \lambda) = \overline{X}(x_s)\overline{B}(\mu,\lambda)\overline{D}(\overline{M})^n A,$$
$$s = 0, 1, \ldots, N, \quad n = 0, 1, \ldots m, \tag{17}$$

which can be written as

$$Y^{(n)} = X\overline{B}(\mu,\lambda)\bar{D}\bar{M}^n A; \quad \mathbf{X} = \begin{bmatrix} \overline{X}(x_0) \\ \overline{X}(x_1) \\ \vdots \\ \overline{X}(x_N) \end{bmatrix},$$
$$\overline{X}(x_s) = \begin{bmatrix} X(x_s) & 0 & \cdots & 0 \\ 0 & X(x_s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X(x_s) \end{bmatrix}. \tag{18}$$

Substituting relation (18) into Eq. (3), we have the fundamental matrix equation

$$\left\{\sum_{n=0}^{m} F_n X\overline{B}(\mu,\lambda)\bar{D}\bar{M}^n\right\}A = G. \tag{19}$$

Briefly, the fundamental matrix Eq. (19) corresponding to Eq. (1) can be written in the form

$$WA = G, \quad or \ [W;G]. \tag{20}$$

To obtain the solution of Eq. (1) under conditions (2), we replace the row matrices (14) by the last mk rows of the matrix (20) and obtain the new augmented matrix as follows

$$[\overline{\overline{W}};\overline{\overline{G}}] = \begin{bmatrix} \overline{W}; & \overline{G} \\ U; & \bar{\mu} \end{bmatrix} \quad \text{then} \quad A = (\overline{\overline{W}})^{-1}\overline{\overline{G}}. \tag{21}$$

4. Study of error

Suppose that $H = L^2[0,1]$, $Y = span\{E_0(x), E_1(x), \ldots, E_N(x)\}$ and g be an arbitrary element in H. Since Y is a finite dimensional vector space, g has the unique best approximation out of Y such as $\bar{g} \in Y$, that is for all $y \in Y$, $\|g - \bar{g}\| \leqslant \|g - y\|$. Since $\bar{g} \in Y$, there exists the unique coefficients $g_0, g_1, \cdots, g_N$ such that $g(x) \simeq \bar{g}(x) = \sum_{n=0}^{N} g_n E_n(x) = E(x)G^T$, where $E(x) = [E_0(x), E_1(x), \cdots, E_N(x)]$ and $G = [g_0, g_1, \ldots, g_N]$.

Theorem 1. *Assume that $g \in H$ be an arbitrary function and also is approximated by the truncated Euler series $\sum_{n=0}^{N} g_n E_n(x)$, then the coefficients g_n for all $n = 0, 1, \ldots, N$ can be calculated from the following relation*

$$g_n = \begin{cases} \frac{1}{N!}\int_0^1 g^{(N)}(x)dx, n = N \\ \frac{1}{n!}\left(\int_0^1 g^{(n)}(x)dx + \sum_{k=0}^{N-n-1} \frac{2(n!)}{k+2}\binom{k+n+1}{k+1} E_{k+2}(0) g_{n+k+1}\right), \\ n = N-1, \quad N-2, \ldots, 0. \end{cases}$$

Proof. Suppose that g is approximated by the truncated Euler series $\sum_{n=0}^{N} g_n E_n(x)$; in other words

$$g(x) \simeq \sum_{n=0}^{N} g_n E_n(x) = g_0 E_0(x) + g_1 E_1(x) + \dots g_N E_N(x). \quad (22)$$

By the following familiar expansion [15]

$$E_n(x) = \sum_{k=0}^{n} \frac{-2}{k+1} \binom{n}{k} E_{k+1}(0) B_{n-k}(x), \quad (23)$$

Euler polynomials $(E_n(x), n = 0, 1, \dots N)$ can be expressed in terms of the Bernoulli polynomials. Using Eq. (23), and substituting results into Eq. (22), we get

$$\begin{aligned} g(x) \simeq & -\frac{2}{1}\binom{0}{0} g_0 E_1(0) B_0(x) \\ & - g_1 \left(\frac{2}{2}\binom{1}{1} E_2(0) B_0(x) + \frac{2}{1}\binom{1}{0} E_1(0) B_1(x) \right) \\ & - g_2 \left(\frac{2}{3}\binom{2}{2} E_3(0) B_0(x) + \frac{2}{2}\binom{2}{1} E_2(0) B_1(x) + \frac{2}{1}\binom{2}{0} E_1(0) B_2(x)) \right) \\ & - \dots - g_N \left(\frac{2}{N+1}\binom{N}{N} E_{N+1}(0) B_0(x) + \frac{2}{N}\binom{N}{N-1} E_N(0) B_1(x) + \cdots \right. \\ & \left. + \frac{2}{2}\binom{N}{1} E_2(0) B_{N-1}(x) + \frac{2}{1}\binom{N}{0} E_1(0) B_N(x)) \right) \\ = & \left(-\frac{2}{1}\binom{0}{0} g_0 E_1(0) - \dots - \frac{2}{N+1}\binom{N}{N} g_N E_{N+1}(0) \right) B_0(x) \\ & + \left(-\frac{2}{1}\binom{1}{0} g_1 E_1(0) - \dots - \frac{2}{N}\binom{N}{N-1} g_N E_N(0) \right) B_1(x) + \cdots \\ & + \left(-\frac{2}{1}\binom{N-1}{0} g_{N-1} E_1(0) - \frac{2}{2}\binom{N}{1} g_N E_2(0) \right) B_{N-1}(x) \\ & + \left(-\frac{2}{1}\binom{N}{0} E_1(0) g_N \right) B_N(x). \end{aligned} \quad (24)$$

By differentiating the above equation N times, applying the differentiation means condition property of Bernoulli polynomials (i.e. $B'_n(x) = nB_{n-1}(x),\ n \geqslant 1$) and integrating the obtained equation in the interval $[0, 1]$ and applying the integral means condition property of Bernoulli polynomials (i.e. $\int_0^1 B_n(x)dx = 0,\ n \geqslant 1$)

$$\int_0^1 g^{(N)}(x)dx \simeq \int_0^1 \left(\sum_{n=0}^{N} g_n E_n^{(N)}(x) \right) dx = -\frac{2}{1}\binom{N}{0} N! E_1(0) g_N \overbrace{\int_0^1 B_0(x)dx}^{1 \text{since} B_0(x)=1},$$

since $E_1(0) = \frac{-1}{2}$, hence $g_N = \frac{1}{N!}\int_0^1 g^{(N)}(x)dx$. Now, by differentiating Eq. (24) $(N-1)$ times and again integrating the obtained equation in the interval $[0, 1]$,

$$\int_0^1 g^{(N-1)}(x)dx \simeq \left(-\frac{2}{1}\binom{N-1}{0} g_{N-1} E_1(0) - \frac{2}{2}\binom{N}{1} g_N E_2(0) \right)$$
$$(N-1)! \overbrace{\int_0^1 B_0(x)dx}^{1 \text{since} B_0(x)=1} + (N-1)! g_N \overbrace{\int_0^1 B_1(x)dx}^{\text{equals to zero}},$$

and hence $g_{N-1} = \frac{\int_0^1 g^{(N-1)}(x)dx + \frac{2}{2}\binom{N}{1}(N-1)! E_2(0) g_N}{(N-1)!}$. Repeating this procedure n times for $n = (N-2), (N-3), \dots, 0$ yields $g_n = \frac{\int_0^1 g^{(n)}(x)dx + \sum_{k=0}^{N-n-1} \frac{2(n!)}{k+2}\binom{k+n+1}{k+1} E_{k+2}(0) g_{n+k+1}}{n!}$. □

The above Theorem implies that Euler coefficients are decayed rapidly with increasing of n. Also we succeed to convert the approximate polynomial of $g(x)$ in terms of Euler polynomials with the Eq. (23) to the corresponding approximate polynomial of $g(x)$ in terms of Bernoulli polynomials [16].

Theorem 2. *([17]). Suppose that $g(x)$ be an enough smooth function in the interval $[0, 1]$ and $P_N[g](x)$ is the approximate polynomial of $g(x)$ in terms of Bernoulli polynomials and $R_N[g](x)$ is the remainder term. Then, the associated formulae are stated for $x \in [0, 1]$ as $g(x) = P_N[g](x) + R_N[g](x)$*

$$P_N[g](x) = \int_0^1 g(x)dx + \sum_{j=1}^{N} \frac{B_j(x)}{j!} (g^{(j-1)}(1) - g^{(j-1)}(0)),$$

$$R_N[g](x) = -\frac{1}{N!} \int_0^1 B_N^*(x-t) g^{(N)}(t)dt,$$

where $B_N^*(x) = B_N(x - [x])$ and $[x]$ denotes the largest integer not greater than n.

Theorem 3. *Suppose $g(x) \in C^\infty[0, 1]$ and $P_N[g](x)$ is the approximate polynomial using Bernoulli polynomials. Then the error bound would be obtained as follows*

$$\|\text{Error}(g(x))\|_\infty \leqslant \frac{1}{N!} B_N G_N,$$

where B_N and G_N denote the maximum value of $B_N^*(x)$ and $g^{(N)}(x)$ in the interval $[0, 1]$ respectively.

Proof. By considering $R_N[g](x) = -\frac{1}{N!}\int_0^1 B_N^*(x-t) g^{(N)}(t)dt$, the proof is clear. □

5. Test problems

Example 1. Consider first the system of initial value problems given by

$$\begin{cases} y'_1(x-1) + y'_2(x-1) = 2x & y_1(0) = 0 \\ y'_1(x-1) - y'_3(x-1) = 2x - 1 & y_2(0) = 0 \quad -3 \leqslant x \leqslant 4, \\ y'_1(x-1) + y_3(x-1) = x - 1 & y_3(0) = 0 \end{cases} \quad (25)$$

with exact solutions $y_1(x) = x^2,\ y_2(x) = 2x,\ y_3(x) = -x$. By applying the technique described in Section 3 with $N = 3$, we may write the approximate solution in the form

$$y_i(x) = \sum_{n=0}^{3} a_{i,n} E_n(x), \quad i = 1, 2, 3,$$

where

$$F_0(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_1(x) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 2x \\ 2x - 1 \\ x - 1 \end{bmatrix}.$$

Table 1 Comparison of the absolute errors for $N = 8, 10$ of $y_1(x)$ and $y_2(x)$ of Eq. (26).

x_i	Absolute errors of $y_1(x)$			Absolute errors of $y_2(x)$		
	TCM [9] ($N = 10$)	CMEP method ($N = 8$)	CMEP method ($N = 10$)	TCM [9] ($N = 10$)	CMEP method ($N = 8$)	CMEP method ($N = 10$)
0	0	0	0	0	0	0
0.1	3.226e−11	4.9552e−11	4.8850e−14	3.1571e−11	2.4766e−11	3.5527e−14
0.2	3.681e−11	4.9312e−11	4.8850e−14	3.139e−11	2.4641e−11	3.5527e−14
0.5	9.215e−11	6.4881e−11	1.1369e−13	7.246e−11	3.2397e−11	1.0747e−13
0.8	1.3808e−10	9.1289e−11	2.8244e−13	1.0270e−10	4.5579e−11	2.2737e−13
1	2.7443e−10	1.1536e−09	1.1253e−12	3.9660e−10	5.7658e−10	4.7373e−13

Table 2 Comparison of the absolute errors for $N = 6, 10$ of $y_1(x)$ and $y_2(x)$ of Eq. (27).

x_i	Absolute errors of $y_1(x)$			Absolute errors of $y_2(x)$		
	Transform method [6] ($N = 6$)	CMEP method ($N = 6$)	CMEP method ($N = 10$)	Transform method [6] ($N = 6$)	CMEP method ($N = 6$)	CMEP method ($N = 10$)
0	0	0	0	0	0	0
0.2	1.5575e−02	2.8460e−08	3.1863e−14	2.3262e−03	2.8460e−08	3.3529e−14
0.4	5.1209e−02	1.7820e−08	3.3196e−14	1.6867e−02	1.7820e−08	4.8184e−14
0.6	1.0150e−01	1.2668e−08	4.0967e−14	5.4499e−02	1.2668e−08	9.2815e−14
0.8	1.6630e−01	3.3538e−08	5.3846e−14	1.3194e−01	3.3538e−08	1.7519e−13
1	2.3351e−01	6.8657e−07	1.6325e−12	2.8038e−01	6.8657e−07	1.4025e−12

The augmented matrix for treating Eq. (25) with the collocation points $\{x_0 = -3, x_1 = -\frac{2}{3}, x_2 = \frac{5}{3}, x_3 = 4\}$, is

$$[W;G] = \begin{bmatrix} 0 & 1 & -9 & 60 & 0 & 1 & -9 & 60 & 0 & 0 & 0 & 0; & -6 \\ 0 & 1 & -9 & 60 & 0 & 0 & 0 & 0 & 0 & -1 & 9 & -60; & -7 \\ 0 & 1 & -9 & 60 & 0 & 0 & 0 & 0 & 1 & -9/2 & 20 & -351/4; & -4 \\ 0 & 1 & -13/3 & 40/3 & 0 & 1 & -13/3 & 40/3 & 0 & 0 & 0 & 0; & -4/3 \\ 0 & 1 & -13/3 & 40/3 & 0 & 0 & 0 & 0 & 0 & -1 & 13/3 & -40/3; & -7/3 \\ 0 & 1 & -13/3 & 40/3 & 0 & 0 & 0 & 0 & 1 & -13/6 & 40/9 & -923/108; & -5/3 \\ 0 & 1 & 1/3 & -2/3 & 0 & 1 & 1/3 & -2/3 & 0 & 0 & 0 & 0; & 10/3 \\ 0 & 1 & 1/3 & -2/3 & 0 & 0 & 0 & 0 & 0 & -1 & -1/3 & 2/3; & 7/3 \\ 0 & 1 & 1/3 & -2/3 & 0 & 0 & 0 & 0 & 1 & 1/6 & -2/9 & -13/108; & 2/3 \\ 0 & 1 & 5 & 18 & 0 & 1 & 5 & 18 & 0 & 0 & 0 & 0; & 8 \\ 0 & 1 & 5 & 18 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & -18; & 7 \\ 0 & 1 & 5 & 18 & 0 & 0 & 0 & 0 & 1 & 5/2 & 6 & 55/4; & 3 \end{bmatrix}.$$

and the matrix forms for conditions are

$$[U_0 : \mu_1] = [1 \ -1/2 \ 0 \ 1/4 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ ; \ 0],$$

$$[U_1 : \mu_2] = [0 \ 0 \ 0 \ 0 \ 1 \ -1/2 \ 0 \ 1/4 \ 0 \ 0 \ 0 \ 0 \ ; \ 0],$$

$$[U_2 : \mu_3] = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1/2 \ 0 \ 1/4 \ ; \ 0].$$

Solving the previous system, the unknown Euler coefficients vector is $A = [1/2 \ 1 \ 1 \ 0 \ 1 \ 2 \ 0 \ 0 \ -1/2 \ -1 \ 0 \ 0]^T$.

Hence, the solutions of the problem for $N = 3$ become $y_1(x) = x^2$, $y_2(x) = 2x$ and $y_3(x) = -x$ which are the exact solutions. Moreover, if higher values of N be chosen, we obtain the exact solution again.

Example 2 ([18]). Consider the system of initial value problems given by

$$\begin{cases} y_1'(x) + 4y_2'(x) + y_1(x) = 1 + 2e^{-x} & y_1(0) = 1 \\ y_1'(x) + y_2'(x) + y_2(x) = x - e^{-x} & y_2(0) = 0 \quad 0 \leqslant x \leqslant 1, \end{cases} \tag{26}$$

with the exact solutions $y_1(x) = e^{-x} + 3e^{-x/3} - 3$, $y_2(x) = -\frac{1}{2}e^{-x} + \frac{3}{2}e^{-x/3} + x - 1$. Tables 1 shows the numerical results obtained by collocation method based on Euler polynomials (CMEP) and the numerical results of [9] using Taylor collocation method (TCM).

Example 3 ([19]). Consider the following linear differential system

$$\begin{cases} y_2'(x) - 2y_1(x) - y_2(x) = 0 & y_1(0) = 0 \\ y_1'(x) + y_2'(x) + y_1(x) + y_2(x) = 1 & y_2(0) = 0 \quad 0 \leqslant x \leqslant 1, \end{cases} \tag{27}$$

with the exact solutions $y_1(x) = e^{-x} - 1$, $y_2(x) = 2 - e^{-x}$. In Table 2 the numerical results for this example with $N = 6, 10$ are displayed together with the results obtained in [6] using transform method.

6. Conclusion

In this article, we introduced a new collocation method based on the Euler polynomials and used it for solving systems of high-order linear differential–difference equations with variable coefficients. One of the advantages of this method is that the proposed problem is transformed to a system of algebraic equations. Another considerable advantage of this method is to obtain the analytical solutions if the system has an exact solution that is a polynomial function. The numerical results show that the algorithm converges as the number of N terms is increased. The method proposed in this work can be extended to solve the important nonlinear partial differential equations investigated in [20,21], but some modifications are required.

Acknowledgement

The authors are extending their heartfelt thanks to the anonymous referees for their valuable comments and suggestions.

References

[1] A. Bellen, M. Zennaro, Numerical Methods for Delay Differential Equations, Clarendon Press, 2003.
[2] J. Saberi-Nadjafi, M. Tamamgar, The variational iteration method, a highly promising method for solving the system of integro-differential equations, Comput. Math. Appl. 56 (2008) 346–351.
[3] M. Tatari, M. Dehghan, Improvement of he's variational iteration method for solving systems of differential equations, Comput. Math. Appl. 58 (2009) 2160–2166.
[4] F. Mahomed, I. Naeem, A. Qadir, Conditional linearizability criteria for a system of third-order ordinary differential equations, Nonl. Anal. Real World Appl. 10 (2009) 3404–3412.
[5] A. Saadatmandi, M. Dehghan, A. Eftekhari, Application of he's homotopy perturbation method for non-linear system of second-order boundary value problems, Nonl. Anal. Real World Appl. 10 (2009) 1912–1922.
[6] M. Thangmoon, S. Pusjuso, The numerical solution of differential transform method and Laplace transform method for a system of differential equations, Nonl. Anal. Hybrid Syst. 4 (2010) 425–431.
[7] Ş. Yüzbaşı, N. Şahin, A. Yıldırım, Numerical solutions of systems of high-order linear differential–difference equations with Bessel polynomial bases, Zeit. Natu. A. J. Phys. Sci. 66 (2011) 519–532.
[8] Ş. Yüzbaşı, On the solutions of a system of linear retarded and advanced differential equations by the Bessel collocation approximation, Comput. Math. Appl. 63 (10) (2012) 1442–1455.
[9] E. Gökmen, M. Sezer, Taylor collocation method for systems of high-order linear differential–difference equations with variable coefficients, Ain Shams Eng. J. 4 (2013) 117–125.
[10] H.M. Ahmed, Solutions of 2nd-order linear differential equations subject to Dirichlet boundary conditions in a Bernstein polynomial basis, J. Egypt. Math. Soc. 22 (2) (2014) 227–237.
[11] S.H. Behiry, Solution of nonlinear Fredholm integro-differential equations using a hybrid of block pulse functions and normalized Bernstein polynomials, J. Comput. Appl. Math. 260 (2014) 258–265.
[12] M.M. Khader, Numerical treatment for solving fractional Riccati differential equation, J. Egypt. Math. Soc. 21 (2013) 32–37.
[13] F. Mirzaee, S. Bimesl, A new approach to numerical solution of second-order linear hyperbolic partial differential equations arising from physics and engineering, Res. Phys. 3 (2013) 241–247.
[14] F. Mirzaee, S. Bimesl, Application of Euler matrix method for solving linear and a class of nonlinear Fredholm integro-differential equations, Mediter. J. Math. (2014) in press.
[15] F. Mirzaee, S. Bimesl, A new Euler matrix method for solving systems of linear Volterra integral equations with variable coefficients, J. Egypt. Math. Soc. 22 (2) (2014) 238–248.
[16] F. Mirzaee, S. Bimesl, An efficient numerical approach for solving systems of high-order linear Volterra integral equations, Sci. Iran. (2014) in press.
[17] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Wiley, New York, 1972.
[18] A. Davies, D. Crann, The solution of differential equations using numerical Laplace transforms, Int. J. Math. Educ. Sci. Technol. 30 (1999) 65–79.
[19] Ş. Yüzbaşı, A numerical approach for solving the high-order linear singular differential–difference equations, Comput. Math., Appl. 62 (5) (2011) 2289–2303.
[20] M. Dehghan, A. Ghesmati, Solution of the second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation (DRBIE) method, Eng. Ana. Bound. Elem. 34 (1) (2010) 51–59.
[21] M. Dehghan, A. Ghesmati, Application of the dual reciprocity boundary integral equation technique to solve the nonlinear Klein–Gordon equation, Comput. Phys. Commun. 181 (8) (2010) 1410–1418.