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**ORIGINAL ARTICLE**

# Transient analysis of a two-heterogeneous servers queue with impatient behavior

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**Abstract** Recently, [1] have obtained the transient solution of multi-server queue with balking and reneging. In this paper, a similar technique is used to drive a new elegant explicit solution for a two heterogeneous servers queue with impatient behavior. In addition, steady-state probabilities of the system size are studied and some important performance measures are discussed for the considered system.

**2000 MATHEMATICS SUBJECT CLASSIFICATION:** 60K25; 68M20; 90B22



## 1. Introduction

In most studies on queueing systems, the customers always wait in the system until service is completed. In many practical systems, such as telephone switchboard customers, hospital emergency rooms' handling of critical patients, and perishable goods storage inventory systems, the customers may become impatient and leave (i.e., balk or renege) the system without getting services when the waiting time is intolerable. For example, for a call-in customer who cannot get service immediately by the server, he/she is told how long he/she needs to wait. The customer might hang up (balk) or hold on (non-balk and waiting). This is a balking behavior of the customer when the queue length or waiting time is too long. In addition, a waiting customer might hang up (renege) if he/she becomes impatient. Someone who wants to buy a train ticket (or meal ticket) might decide not entering the system (balk) if the waiting line is too long. As a customer waiting in the queue, he/she might leave the queue (renege) and choose an automat (or instant food). Queueing models with balking, or reneging, or both have attracted much attention from numerous researchers. For related literature, interested readers may refer to [1,2] and references therein.

On the other hand, heterogeneity of service is a common feature of many real multi-server queueing situations. The heterogeneous service mechanisms are invaluable scheduling methods that allow customers to receive different quality of service. Heterogeneous service is clearly a main feature of the operation of almost any manufacturing system. The role of quality and service performance is crucial aspects in customer perceptions and firms must dedicate special attention to them

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when designing and implementing their operations. For this reason, queues with heterogeneous servers have received considerable attention in the literature [3].

In [4], the authors have pointed out that for the queueing systems with more than two heterogeneous servers, analytical results are intractable. Therefore, many researchers have studied queueing systems with two heterogeneous servers. In [5], the author studied an M/M/2 queueing system with balking and two heterogeneous servers. In [6], the authors have discussed the two-channels queue M/M/2 with both balking, heterogeneity and considering different probability in choosing the server. In [7], the author proposed a transient solution of the non-truncated queue M/M/2 involving balking, and an additional server for longer queues. In [8], the authors have analyzed an M/M/2 queueing system with two heterogeneous servers and multiple vacations by using the matrix-geometric solution method. In [3], the authors have derived the transient solution for the probabilities in the two-server queueing system subject to catastrophes, where one server is faster than the other, by defining a suitable probability generating function. In [9], the authors have presented the transient solution of the M/M/2 queue with catastrophe at the service station

The authors in [10] further considered the model in [8]. They have obtained the explicit expression of the rate matrix and presented the conditional stochastic decomposition results for the queue length and the waiting time. In [11], the authors have investigated an M/M/2 queue with Bernoulli schedules and a single vacation policy where the two servers provide heterogeneous exponential service to customers. They obtained the steady-state probability generating functions of the system size for various states of the servers. In [12], the authors introduced an M/M/2 queueing system with balking and two heterogeneous servers under Bernoulli schedules and a single vacation policy. They presented a generalization of Model B in [11] and obtained the explicit expressions of the steady-state condition, the stationary distribution of the system size, and the mean system size. In [13], the authors have discussed an M/M/2 queueing system with two heterogeneous servers under a variant vacation policy, where the two servers may take together at most J vacations when the system is empty. In [14], the authors have displayed an M/M/2 queueing model with heterogeneous servers where one server remains idle but the other goes on vacation in the absence of waiting customers.

The two heterogeneous servers are extensively studied as mentioned above, however, in the literature is no work on an M/M/2 queue with heterogeneous servers subject to balking and reneging. Based on this observation, we have investigated the transient solution for the probabilities in the two server subject to balking and reneging.

The rest of the paper is organized as follows. Section 2 presents a model description and obtains the time dependent state probabilities for the number in the system.. In Section 3, we get the solution for the steady-state probabilities. In Section 4, we give some performance measures of the system.

## 2. Model description and main results

In this paper, we consider an M/M/2 queueing system with impatient customers, where two servers have different rates. The assumptions of the system model are as follows:

(a) Customers arrive at the system one by one according to a Poisson process with rate $\lambda$.
(b) The two servers provide heterogeneous exponential service to customer on a first-come, first served (FCFS) basis with service rate$\mu_j$ for $j$th server, $j = 1, 2$.
(c) A customer who on arrival finds at least two customers in the system, either decides to enter the queue with probability $p$ or balk with probability $1-p$. Let $\lambda_p = \lambda p$.
(d) After joining the queue, each customer will wait a certain length of time $T$ for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service. This time $T$ is assumed to be distributed according to an exponential distribution with mean $1/\alpha$. Since the arrival and the departure of the impatient customers without service are independent, the reneging rate when there are $n$ customers is $(n-2)\alpha$.
(e) Let $\{X(t), t \in R^+\}$ be the number of customers in the system at time $t$, let $P_n(t) = P(X(t) = n)$, $n = 2, 3, 4, \ldots$ denote the probability that there are $n$ customers in the system at time $t$.
(f) Let $P_{0,0}(t) = P(X(t) = 0)$ be the probability that the system is empty at time $t$, $P_{1,0}(t) = P(X(t) = 1)$ be the probability that there is one customer in the system and he is served by server 1 and $P_{0,1}(t) = P(X(t) = 1)$ be the probability that there is one customer in the system and he is served by server 2.
(g) If a customer arrives to an empty system, it always joins server 1.

From the above assumptions, the forward equations for the system

$$\frac{dP_{0,0}(t)}{dt} = -\lambda P_{0,0}(t) + \mu_1 P_{1,0}(t) + \mu_2 P_{0,1}(t), \tag{2.1}$$

$$\frac{dP_{1,0}(t)}{dt} = -(\lambda + \mu_1)P_{1,0}(t) + \lambda P_{0,0}(t) + \mu_2 P_2(t), \tag{2.2}$$

$$\frac{dP_{0,1}(t)}{dt} = -(\lambda + \mu_2)P_{0,1}(t) + \mu_1 P_2(t), \tag{2.3}$$

$$\frac{dP_2(t)}{dt} = -(\lambda_p + \mu)P_2(t) + \lambda(P_{0,1}(t) + P_{1,0}(t)) + (\mu + \alpha)P_3(t) \tag{2.4}$$

and

$$\frac{dP_n(t)}{dt} = -(\lambda_p + \mu + (n-2)\alpha)P_n(t) + \lambda_p P_{n-1}(t) + (\mu + (n-1)\alpha)P_{n+1}(t), \quad n \geqslant 3 \tag{2.5}$$

where $\mu = \mu_1 + \mu_2$

We assume that there is no customer in the system at time $t = 0$, so that $P_{0,0}(0) = 1$.

We define the probability generating function

$$P(z,t) = R_0(t) + \sum_{n=0}^{\infty} P_{n+3}(t) z^{n+1} \tag{2.6}$$

where

$$R_0(t) = P_{0,0}(t) + P_{0,1}(t) + P_{1,0}(t) + P_2(t)$$

with initial condition $P(z,0) = 1$

The system of Eqs. (2.1)–(2.6) yields the following partial differential equation

$$\frac{\partial P(z,t)}{\partial t} - \alpha(1-z)\frac{\partial P(z,t)}{\partial z} = [(\mu-\alpha)(z^{-1}-1) + \lambda_p(z-1)] \times [P(z,t) - R_0(t)] + \lambda_p(z-1)P_2(t) \tag{2.7}$$

The solution of (2.7) is easily obtained as

$$\begin{aligned} P(z,t) = \exp\{-[(\mu-\alpha)(z^{-1}-1) + \lambda_p(z-1)]t\} \\ + \int_0^t [\lambda_p(z-1)P_2(u) + \{(\mu-\alpha)(z^{-1}-1) \\ + \lambda_p(z-1)\}R_0(u)].\exp\{-[(\mu-\alpha)(z^{-1}-1) \\ + \lambda_p(z-1)](t-u)\}du. \end{aligned} \tag{2.8}$$

It is well known that if $r = 2\sqrt{\lambda_p(\mu-\alpha)}$ and $\beta = \sqrt{\lambda_p/(\mu-\alpha)}$, then

$$\exp\left\{\left(\lambda_p z + \frac{\mu-\alpha}{z}\right)t\right\} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(rt),$$

where $I_n(\cdot)$ is the modified Bessel function of first kind order $n$. Substituting this in Eq. (2.8), expanding $P(z,t)$ as a series in $z$ and comparing the coefficient of $z^n$ on either side, we get, for $n = 1, 2, 3, \ldots,$

$$\begin{aligned} P_{n+2}(t) = \exp\{-[\lambda_p + (\mu-\alpha)]t\}\beta^n I_n(rt) \\ + \beta^{n-1}\int_0^t \exp(-(\lambda_p + (\mu-\alpha))(t-u)) \\ \times [\lambda_p\{I_{n-1}(r(t-u)) - \beta I_n(r(t-u))\}P_2(u)du \\ - \beta^{n-1}\int_0^t \exp(-(\lambda_p + (\mu-\alpha))(t-u)) \\ \times [\lambda_p\{I_{n-1}(r(t-u)) + I_{n+1}(r(t-u))\} \\ - \beta(\lambda_p + (\mu-\alpha))I_n(r(t-u))R_0(u)]du \end{aligned} \tag{2.9}$$

and for $n = 0$,

$$\begin{aligned} \beta R_0(t) = \exp\{-[\lambda_p + (\mu-\alpha)]t\}\beta I_0(rt) \\ + \int_0^t \exp(-(\lambda_p + (\mu-\alpha))(t-u)) \\ \times [\lambda_p\{I_1(r(t-u)) - \beta I_0(r(t-u))\}P_2(u)du \\ - \int_0^t R_0(u)\exp(-(\lambda_p + (\mu-\alpha))(t-u)) \\ \times [2\lambda_p I_1(r(t-u)) - \beta(\lambda_p + (\mu-\alpha))I_0(r(t-u))]du \end{aligned} \tag{2.10}$$

As $P(z,t)$ does not contain terms with negative powers of $z$, the right hand side of (2.9) with $n$ replaced by $-n$, must be zero. Thus,

$$\begin{aligned} \beta^{n-1}\int_0^t \exp(-(\lambda_p + (\mu-\alpha))(t-u))[\lambda_p\{I_{n+1}(r(t-u)) \\ + I_{n-1}(r(t-u))\} - \beta(\lambda_p + (\mu-\alpha))I_n(r(t-u))R_0(u)]du \\ = \exp\{-[\lambda_p + (\mu-\alpha)]t\}\beta^n I_n(rt) \\ + \beta^{n-1}\int_0^t \exp(-(\lambda_p + (\mu-\alpha))(t-u))[\lambda_p\{I_{n+1}(r(t-u)) \\ - \beta I_n(r(t-u))\}P_2(u)]du \end{aligned} \tag{2.11}$$

where we have used $I_{-k}(\cdot) = I_k(\cdot)$. The usage of (2.11) in (2.9) considerably simplifies the working and results in elegant expression for$P_n(t)$. This yields, for $n = 1, 2, \ldots$

$$\begin{aligned} P_{n+2}(t) = \beta^{n-1}\int_0^t \exp(-(\lambda_p + (\mu-\alpha))(t-u)) \\ \times [\lambda_p\{I_{n-1}(r(t-u)) - I_{n+1}(r(t \\ -u))\}P_2(u)]du \end{aligned} \tag{2.12}$$

Now, the probabilities $P_{0,0}(t)$, $P_{0,1}(t)$, $P_{1,0}(t)$ and $P_2(t)$ remain to be found. For this, we consider the system of Eqs. (2.1)–(2.3) subject to the condition (2.10). Eqs. (2.1)–(2.3) can be expressed in matrix form as

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{AP(t)} + \mu_2 P_2(t)\mathbf{e}_1 + \mu_1 P_2(t)\mathbf{e}_2 \tag{2.13}$$

where

$$P(t) = (P_{00}(t), P_{01}(t), P_{10}(t))^T, A = \begin{pmatrix} -\lambda & \mu_1 & \mu_2 \\ \lambda & -(\lambda+\mu_1) & 0 \\ 0 & 0 & -(\lambda+\mu_2) \end{pmatrix}$$

$\mathbf{e}_1 = (0,1,0)^T$, and $bf\ e_2 = (0,0,1)^T$.

In the sequel, let $P_n^*(s)$ denote the Laplace transform of $P_n(t)$. Now, by taking Laplace transform, the solution of (2.13) is obtained as

$$\mathbf{P}^*(s) = [sI - A]^{-1}\left[\mu_2 P_2^*(s)\mathbf{e}_1 + \mu_1 P_2^*(s)\mathbf{e}_2 + \mathbf{P}(0)\right] \tag{2.14}$$

with

$\mathbf{P}(0) = (1,0,0)^T$.

Thus, only $P_2^*(s)$ is to be found. We note that, if $e = (1,1,1)^T$,

$R_0^*(s) = \mathbf{e}^T\mathbf{P}^*(s) + P_2^*(s)$.

Using (2.10) in the above equation and simplifying, we get, with $q = s + \lambda_p + (\mu - \alpha)$,

$$P_2^*(s) = \frac{1 - s\mathbf{e}^T(sI-A)^{-1}\mathbf{P}(0)}{s + \lambda_p - \left(\frac{q-\sqrt{q^2-r^2}}{2}\right) + s\mathbf{e}^T(sI-A)^{-1}[\mu_2\mathbf{e}_2 + \mu_1\mathbf{e}_1]} \tag{2.15}$$

Let

$(sI - A)^{-1} = (a_{ij}^*(s))_{3\times 3}$

We can use the usual method to find $(sI - A)^{-1}$ and is given by

$$\frac{1}{|D(s)|}\begin{pmatrix} (s+\lambda+\mu_1)(s+\lambda+\mu_2) & \mu_1(s+\lambda+\mu_2) & \mu_2(s+\lambda+\mu_1) \\ \lambda(s+\lambda+\mu_2) & (s+\lambda)(s+\lambda+\mu_2) & \lambda\mu_2 \\ 0 & 0 & (s+\lambda)(s+\lambda+\mu_1) - \lambda\mu_1 \end{pmatrix} \tag{2.16}$$

where $|D(s)| = s^3 + (3\lambda + \mu)s^2 + (3\lambda^2 + \lambda(\mu + \mu_2) + \mu_1\mu_2)s + \lambda^2(\lambda + \mu_2)$

The characteristic roots of the matrix $\boldsymbol{A}$ are given by

$$|D(s)| = 0 \tag{2.17}$$

Let $s_k$, $k = 1, 2, 3$ the characteristic roots of (2.17). Then,

$$s_1 = -(\lambda + \mu_2),$$

$$s_2, s_3 = \frac{-(2\lambda + \mu_1) \pm \sqrt{4\lambda\mu_1 + \mu_1^2}}{2}.$$

We observe that $a_{kj}^*(s)$ are rational algebraic functions in $s$. The cofactor of the $(i,j)^{th}$ element of $(sI - A)$ is a polynomial of degree $2 - |i - j|$. Since the characteristic roots of $A$ are all real and distinct, the inverse transform $a_{ij}(t)$ of $a_{ij}^*(s)$ can be obtained by partial fraction decomposition and are given below.

$$a_{11}(t) = \frac{\mu_1 + \sqrt{4\lambda\mu_1 + \mu_1^2}}{2\sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_2 t} - \frac{\mu_1 - \sqrt{4\lambda\mu_1 + \mu_1^2}}{2\sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_3 t},$$

$$a_{12}(t) = \frac{\mu_1}{\sqrt{4\lambda\mu_1 + \mu_1^2}} (e^{s_2 t} - e^{s_3 t}),$$

$$a_{13}(t) = \frac{\mu_2(\mu_1 - \mu_2)}{\mu_2(\mu_2 - \mu_1) - \lambda\mu_1} e^{s_1 t} - \frac{\mu_2}{\sqrt{4\lambda\mu_1 + \mu_1^2}} \times \left( \frac{\mu_1 + \sqrt{4\lambda\mu_1 + \mu_1^2}}{\mu_1 - 2\mu_2 - \sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_2 t} - \frac{\mu_1 - \sqrt{4\lambda\mu_1 + \mu_1^2}}{\mu_1 - 2\mu_2 + \sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_3 t} \right),$$

$$a_{21}(t) = \frac{\lambda}{\sqrt{4\lambda\mu_1 + \mu_1^2}} (e^{s_2 t} - e^{s_3 t}),$$

$$a_{22}(t) = \frac{\sqrt{4\lambda\mu_1 + \mu_1^2} - \mu_1}{2\sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_2 t} - \frac{\sqrt{4\lambda\mu_1 + \mu_1^2} + \mu_1}{2\sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_3 t},$$

$$a_{13}(t) = \frac{\lambda\mu_2}{\mu_2(\mu_2 - \mu_1) - \lambda\mu_1} e^{s_1 t} - \frac{2\lambda\mu_2}{\sqrt{4\lambda\mu_1 + \mu_1^2}} \times \left( \frac{1}{\mu_1 - 2\mu_2 - \sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_2 t} - \frac{1}{\mu_1 - 2\mu_2 + \sqrt{4\lambda\mu_1 + \mu_1^2}} e^{s_3 t} \right),$$

$$a_{33}(t) = e^{s_1 t},$$

and $a_{31}(t) = a_{32}(t) = 0$.

Now using (2.16), we get

$$s\mathbf{e}^T(sI - A)^{-1}\mathbf{P}(0) = s\sum_{j=1}^{3} a_{j1}^*(s) \tag{2.18}$$

and

$$s\mathbf{e}^T[sI - A]^{-1}[\mu_2\mathbf{e}_1 + \mu_1\mathbf{e}_2] = s[\mu_2\sum_{j=1}^{3} a_{j2}^*(s) + \mu_1\sum_{j=1}^{3} a_{j3}^*(s)] \tag{2.19}$$

Substituting (2.18) and (2.19) in (2.15), we obtain

$$P_2^*(s) = \frac{1 - b_1^*(s)}{s + \lambda_p - \frac{1}{2}(q - \sqrt{q^2 - r^2}) + b_2^*(s)} \tag{2.20}$$

where

$$b_1^*(s) = s\left[a_{11}^*(s) + a_{21}^*(s)\right],$$
$$b_2^*(s) = s\left[\mu_2\left(a_{12}^*(s) + a_{22}^*(s)\right) + \mu_1\left(a_{13}^*(s) + a_{23}^*(s) + a_{33}^*(s)\right)\right].$$

Using Eq. (2.16) in (2.14), we have

$$P_{0,0}^*(s) = \frac{1}{|D(s)|}[(s + \lambda + \mu_1)(s + \lambda + \mu_2) + \mu_1\mu_2(s + \lambda + \mu_2)P_2^*(s) + \mu_1\mu_2(s + \lambda + \mu_1)P_2^*(s)] \tag{2.21}$$

$$P_{1,0}^*(s) = \frac{1}{|D(s)|}\left[\lambda(s + \lambda + \mu_2) + \mu_2(s + \lambda)(s + \lambda + \mu_2)P_2^*(s) + \mu_1\mu_2\lambda P_2^*(s)\right] \tag{2.22}$$

$$P_{0,1}^*(s) = \frac{1}{|D(s)|}\left[(\mu_1(s + \lambda)(s + \lambda + \mu_1) - \lambda\mu_1)P_2^*(s)\right] \tag{2.23}$$

After considerable simplification, (2.20) reduces to

$$P_2^*(s) = \frac{2}{q + \sqrt{q^2 - r^2}}\{1 - b_1^*(s)\}\left[1 - \left(\frac{\mu - \alpha}{\lambda_p}\right)^{1/2}\frac{q - \sqrt{q^2 - r^2}}{r}\left(1 - \frac{b_2^*(s)}{\mu - \alpha}\right)\right]^{-1}$$
$$= \frac{2}{r}\left(\frac{q - \sqrt{q^2 - r^2}}{r}\right)\{1 - b_1^*(s)\}\left[1 - \left(\frac{\mu - \alpha}{\lambda_p}\right)^{1/2}\frac{q - \sqrt{q^2 - r^2}}{r}\left(1 - \frac{b_2^*(s)}{\mu - \alpha}\right)\right]^{-1}$$

On inversion, we get an explicit expression for $P_2(t)$ as

$$P_2(t) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\sqrt{\frac{\mu - \alpha}{\lambda_p}}\right)^m\left(\frac{1}{\mu - \alpha}\right)^k \times \int_0^t b_2^{*k}(u - v)[exp(-(\lambda_p + (\mu - \alpha))u)\{I_m(ru) - I_{m+2}(ru)\} - \int_0^u b_1(u - v)\exp(-(\lambda_p + (\mu - \alpha))v)\{I_m(rv) - I_{m+2}(rv)\}dv]du \tag{2.24}$$

where $b_2^{*k}$ is the k-fold convolution of $b_2(t)$ with itself. We note $b_2^{*0}(t) = \delta(t)$.

Using (2.21)–(2.23) and inverting, we obtain

$$P_{00}(t) = a_{11}(t) + \int_0^t (\mu_2 a_{12}(u) + \mu_1 a_{13}(u))P_2(t - u)du, \tag{2.25}$$

$$P_{10}(t) = a_{21}(t) + \int_0^t (\mu_2 a_{22}(u) + \mu_1 a_{23}(u))P_2(t - u)du, \tag{2.26}$$

$$P_{01}(t) = a_{31}(t) + \int_0^t (\mu_2 a_{32}(u) + \mu_1 a_{33}(u))P_2(t - u)du, \tag{2.27}$$

Thus, (2.12),(2.24),(2.25),(2.26) and (2.27) completely determine all the system size probabilities.

**Remark.** It is observed that our results are coincident with those of Kumar and Madheswari [9] when $\gamma = 0$ for $\mu_1 = \mu_2, \beta = 1$ and $\alpha = 0$.

### 3. Steady-state probabilities

In this section, we shall investigate the behavior of the steady-state probabilities of the M/M/2 queueing system with heterogeneous servers, balking and reneging. It is worthy to mention that the system is always stable for $\alpha > 0$.

**Theorem 1.** *The steady-state distribution of the M/M/2 queue with heterogeneous service rate, balking and reneging is obtained as follows:*

(i) *For $\lambda \neq \mu$, then*

$$P_2 = \frac{1}{\frac{1}{2}(\lambda_p - \mu + \alpha + \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2})} \tag{3.1}$$

$$P_{n+2} = \left(\frac{\beta}{r}\right)^n[\lambda_p + \mu - \alpha + \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2}]^n P_2, n = 1, 2, 3, \ldots \tag{3.2}$$

$$P_{0,0} = \frac{\mu_1\mu_2(2\lambda + \mu)}{\lambda^2(\lambda + \mu_2)}P_2 \tag{3.3}$$

$$P_{1,0} = \frac{\mu_2(\lambda + \mu)}{\lambda(\lambda + \mu_2)}P_2 \tag{3.4}$$

*and*

$$P_{0,1} = \frac{\mu_1}{(\lambda + \mu_2)}P_2 \tag{3.5}$$

(ii) *For* $\lambda = \mu$, *then*

$$P_2 = \frac{1}{\frac{1}{2}(\lambda_p - \mu + \alpha + \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2})} \tag{3.6}$$

$$P_{n+2} = \left(\frac{\beta}{r}\right)^n [\lambda_p + \mu - \alpha + \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2}]^n P_2,$$
$$n = 1, 2, 3, \ldots \tag{3.7}$$

$$P_{0,0} = \frac{3\mu_1\mu_2}{\mu(\mu + \mu_2)} P_2 \tag{3.8}$$

$$P_{1,0} = \frac{2\mu_2}{\mu + \mu_2} P_2 \tag{3.9}$$

*and*

$$P_{0,1} = \frac{\mu_1}{(\mu + \mu_2)} P_2 \tag{3.10}$$

**Proof.** For $\lambda \neq \mu$, from (2.20), we have

$$P_2^*(s) = \frac{1 - b_1^*(s)}{s + \lambda_p - \frac{1}{2}(q - \sqrt{q^2 - r^2}) + b_2^*(s)}$$

Multiplying the above equation by $s$ on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} s P_2^*(s) = \frac{1}{\frac{1}{2}(\lambda_p - \mu + \alpha + \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2})}$$

The result (3.1) follows directly from (3.11) by Tauberian theorem. Taking Laplace transform of (2.12), we have

$$P_{n+2}^*(s) = \left(\frac{\beta}{r}\right)^n [q + \sqrt{q^2 - r^2}]^n P_2^*(s), \quad n = 1, 2, 3, \ldots \tag{3.11}$$

As before, multiplying (3.12) by $s$ on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} s P_{n+2}^*(s) = \lim_{s\to 0} \left(\frac{\beta}{r}\right)^n [q + \sqrt{q^2 - r^2}]^n s P_2^*(s), n$$
$$= 1, 2, 3, \ldots \tag{3.12}$$

Which yields (3.2), by applying Tauberian theorem again.

Similarly, the results (3.3)–(3.5) can be obtained from (2.21)–(2.23).

For $\lambda = \mu$,the results (3.6)–(3.10) can be obtained directly by putting $\lambda = \mu$ in (3.1)–(3.5). □

## 4. Performance measures

Let$N(t)$ be the number of customers in the system at time $t$. The average number of customers in the system at time $t$ is given by

$$E(N(t)) = P_{1,0}(t) + P_{0,1}(t) + \sum_{n=0}^{\infty} (n+2) P_{n+2}(t).$$

Using (2.12), (2.19) and (2.20), the above relation can be written as

$$E(N(t)) = a_{21}(t) + \int_0^t [\mu_2 a_{22}(t-u) + \mu_1(a_{23}(t-u)$$
$$+ a_{33}(t-u))] P_2(u) du + 2P_2(t)$$
$$+ \sum_{n=1}^{\infty} (n+1)\beta^{n-1} \int_0^t \exp(-(\lambda_p + (\mu - \alpha))(t-u))$$
$$\times [\lambda_p \{I_{n-1}(r(t-u)) - I_{n+1}(r(t-u))\} P_2(u)] du \tag{4.1}$$

where $P_2(t)$ is given in (2.24).

If $\lambda \neq \mu$, the mean number of customers in the system under steady state is computed as

$$E(N) = \frac{2\mu\{4\mu - (\lambda_p + \mu - \alpha - \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2})\}}{\{2\mu - (\lambda_p + \mu - \alpha - \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2})\}^2} P_2$$
$$+ \frac{\lambda(\mu + \mu_1\mu_2) + \mu_2^2}{\lambda(\lambda + \mu_2)} P_2 \tag{4.2}$$

and for $\lambda = \mu$,

$$E(N) = \frac{2\mu\{4\mu - (\lambda_p + \mu - \alpha - \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2})\}}{\{2\mu - (\lambda_p + \mu - \alpha - \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2})\}^2} P_2$$
$$+ \frac{\mu(\mu + \mu_1\mu_2) + \mu_2^2}{\mu(\mu + \mu_2)} P_2 \tag{4.3}$$

where $P_2$ is given in (3.1) for $\lambda \neq \mu$ and in (3.6) for $\lambda = \mu$.

The probability that an arriving customer is required to join the queue at time $t$ is given by

$$P(N(t) \geqslant 2) = \sum_{n=0}^{\infty} P_{n+2}(t)$$
$$= P_2(t) + \sum_{n=1}^{\infty} \beta^{n-1} \int_0^t \exp(-(\lambda_p + (\mu - \alpha))(t-u))$$
$$\times [\lambda_p \{I_{n-1}(r(t-u)) - I_{n+1}(r(t-u))\} P_2(u)] du. \tag{4.4}$$

Similarly, the steady-state probability that an arriving customer joins the queue is

$$P(N \geqslant 2) = \sum_{n=0}^{\infty} P_{n+2}$$
$$= \frac{2\mu P_2}{\lambda_p + \mu - \alpha - \sqrt{(\lambda_p + \mu - \alpha)^2 - r^2}} \tag{4.5}$$

Let the random variable$M(t)$ denote the number of busy servers at time $t$. The probability that the system has $n$ busy servers is given as,

$$P(M(t) = n) = \begin{cases} P(N(t) = 1) = P_{1,0}(t) + P_{0,1}(t), n = 1 \\ P(N(t) > 1) = \sum_{n=0}^{\infty} P_{n+2}(t), n = 2 \end{cases} \tag{4.6}$$

and the corresponding steady-state probability is obtained for $\lambda \neq \mu$ as

$$P(M=n)=\begin{cases} P(N(t)=1)=\left(\frac{\mu}{\lambda}\right)P_2, & n=1 \\ \frac{2\mu P_2}{\lambda_p+\mu-\alpha-\sqrt{(\lambda_p+\mu-\alpha)^2-r^2}}, & n=2 \end{cases} \quad (4.7)$$

Similarly, the above probability can be obtained directly, for $\lambda = \mu$ by substituting $\lambda = \mu$ in (4.7).

Furthermore, the mean number of busy servers at time at time $t$ is given by

$$E(M(t)) = P_{1,0}(t) + P_{0,1}(t) + 2\sum_{n=0}^{\infty} P_{n+2}(t)$$

This can be simplified as

$$E(M(t)) = 2[1 - P_{0,0}(t)] - [P_{1,0}(t) + P_{0,1}(t)] \quad (4.8)$$

For $\lambda \neq \mu$, the corresponding steady-state result is given as

$$E(M) = \left\{\frac{2[\lambda^2(\lambda+\mu_2) - \mu_1\mu_2(2\lambda+\mu)] + \lambda\mu(\lambda+\mu_2)}{\lambda^2(\lambda+\mu_2)}\right\}P_2 \quad (4.9)$$

and for $\lambda = \mu$,

$$E(M) = \left\{\frac{2[\mu(\mu+\mu_2) - 3\mu_1\mu_2] + \mu(\mu+\mu_2)}{\mu(\lambda+\mu_2)}\right\}P_2 \quad (4.10)$$


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