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ORIGINAL ARTICLE

Some fixed point results without monotone property in partially ordered metric-like spaces

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Abstract The purpose of this paper is to obtain the fixed point results for F-type contractions which satisfies a weaker condition than the monotonicity of self-mapping of a partially ordered metric-like space. A fixed point result for F-expansive mapping is also proved. Therefore, several well known results are generalized. Some examples are included which illustrate the results.

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1. Introduction and preliminaries

Ran and Reurings [1] and Nieto and Lopez [2,3] obtained the existence of fixed points of a self-mapping of a metric space equipped with a partial order. The fixed point results in spaces equipped with a partial order can be applied in proving existence and uniqueness of solutions for matrix equations as well as for boundary value problems of ordinary differential equations, integral equations, fuzzy equations, of problems in L-spaces, etc. (see [1–11]). The results of Ran and Reurings [1] and Nieto and Lopez [2,3] were generalized by several authors (see, e.g., [4,5,8,12–17]).

In all these papers, the condition of monotonicity with respect to the partial order defined on space is required. Following is a typical result among these.

Theorem 1 ([1,2]). *Let $(X,\sqsubseteq)$ be a partially ordered set which is directed (upward or downward) and let d be a metric on X such that (X,d) is a complete metric space. Let $f:X\to X$ be a mapping such that the following conditions hold:*

(i) *f is monotone (nondecreasing or nonincreasing) on X with respect to "$\sqsubseteq$";*
(ii) *there exists $x_0\in X$ such that $x_0\sqsubseteq fx_0$ or $fx_0\sqsubseteq x_0$;*
(iii) *there exists $k\in(0,1)$ such that $d(fx,fy)\leqslant kd(x,y)$ for all $x,y\in X$ with $y\sqsubseteq x$;*
(iv) (a) *f is continuous, or*

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(b) *if a nondecreasing sequence* $\{x_n\}$ *converges to* $x \in X$, *then* $x_n \sqsubseteq x$ *for all* n.

Then, f has a fixed point $x^* \in X$.

Recently, the fixed point results on partially ordered sets are investigated via a weaker property than the monotonicity of f (see [8,13,18,19]). We state following facts from these papers.

Let $(X, \sqsubseteq)$ be a partially ordered set and $x, y \in X$. If x, y are comparable (i.e., $x \sqsubseteq y$ or $y \sqsubseteq x$ holds), then we will write $x \asymp y$.

Lemma 2 [18]. *Consider the following properties for a self-map f on a partially ordered set* $(X, \sqsubseteq)$:

1. *f is monotone (nondecreasing or nonincreasing), i.e.,* $x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$ *for all* $x, y \in X$ *or* $y \sqsubseteq x \Rightarrow fx \sqsubseteq fy$ *for all* $x, y \in X$;
2. $x \asymp y \Rightarrow fx \asymp fy$ for $x, y \in X$;
3. $x \asymp fx \Rightarrow fx \asymp ffx$ for $x \in X$.

Then $1 \Rightarrow 2 \Rightarrow 3$. *The reverse implications do not hold in general.*

On the other hand, Matthews [20] introduced the notion of partial metric space as a part of the study of denotational semantics of data flow network. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of space may not be zero. Further, Matthews showed that the Banach contraction principle is valid in partial metric space and can be applied in program verification.

Very recently, Amini-Harandi [21] generalized the partial metric spaces by introducing the metric-like spaces and proved some fixed point theorems in such spaces. In [22], Wardowski introduced a new concept of an F-contraction and proved a fixed point theorem which generalizes Banach contraction principle in a different way than in the known results from the literature in complete metric spaces. In this paper, we consider a more generalized type of F-contractions and prove some common fixed point theorems for such type of mappings in metric-like spaces. We generalize the result of Wardowski [22], Matthews [20], Ran and Reurings [1], Nieto and Lopez [2], and the recent result of Đorić et al. [18] by proving the fixed point results for $F-g-$weak contractions in metric-like spaces equipped with a partial order. Results of this paper are new not only in the setting of metric-like spaces but also in the setting of metric and partial metric spaces.

First, we recall some definitions and facts about partial metric and metric-like spaces.

Definition 1 [20]. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that, for all $x, y, z \in X$:

(p1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(p2) $p(x,x) \leqslant p(x,y)$;
(p3) $p(x,y) = p(y,x)$;
(p4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$. A sequence $\{x_n\}$ in (X,p) is called p-Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n,x_m)$ and is finite. (X,p) is said to be complete if every p-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 2 [21]. A metric-like on a nonempty set X is a function $\sigma : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(σ1) $\sigma(x,y) = 0$ implies $x = y$;
(σ2) $\sigma(x,y) = \sigma(y,x)$;
(σ3) $\sigma(x,y) \leqslant \sigma(x,z) + \sigma(z,y)$.

A metric-like space is a pair (X,σ) such that X is a nonempty set and σ is a metric-like on X. Note that, a metric-like satisfies all the conditions of metric except that $\sigma(x,x)$ may be positive for $x \in X$. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x,\epsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \epsilon\}, \quad \text{for all } x \in X \text{ and } \epsilon > 0.$$

A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x)$. Sequence $\{x_n\}$ is said to be σ-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite. The metric-like space (X,σ) is called σ-complete if for each σ-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{m,n\to\infty} \sigma(x_n,x_m).$$

Note that every partial metric space is a metric-like space, but the converse may not be true.

Example 1 [21]. Let $X = \{0,1\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2, & \text{if } x = y = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a metric-like space, but it is not a partial metric space, as $\sigma(0,0) \nleq \sigma(0,1)$.

Example 2. Let $X = \mathbb{R}, k \geqslant 0$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2k, & \text{if } x = y = 0; \\ k, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a metric-like space, but for $k > 0$, it is not a partial metric space, as $\sigma(0,0) \nless \sigma(0,1)$.

Example 3. Let $X = \mathbb{R}^+$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2x, & \text{if } x = y; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a metric-like space, it is not a partial metric space, as $\sigma(1,1) = 2 \nleq \sigma(0,1) = 1$.

Definition 3. If a nonempty set X is equipped with a partial order "$\sqsubseteq$" such that (X,σ) is a metric-like space, then the $(X,\sigma,\sqsubseteq)$ is called a partially ordered metric-like space. A subset

$\mathcal{A}$ of X is called well ordered if all the elements of $\mathcal{A}$ are comparable, i.e., for all $x, y \in \mathcal{A}$, we have $x \asymp y$. $\mathcal{A}$ is called g-well ordered if all the elements of $\mathcal{A}$ are g-comparable, i.e., for all $x, y \in \mathcal{A}$, we have $gx \asymp gy$.

In the trivial case, i.e., for $g = I_X$ (the identity mapping of X), the g-well orderedness reduces into well orderedness. But, for nontrivial cases, i.e., when $g \neq I_X$ the concepts of g-well orderedness and well orderedness are independent.

Example 4. Let $X = \{0,1,2,3,4\}$, "$\sqsubseteq$" a partial order relation on X defined by $\sqsubseteq = \{(0,0),(1,1),(2,2),(3,3),(4,4),(1,2),(2,3),(1,3),(1,4)\}$. Let $\mathcal{A} = \{0,1,3\}, \mathcal{B} = \{1,4\}$ and $g:X \to X$ be defined by $g0 = 1$, $g1 = 2$, $g2 = 3$, $g3 = 3$, $g4 = 0$. Then it is clear that $\mathcal{A}$ is not well ordered but it is g-well ordered, while $\mathcal{B}$ is not g-well ordered but it is well ordered.

The proof of following lemma is similar as for the metric case, and for the sake of completeness, we give the proof.

Lemma 3. *Let (X,σ) be a metric-like space and $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ converges to some $x \in X$ with $\sigma(x,x) = 0$ then $\lim_{n\to\infty}\sigma(x_n,y) = \sigma(x,y)$ for all $y \in X$.*

Proof. Let $\lim_{n\to\infty}\sigma(x_n,x) = \sigma(x,x) = 0$, i.e., for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$\sigma(x_n,x) < \varepsilon$ for all $n > n_0$.

Now by (σ_3) we have $\sigma(x_n,y) \leqslant \sigma(x_n,x) + \sigma(x,y)$ and $\sigma(x,y) \leqslant \sigma(x,x_n) + \sigma(x_n,y)$, i.e.,

$\sigma(x_n,y) - \sigma(x,y) \leqslant \sigma(x_n,x)$ and $\sigma(x,y) - \sigma(x_n,y) \leqslant \sigma(x,x_n)$.

Therefore $|\sigma(x_n,y) - \sigma(x,y)| \leqslant \sigma(x_n,x)$ and so $|\sigma(x_n,y) - \sigma(x,y)| < \varepsilon$ for all $n > n_0$ and the result follows. □

Analogous to [22], we have following definitions.

Definition 4. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty}\alpha_n = 0$ if and only if $\lim_{n\to\infty}F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha) = 0$.

For examples of functions F, we refer to [22]. We denote the set of all functions satisfying properties (F1)–(F3), by $\mathcal{F}$.

Wardowski in [22], defined the F-contraction as follows.

Definition 5. Let (X,ρ) be a metric space. A mapping $T:X \to X$ is said to be an F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$, $\rho(Tx,Ty) > 0$ we have

$$\tau + F(\rho(Tx,Ty)) \leqslant F(\rho(x,y)).$$

Definition 6. Let $(X,\sigma,\sqsubseteq)$ be a partially ordered metric-like space, f, $g:X \to X$ be mappings. Suppose $\tau > 0$ and $F \in \mathcal{F}$ are such that:

$$\sigma(fx,fy) > 0 \Rightarrow \tau + F(\sigma(fx,fy)) \leqslant F(\max\{\sigma(gx,gy),\sigma(gx,fx),\sigma(gy,fy)\}) \quad (1)$$

for all $x, y \in X$ with $gx \asymp gy$. Then mapping f is called an ordered F–g-weak contraction. For $g = I_X$ the F–g-weak contraction reduces in to an ordered F-weak contraction. It is clear that the concept of an ordered F–g-weak contraction and ordered F-weak contraction are more general than the F-contraction.

Definition 7 [23]. Let f and g be self-mappings of a nonempty set X and $C(f,g) = \{x \in X: fx = gx\}$. The pair (f,g) is called weakly compatible if $fgx = gfx$ for all $x \in C(f,g)$. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

The following lemma will be useful in proving our main result.

Lemma 4. *Let $(X,\sigma,\sqsubseteq)$ be a partially ordered metric-like space, $f,g:X \to X$ be mappings such that f is an ordered F–g-weak contraction. If $v \in X$ be a point of coincidence of f and g, then $\sigma(v,v) = 0$.*

Proof. Let $v \in X$ be a point of coincidence of f and g, then there exists $u \in X$ such that $fu = gu = v$. If $\sigma(v,v) > 0$, then from (1) it follows that

$$\begin{aligned}\tau + F(\sigma(v,v)) &= \tau + F(\sigma(fu,fu)) \\ &\leqslant F(\max\{\sigma(gu,gu),\sigma(gu,fu),\sigma(gu,fu)\}) \\ &\leqslant F(\max\{\sigma(v,v),\sigma(v,v),\sigma(v,v)\}) \\ &= F(\sigma(v,v)).\end{aligned}$$

This is a contradiction. Therefore, we must have $\sigma(v,v) = 0$. □

2. Main results

The following theorem is the fixed point result for an ordered F–g-weak contraction in partially ordered metric-like spaces.

Theorem 5. *Let $(X,\sigma,\sqsubseteq)$ be a partially ordered metric-like space and let f, $g:X \to X$ be mappings such that $f(X) \subset g(X)$ and $g(X)$ is σ-complete. Suppose that the following hold:*

(i) if x, $y \in X$ such that $gx \asymp fx = gy$, then $fx \asymp fy$;
(ii) *there exists $x_0 \in X$ such that $gx_0 \asymp fx_0$;*
(iii) *there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $gx \asymp gy$, we have*

$$\sigma(fx,fy) > 0 \Rightarrow \tau + F(\sigma(fx,fy)) \leqslant F(\max\{\sigma(gx,gy),\sigma(gx,fx),\sigma(gy,fy)\}) \quad (2)$$

(iv) *if $\{x_n\}$ is sequence in (X,σ) converging to $x \in X$ and $\{x_n : n \in \mathbb{N}\}$ is well ordered, then $x_n \asymp x$ for sufficiently large n.*

Suppose F is continuous, then the pair (f,g) have a point of coincidence $v \in X$ and $\sigma(v,v) = 0$. Furthermore, if the set of coincidence points of the pair (f,g) is g-well ordered then the pair (f,g) have a unique point of coincidence. If in addition, the pair (f,g) is weakly compatible, then there exists a unique common fixed point of the pair (f,g).

Proof. Starting with given $x_0 \in X$ and using the fact that $f(X) \subset g(X)$ we define a sequence $\{y_n\} = \{gx_n\}$ by

$y_n = gx_n = fx_{n-1}$ for all $n \in \mathbb{N}$.

As, $gx_0 \asymp fx_0 = gx_1$ and (i) holds, we have $fx_0 \asymp fx_1$, i.e., $gx_1 \asymp fx_1 = gx_2$ and so $fx_1 \asymp fx_2$. On repeating this process, we obtain $gx_n \asymp gx_{n+1}$, i.e., $y_n \asymp y_{n+1}$ for all $n \in \mathbb{N}$.

We shall show that the pair (f,g) has a point of coincidence. If $\sigma(y_n,y_{n+1}) = 0$ for any $n \in \mathbb{N}$, then $y_n = y_{n+1}$, i.e., $gx_n = fx_{n-1} = gx_{n+1} = fx_n$ and so x_n is a coincidence point and $gx_n = fx_n$ is point of coincidence of pair (f,g).

Suppose $\sigma(y_n,y_{n+1}) > 0$ for all $n \in \mathbb{N}$. As, $gx_n \asymp gx_{n+1}$ for all $n \in \mathbb{N}$, we obtain from (2) that

$$\begin{aligned}\tau + F(\sigma(y_n,y_{n+1})) &= \tau + F(\sigma(fx_{n-1},fx_n))\\ &\leqslant F(\max\{\sigma(gx_{n-1},gx_n),\sigma(gx_{n-1},fx_{n-1}),\sigma(gx_n,fx_n)\})\\ &= F(\max\{\sigma(y_{n-1},y_n),\sigma(y_{n-1},y_n),\sigma(y_n,y_{n+1})\})\\ &= F(\max\{\sigma(y_{n-1},y_n),\sigma(y_n,y_{n+1})\}). \qquad (3)\end{aligned}$$

If there exists $n \in \mathbb{N}$ such that $\max\{\sigma(y_{n-1},y_n),\sigma(y_n,y_{n+1})\} = \sigma(y_n, y_{n+1})$ then it follows from (3) that

$\tau + F(\sigma(y_n,y_{n+1})) < F(\sigma(y_n,y_{n+1}))$ (as $\tau > 0$).

As, $F \in \mathcal{F}$, by (F1) we have $\sigma(y_n,y_{n+1}) < \sigma(y_n,y_{n+1})$. This is a contradiction. Therefore we must have $\max\{\sigma(y_{n-1},y_n), \sigma(y_n,y_{n+1})\} = \sigma(y_{n-1},y_n)$ for all $n \in \mathbb{N}$, and then, we obtain from (3) that

$$F(\sigma(y_n,y_{n+1})) \leqslant F(\sigma(y_{n-1},y_n)) - \tau. \qquad (4)$$

On repeating this process, we obtain

$$\begin{aligned}F(\sigma(y_n,y_{n+1})) &\leqslant F(\sigma(y_{n-1},y_n)) - \tau \leqslant F(\sigma(y_{n-2},y_{n-1})) - 2\tau\\ &\leqslant \cdots \leqslant F(\sigma(y_0,y_1)) - n\tau. \qquad (5)\end{aligned}$$

Letting $n \to \infty$ in above inequality, we obtain $\lim_{n\to\infty}F(\sigma(y_n,y_{n+1})) = -\infty$ and as $F \in \mathcal{F}$, by (F2) we have

$$\lim_{n\to\infty}\sigma(y_n,y_{n+1}) = 0. \qquad (6)$$

Again, by (F3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty}[\sigma(y_n,y_{n+1})]^k F(\sigma(y_n,y_{n+1})) = 0. \qquad (7)$$

From (5), we obtain

$$[\sigma(y_n,y_{n+1})]^k[F(\sigma(y_n,y_{n+1})) - F(\sigma(y_0,y_1))] \leqslant -n[\sigma(y_n,y_{n+1})]^k\tau \leqslant 0.$$

As, $\tau > 0$, using (6) and (7) in above inequality we have

$$\lim_{n\to\infty} n[\sigma(y_n,y_{n+1})]^k = 0,$$

therefore there exists $n_0 \in \mathbb{N}$ such that $n[\sigma(y_n,y_{n+1})]^k < 1$ for all $n > n_0$, i.e.,

$$\sigma(y_n,y_{n+1}) < \frac{1}{n^{1/k}} \text{ for all } n > n_0. \qquad (8)$$

For $m,n \in \mathbb{N}$ with $m > n$, it follows from (8) that

$$\begin{aligned}\sigma(y_n,y_m) &\leqslant \sigma(y_n,y_{n+1}) + \sigma(y_{n+1},y_{n+2}) + \cdots + \sigma(y_{m-1},y_m)\\ &\leqslant \frac{1}{n^{1/k}} + \frac{1}{(n+1)^{1/k}} + \cdots + \frac{1}{(m-1)^{1/k}} \leqslant \sum_{i=n}^{\infty}\frac{1}{i^{1/k}}.\end{aligned}$$

As, $k \in (0,1)$ it follows that the series $\sum_{i=n}^{\infty}\frac{1}{i^{1/k}}$ converges and so by above inequality we have

$$\lim_{n,m\to\infty}\sigma(y_n,y_m) = 0.$$

Therefore the sequence $\{y_n\} = \{gx_n\}$ is a σ-Cauchy sequence in $g(X)$. By σ-completeness of $g(X)$, there exist $u,v \in X$ such that $v = gu$ and

$$\lim_{n\to\infty}\sigma(y_n, v) = \lim_{n\to\infty}\sigma(gx_n, gu) = \lim_{n,m\to\infty}\sigma(y_n,y_m) = \sigma(v,v) = 0. \qquad (9)$$

We shall show that v is point of coincidence of f and g. If $\sigma(fu,v) = 0$ then $fu = v = gu$ and v is a point of coincidence of f and g. Suppose $\sigma(fu,v) > 0$ then without loss of generality we can assume that there exists $n_1 \in \mathbb{N}$ such that $\sigma(fx_n,fu) > 0$ for all $n > n_1$. From the assumption (iv), there exists $n_2 \in \mathbb{N}$ such that $y_n \asymp v$, i.e. $gx_n \asymp gu$, for all $n > n_2$, therefore using (2), we obtain

$$\begin{aligned}\tau + F(\sigma(y_{n+1},fu)) &= \tau + F(\sigma(fx_n,fu))\\ &\leqslant F(\max\{\sigma(gx_n,gu),\sigma(gx_n,fx_n),\sigma(gu,fu)\})\\ &= F(\max\{\sigma(y_n,gu),\sigma(y_n,y_{n+1}),\sigma(v,fu)\}),\end{aligned}$$

for all $n > n_2$. As, $\sigma(fu,v) > 0$, in view of (9), there exists $n_3 \in \mathbb{N}$ such that

$\max\{\sigma(y_n,gu),\sigma(y_n,y_{n+1}),\sigma(v,fu)\} = \sigma(v,fu)$ for all $n > n_3$.

Therefore, for all $n > \max\{n_1,n_2,n_3\}$ we have

$\tau + F(\sigma(y_{n+1},fu)) \leqslant F(\sigma(v,fu))$.

Using continuity of F, (9) and Lemma 3, we obtain from above inequality that

$\tau + F(\sigma(v,fu)) \leqslant F(\sigma(v,fu))$.

This is a contradiction. Therefore we must have $\sigma(fu,v) = 0$, i.e., $fu = gu = v$. Thus, v is a point of coincidence of f and g.

Suppose, the set of coincidence points of f and g, i.e., $C(f,g)$ is g-well ordered and v' is another point of coincidence of f and g, then there exists $u' \in X$ such that $fu' = gu' = v'$. By Lemma 4 we have $\sigma(v',v') = 0$, also as $C(f,g)$ is g-well ordered we have $gu \asymp gu'$. So, if $\sigma(v,v') > 0$, it follows from (2) that

$$\begin{aligned}\tau + F(\sigma(v,v')) &= \tau + F(\sigma(fu,fu'))\\ &\leqslant F(\max\{\sigma(gu,gu'),\sigma(gu,fu),\sigma(gu',fu')\})\\ &\leqslant F(\max\{\sigma(v,v'),\sigma(v,v),\sigma(v',v')\}) = F(\sigma(v,v'))\end{aligned}$$

This is a contradiction. Therefore we must have $\sigma(v,v') = 0$, i.e., $v = v'$. Thus, point of coincidence is unique.

In addition, if the pair (f,g) is weakly compatible, then there exists $w \in X$ such that $fu = fgu = \cdots = w$. So, w is another point of coincidence of pair (f,g) and by uniqueness $w = v$, i.e., $fv = gv = v$. Thus, the pair (f,g) have a unique common fixed point v and $\sigma(v,v) = 0$. □

Taking $g = I_X$ in above inequality, we obtain the following fixed point result for an F-weak contraction in ordered metric-like spaces.

Corollary 6. *Let $(X,\sigma,\sqsubseteq)$ be a partially ordered σ-complete metric-like space and let $f{:}X \to X$ be a mapping such that the following hold:*

(i) *if* $x, y \in X$ *such that* $x \asymp fx$, then $fx \asymp ffx$;
(ii) *there exists* $x_0 \in X$ *such that* $x_0 \asymp fx_0$;
(iii) *there exist* $F \in \mathcal{F}$ and $\tau > 0$ *such that for all* $x, y \in X$ *satisfying* $x \asymp y$, *we have*

$$\sigma(fx,fy) > 0 \Rightarrow \tau + F(\sigma(fx,fy)) \leqslant F(\max\{\sigma(x,y), \sigma(x,fx), \sigma(y,fy)\}); \tag{10}$$

(iv) *if* $\{x_n\}$ *is sequence in* (X,σ) *converging to* $x \in X$ *and* $\{x_n : n \in \mathbb{N}\}$ *is well ordered, then* $x_n \asymp x$ *for sufficiently large* n.

Then f has a fixed point $v \in X$ *and* $\sigma(v,v) = 0$. *Furthermore, the set of fixed points of f is well ordered if and only if f has a unique fixed point.*

Corollary 7. *Let* $(X,\sigma,\sqsubseteq)$ *be a partially ordered metric-like space and let* $f, g: X \to X$ *be mappings such that* $f(X) \subset g(X)$ *and* $g(X)$ *is σ-complete. Suppose that the following hold:*

(i) *if* $x, y \in X$ *such that* $gx \asymp fx = gy$, *then* $fx \asymp fy$;
(ii) *there exists* $x_0 \in X$ *such that* $gx_0 \asymp fx_0$;
(iii) *there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that for all* $x, y \in X$ *satisfying* $gx \asymp gy$, *we have*

$$\sigma(fx,fy) > 0 \Rightarrow \tau + F(\sigma(fx,fy)) \leqslant F(\sigma(gx,gy)); \tag{11}$$

(iv) *if* $\{x_n\}$ *is sequence in* (X,σ) *converging to* $x \in X$ *and* $\{x_n : n \in \mathbb{N}\}$ *is well ordered, then* $x_n \asymp x$ *for sufficiently large* n.

Suppose F is continuous, then the pair (f,g) have a point of coincidence $v \in X$ *and* $\sigma(v,v) = 0$. *Furthermore, if the set of coincidence points of the pair (f,g) is g-well ordered then the pair (f,g) have a unique point of coincidence. If in addition, the pair (f,g) is weakly compatible, then there exists a unique common fixed point of the pair (f,g).*

Following example illustrates the case when usual metric version of Corollary 7 as well as the main result of [18] is not applicable, while our Corollary 7 is applicable.

Example 5. Let $X = [0,2]$ and define $\sigma : X \times X \to \mathbb{R}^+$ by

$$\sigma(x,y) = \begin{cases} 0, & \text{if } x = y = 2; \\ 2x, & \text{if } x = y; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a σ-complete metric-like space. Define a partial order relation "$\sqsubseteq$" on X and $f, g: X \to X$ by

$$\sqsubseteq = \{(x,y) : x,y \in [0,1] \text{with} x \geqslant y\} \cup \{(x,y) : x,y \in (1,2] \text{with} x \leqslant y\},$$

$$fx = \begin{cases} \frac{x}{3}, & \text{if } x \in [0,1); \\ 0, & \text{if } x = 1; \\ 2, & \text{if } x \in (1,2]. \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{2x}{3}, & \text{if } x \in [0,1]; \\ 2, & \text{if } x \in (1,2]. \end{cases}$$

Then, the set $C(f,g)$ is not g-well ordered and all other conditions of Corollary 7 are satisfied with $\tau \in (0, \log 2]$ and $F(\alpha) = \log \alpha$, and 0, 2 are two points of coincidence as well as common fixed points of pair (f,g). Note that $C(f,g) = \{0\} \cup (1,2]$ and $(g0,gx)$, $(gx,g0) \notin \sqsubseteq$ for all $x \in (1,2]$, therefore $C(f,g)$ is not g-well ordered and the point of coincidence is not unique.

Note that, if d the usual metric on X then at point $(x,y) = \left(1, \frac{7}{10}\right)$ we have, $d(fx,fy) = d\left(f1, f\frac{7}{10}\right) = \frac{7}{30}$, and $d(gx,gy) = d\left(g1, g\frac{7}{10}\right) = \frac{1}{5}$. Therefore, there is no $\tau > 0$ and $F \in \mathcal{F}$ such that $\tau + F(d(fx,fy)) \leqslant F(d(gx,gy))$ for all $x, y \in X$ with $x \asymp y$. Thus, the metric version of Corollary 7 is not applicable. Also, with same point we conclude that there is no $k \in [0,1)$ such that $d(fx,fy) \leqslant kd(gx,gy)$ for all $x, y \in X$ with $x \asymp y$, so the results of [18] are not applicable.

The following example shows that the class of ordered F-weak contractions in metric-like spaces is more general than that in usual metric spaces, and also it provides an example of an ordered F-weak contraction in metric-like spaces which satisfies all the conditions of our Corollary 6 but does not have the monotone property.

Example 6. Let $X = \{0, 1, 2, 3\}$ and define $\sigma : X \times X \to \mathbb{R}^+$ by

$$\sigma(x,y) = \begin{cases} 2x, & \text{if } x = y; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a σ-complete metric-like space. Define a partial order relation "$\sqsubseteq$" on X and $f, g: X \to X$ by

$$\sqsubseteq = \{(0,0),(1,1),(2,2),(3,3),(0,2),(2,1),(0,1)\},$$

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Now, it is easy to verify that the condition (10) is satisfied for $\tau \in (0, \log\frac{3}{2}]$ and $F(\alpha) = \log \alpha$. All other conditions of Corollary 6 are satisfied and 0 is the unique fixed point of f. On the other hand, if d is the usual metric on X, then at point $(x,y) = (1,2)$ we have $d(f1,f2) = 1$ and $d(x,y) = d(1,2) = 1$, $d(x,fx) = d(1,f1) = 1$, $d(y,fy) = d(2,f2) = 1$, therefore, there exists no $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(fx,fy)) \leqslant F(\max\{d(x,y), d(x,fx), d(y,fy)\}).$$

So, f is not an F-weak contraction in the usual metric space (X,d). Note that f is neither monotonic increasing nor monotonic decreasing with respect to "$\sqsubseteq$".

Definition 8. Let $(X,\sigma,\sqsubseteq)$ be a partially ordered metric-like space, $f: X \to X$ be a mapping, $\tau > 0$ and $F \in \mathcal{F}$ are such that

$$\sigma(x,y) > 0 \Rightarrow F(\sigma(fx,fy)) \geqslant F(\sigma(x,y)) + \tau \tag{12}$$

for all $x, y \in X$ with $x \asymp y$, then f is called an ordered F-expansive mapping.

In the next theorem we prove a fixed point result for F-expansive type mappings in σ-complete metric-like spaces.

Theorem 8. *Let* $(X,\sigma,\sqsubseteq)$ *be a partially ordered metric-like space and let* $f, g: X \to X$ *be mappings such that* $f(X) \supset g(X)$ *and* $g(X)$ *is σ-complete. Suppose that the following hold:*

(I) *if* $x, y \in X$ such that $fx \asymp gx = fy$, then $gx \asymp gy$;
(II) *there exists* $x_0 \in X$ *such that* $fx_0 \asymp gx_0$;
(III) *there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that for all* $x, y \in X$ *satisfying* $gx \asymp gy$, *we have*

$$\sigma(gx, gy) > 0 \Rightarrow F(\sigma(fx, fy)) \geqslant F(\sigma(gx, gy)) + \tau; \tag{13}$$

(IV) *if $\{x_n\}$ is sequence in (X, σ) converging to $x \in X$ and $\{x_n : n \in \mathbb{N}\}$ is well ordered, then $x_n \asymp x$ for sufficiently large n.*

Suppose F is continuous, then the pair (f, g) have a point of coincidence $v \in X$ and $\sigma(v, v) = 0$. Furthermore, if the set of coincidence points of the pair (f, g) is g-well ordered then the pair (f, g) have a unique point of coincidence. If in addition, the pair (f, g) is weakly compatible, then there exists a unique common fixed point of the pair (f, g).

Proof. Starting with given $x_0 \in X$ and using the fact that $f(X) \supset g(X)$ we define a sequence $\{y_n\} = \{fx_n\}$ by

$$y_{n-1} = gx_{n-1} = fx_n \text{ for all } n \in \mathbb{N}.$$

As, $fx_0 \asymp gx_0 = fx_1$ and (I) holds, we have $gx_0 \asymp gx_1$, i.e., $fx_1 \asymp gx_1 = fx_2$ and so $gx_1 \asymp gx_2$. On repeating this process, we obtain $gx_{n-1} \asymp gx_n$, i.e, $y_{n-1} \asymp y_n$ for all $n \in \mathbb{N}$.

We shall show that the pair (f, g) has a point of coincidence. If $\sigma(y_n, y_{n+1}) = 0$ for any n, then $y_n = y_{n+1}$ and so $gx_n = fx_{n+1} = gx_{n+1} = fx_{n+2}$ and so x_{n+1} is a coincidence point and gx_{n+1} is a point of coincidence of the pair (f, g). Suppose, $\sigma(y_n, y_{n+1}) > 0$ for all n, then as, $gx_{n-1} \asymp gx_n$, for all $n \in \mathbb{N}$ it follows from (13) that

$$\begin{aligned}\sigma(y_{n-1}, y_n) = \sigma(fx_n, fx_{n+1}) &\geqslant F(\sigma(gx_n, gx_{n+1})) + \tau \\ &= F(\sigma(y_n, y_{n+1})) + \tau,\end{aligned}$$

i.e.,

$$F(\sigma(y_n, y_{n+1})) \leqslant \sigma(y_{n-1}, y_n) - \tau.$$

Now since $g(X)$ is σ-complete and $f(X) \supset g(X)$, following a similar process as in Theorem 5, we obtain that there exist $u, v \in X$ such that $v = fu$ and

$$\lim_{n\to\infty} \sigma(y_n, v) = \lim_{n\to\infty} \sigma(fx_n, fu) = \lim_{n,m\to\infty} \sigma(y_n, y_m) = \sigma(v, v) = 0. \tag{14}$$

We shall show that v is a point of coincidence of f and g. If $\sigma(gu, v) = 0$ then $fu = v = gu$ and v is a point of coincidence of f and g. Suppose $\sigma(gu, v) > 0$ then without loss of generality we can assume that there exists $n_0 \in \mathbb{N}$ such that $\sigma(y_n, gu) > 0$ for all $n > n_0$. From assumption (IV), there exists $n_1 \in \mathbb{N}$ such that $y_n \asymp v$, i.e. $gx_n \asymp gu$, for all $n > n_1$, and so using (13), we obtain

$$\begin{aligned}F(\sigma(y_{n-1}, fu)) = F(\sigma(fx_n, fu)) &\geqslant F(\sigma(gx_n, gu)) + \tau \\ &= F(\sigma(y_n, gu)) + \tau\end{aligned}$$

i.e.,

$$F(\sigma(y_n, gu)) + \tau \leqslant F(\sigma(y_{n-1}, fu)) \text{ for all } n > \max\{n_0, n_1\}.$$

In view of (14) and Lemma 3, for $\varepsilon = \sigma(gu, v) > 0$ there exists $n_2 \in \mathbb{N}$ such that $\sigma(y_{n-1}, fu) < \varepsilon = \sigma(gu, v)$ for all $n > n_2$. Therefore, for all $n > \max\{n_0, n_1, n_2\}$ we have

$$F(\sigma(y_n, gu)) + \tau < F(\sigma(gu, v)).$$

Using the continuity of F, (14) and Lemma 3, we obtain from above inequality that

$$F(\sigma(v, gu)) + \tau < F(\sigma(gu, v)).$$

This is a contradiction. Therefore we must have $\sigma(v, gu) = 0$, i.e., $gu = fu = v$. Thus, v is a point of coincidence of the pair (f, g).

Suppose, the set of coincidence points of f and g, i.e., $C(f, g)$ is g-well ordered and v' is another point of coincidence of f and g, i.e., there exists $u' \in X$ such that $fu' = gu' = v'$. By a similar process as used in Lemma 4 one can prove easily that $\sigma(v', v') = 0$, also as $C(f, g)$ is g-well ordered we have $gu \asymp gu'$. If $\sigma(v, v') > 0$, it follows from (13) that

$$F(\sigma(v, v')) = F(\sigma(fu, fu')) \geqslant F(\sigma(gu, gu')) + \tau = F(\sigma(v, v')) + \tau,$$

i.e., $F(\sigma(v, v')) + \tau \leqslant F(\sigma(v, v'))$, a contradiction. Therefore, we must have $\sigma(v, v') = 0$, i.e., $v = v'$. Thus point of coincidence of pair (f, g) is unique.

In addition, if the pair (f, g) is weakly compatible, then $fv = fgu = gfu = gv = w$ (say). So, w is another point of coincidence of pair (f, g) and by the uniqueness $w = v$, i.e., $fv = gv = v$. Thus, the pair (f, g) have a unique common fixed point v and $\sigma(v, v) = 0$. □

Following is a simple example which illustrates the above result.

Example 7. Let $X = [0, 1]$ and define $\sigma : X \times X \to \mathbb{R}^+$ by

$$\sigma(x, y) = \begin{cases} 0, & \text{if } x = y = 1; \\ 2x, & \text{if } x = y; \\ \max\{x, y\}, & \text{otherwise.}\end{cases}$$

Then (X, σ) is a σ-complete metric-like space. Define a partial order relation "$\sqsubseteq$" on X and f, g: $X \to X$ by, $\sqsubseteq = \{(x, y) : x, y \in [0, 1) \text{ with } x \geqslant y\} \cup \{(1, 1)\}$,

$$fx = \begin{cases} \frac{x}{3}, & \text{if } x \in [0, 1); \\ 1, & \text{if } x = 1,\end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{x}{7}, & \text{if } x \in [0, 1); \\ 1, & \text{if } x = 1.\end{cases}$$

Then, the set $C(f, g)$ is not g-well ordered and all other conditions of Theorem 8 are satisfied with $\tau \in \left(0, \log\frac{7}{3}\right]$ and $F(\alpha) = \log\alpha$ and 0, 1 are two points of coincidence as well as common fixed points of pair (f, g). Note that $C(f, g) = \{0, 1\}$ and $(g0, g1), (g1, g0) \notin \sqsubseteq$, therefore $C(f, g)$ is not g-well ordered and the point of coincidence is not unique.

For $g = I_X$, we obtain the following fixed point result for F-expansive mappings in metric-like spaces.

Corollary 9. *Let $(X, \sigma, \sqsubseteq)$ be a σ-complete partially ordered metric-like space and let $f : X \to X$ be a surjection. Suppose that the following hold:*

(I) *if x, $y \in X$ such that $fx \asymp fy$, then $x \asymp y$;*
(II) *there exists $x_0 \in X$ such that $fx_0 \asymp x_0$;*
(III) *there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $x \asymp y$, we have*

$$\sigma(x, y) > 0 \Rightarrow F(\sigma(fx, fy)) \geqslant F(\sigma(x, y)) + \tau;$$

(IV) *if $\{x_n\}$ is a sequence in X converging to $x \in X$ and $\{x_n : n \in \mathbb{N}\}$ is well ordered, then $x_n \asymp x$ for sufficiently large n.*

Suppose F is continuous, then the mapping f has a fixed point $v \in X$ and $\sigma(v, v) = 0$. Furthermore, the set of fixed points of the *mapping f is well ordered if and only if the mapping f has a unique fixed point.*

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