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The inverse spectral problem of some singular version of one-dimensional Schrödinger operator with explosive factor in finite interval



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Abstract The inverse spectral problem is investigated for some singular version of one-dimensional Schrödinger operator with explosive factor on finite interval $[0, \pi]$. In the present paper the explosive factor subdivides the problem into two parts, with different characteristic, which causes a lot of analytical difficulties. We define the spectral data of the problem, derive the main integral equation and show that the potential is uniquely recovered for both parts of the problem.

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1. Introduction and formulation of the inverse problem

Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. Inverse problems also play an important role in solving nonlinear evolution equations in mathematical physics. Interest in this subject has been increasing permanently because of the appearance of new important applications, and nowadays the inverse problem theory develops intensively all over

the world. The greatest success in spectral theory in general, and in particular in inverse spectral problems has been achieved for the Sturm-Liouville operator $Ly := y'' + q(x)y$; which also is called the one-dimensional Schrödinger operator. The main results on inverse spectral problems appear in the second half of the XX-th century. We mention here the works by R. Beals, G. Borg, L.D. Faddeev, M.G. Gasymov, I.M. Gelfand, B.M. Levitan, I.G. Khachatryan, M.G. Krein, N. Levinson, Z.L. Leibenson, V.A. Marchenko, L.A. Sakhnovich, E. Trubowitz, V.A. Yurko and others. In recent years there appeared new areas for applications of inverse spectral problems. We mention a remarkable method for solving some nonlinear evolution equations of mathematical physics connected with the use of inverse spectral problems (see Ablowitz M.J. and Zakharov V.E.). Another important class of inverse problems, which often appear in applications, is the inverse problem of recovering differential equations from incomplete spectral information when only a part of the spectral information is available for measurement [1]. Many applications are

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connected with inverse problems for differential equations having singularities and turning points, for higher-order differential operators, for differential operators with delay or other types of "aftereffect", see [2] and the references that stated there by in. The first one who tackled the problems with turning point was M.G. Gasymov see [3]. It should be noted that, the author is one of Gasymov disciples [4–6]. Turning points appear in elasticity, optics, geophysics and other branches of natural sciences. Moreover, a wide class of differential equations with Bessel-type singularities and their perturbations can be reduced to differential equations having turning points. Inverse problems for equations with turning points and singularities help to study blow-up solutions for some nonlinear integrable evolution equations of mathematical physics see also [5,7,8,6]. It should be mentioned, here, that one of the first paper tackled the direct and inverse problem of Sturm-Liouville differential operators with explosive factor in the half line is Gasymov [9]. In the last time grew the interest of investigation of the boundary value problem by numerical methods e.g [10] presented an approximate construction of the Jost function for some Sturm-Liouville boundary value problem in case $\rho(x) = 1$ by means of collocation method, in addition, [11] is an application of spectral analysis of one-dimensional Schrödinger operators in magnetic field. Also [12,13] are applications of discontinuous wave speed problem on nonhomogeneous medium as in our case.

In the present paper we deal with inverse problem for equations with turning point. This paper is organized as follows. In Section 1, we state the basic results that are needed in the subsequent investigation and proof. In Section 2, the fundamental equation of the inverse problem is obtained for both $0 \leq x \leq y \leq a$ and $a \leq x \leq y \leq \pi$, and its uniqueness is proved. In Section 3 we prove the uniqueness of solution of the inverse problem by its spectral data $\{\lambda_n^\pm, a_n^\pm\}_{n=0}^\infty$. Finally, Section 4, is a conclusion and comments on the results obtained when it is compared with similar problems.

Consider the boundary value problem

$$-y'' + q(x)y = \lambda\rho(x)y \quad 0 \leq x \leq \pi, \quad (1.1)$$

$$y(0) = 0, \quad y(\pi) = 0, \quad (1.2)$$

where the nonnegative real function $q(x)$ has a second piecewise integrable derivatives on $[0, \pi]$, λ is a spectral parameter and the weight function or the explosive factor $\rho(x)$ is of the form

$$\rho(x) = \begin{cases} 1; & 0 \leq x \leq a < \pi, \\ -1; & a < x \leq \pi. \end{cases} \quad (1.3)$$

Following [14], we state the basic results that are needed in the subsequent investigation, where, the authors proved that the Dirichlet problem (1.1) and (1.2) has a countable number of eigenvalues $\lambda_n^\pm, n = 0, 1, 2, \dots$ where λ_n^+ are the nonnegative eigenvalues and λ_n^- are the negative eigenvalues which admit the asymptotic formulas

$$\begin{aligned} \lambda_n^+ &= \frac{\pi^2}{a^2} \left(n - \frac{1}{4}\right)^2 + \frac{2k_0\pi}{a} + \left(\frac{2k_1\pi}{a}\right) \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \\ \lambda_n^- &= -\frac{\pi^2}{(\pi-a)^2} \left(n - \frac{1}{4}\right)^2 - \frac{2k_0\pi}{\pi-a} - \left(\frac{2h_1\pi}{\pi-a}\right) \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \end{aligned} \quad (1.4)$$

where $k_0, k_1(x)$ and $h_1(x)$ are calculated in terms with the function $q(x)$ and its integration over $[0, \pi]$ where as the normalizing numbers $\{a_n^\pm\}_{n=0}^\infty$ have the asymptotic formulas

$$a_n^+ = \frac{d_1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right), \quad a_n^- = -d_2^2 e^{2nd_2} e^{-d_2} \left[\frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right], \quad (1.5)$$

where $d_1 = \frac{a^3}{2\pi^2}$ and $d_2 = \frac{\pi-a}{\pi}$.

The totality of numbers $\{\lambda_n^\pm, a_n^\pm\}, n = 0, 1, 2, \dots$ are called the spectral data (characteristics) of the problem (1.1) and (1.2).

The formulation of the inverse problem

The formulation of the inverse problem is stated as follows, is it possible to reestablish the boundary value problem (1.1) and (1.2) i.e can we find $q(x)$ by means of the spectral data $\lambda_n^\pm, a_n^\pm, n = 0, 1, 2, \dots$. For this we must answer the following two questions.

1. Is the recovering of the boundary value problem (1.1) and (1.2) by its spectral data is unique.
2. What is the effective method of recovering of the boundary value problem by its spectral data. We begin with the second question.

2. The main integral equation

In this section we construct and prove the uniqueness theorem of the main integral equation, (Gelfand–Levitan) integral equation [15].

2.1. The construction of the main integral equation of the inverse problem

The following two lemmas and theorem are devoted to the construction and introduction of the main integral equation of the inverse problem

Lemma 2.1. *Let*

$$Y_n(x, y) = \sum_{k=0}^n \frac{\varphi(x, \lambda_k^+)}{a_k^+} \sin \eta_k^+ y + \sum_{k=0}^n \frac{\varphi(x, \lambda_k^-)}{a_k^-} \sin \eta_k^- y. \quad (2.6)$$

If we fix x and denote $\alpha(y)$ the smooth finite function on the interval $(0, x)$, then

$$\lim_{n \rightarrow \infty} \int_0^x Y_n(x, y) \alpha(y) dy = 0, \quad (2.7)$$

or

$$\sum_{k=0}^{\infty} \frac{\varphi(x, \lambda_k^+)}{a_k^+} \int_0^x \alpha(y) \sin \eta_k^+ y dy + \sum_{k=0}^{\infty} \frac{\varphi(x, \lambda_k^-)}{a_k^-} \int_0^x \alpha(y) \sin \eta_k^- y dy = 0. \quad (2.8)$$

From which, the series in the left hand side of (2.8) is absolutely convergent.

Proof. let $r(x, \xi, \lambda) = \frac{\partial R(x, \xi, \lambda)}{\partial \xi}$ and Γ_n be the contour defined by

$$\Gamma_n = \left\{ |Re \eta| \leq \frac{\pi}{a} \left(n - \frac{1}{4}\right) + \frac{\pi}{2a}, |Im \eta| \leq \frac{\pi}{\pi-a} \left(n - \frac{1}{4}\right) + \frac{\pi}{2(\pi-a)} \right\}. \quad (2.9)$$

Following [14] the function $r(x, 0, \lambda)$, on the contour Γ_n , takes the form

$$r(x, 0, \lambda) \leq \frac{B}{|\eta|} e^{-|\tau| a}, \eta \in \Gamma_n, \forall n, \quad (2.10)$$

where B is a constant and $\eta = \sigma + i\tau$.

Since $\alpha(y) = 0$ in some neighborhood of x , then there exists $\delta > 0$ such that $\alpha(y) = 0, y \geq x - \delta$. So that, arguing as in [5] we get the inequality

$$\left| \int_0^x \alpha(y) \sin \eta y dy \right| \leq B_1 \frac{e^{|\tau|(x-\delta)}}{|\eta|}. \quad (2.11)$$

From [14] Lemma 2.3 we have for arbitrary zero λ_n of the function $\Psi(\lambda)$ we have

$$r^+(x, 0, \lambda) = \frac{-1}{\lambda - \lambda_n^+} \frac{\varphi(x, \lambda_n^+)}{a_n^+} + r_1^+(x, \lambda), \quad (2.12)$$

$$r^-(x, 0, \lambda) = \frac{-1}{\lambda - \lambda_n^-} \frac{\varphi(x, \lambda_n^-)}{a_n^-} + r_1^-(x, \lambda), \quad (2.13)$$

where $r_1^+(x, \lambda)$ is regular in the neighborhood of $\lambda = \lambda_n^+$, $r_1^-(x, \lambda)$ is regular in the neighborhood of $\lambda = \lambda_n^-$ and $a_n^\pm = \int_0^\pi \rho(\xi) \varphi^2(x, \lambda_n^\pm) dx$. Let L_n be the image of the upper half of the contour Γ_n under the mapping $\lambda = \eta^2$. From residue formula we have

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{L_n} \left\{ r(x, 0, \lambda_n) \int_0^x \alpha(y) \cos \eta y dy \right\} d\lambda, \\ &= - \sum_{k=0}^n \frac{\varphi(x, \lambda_k^+)}{a_k^+} \int_0^x \alpha(y) \sin \eta_k^+ y dy \\ & \quad - \sum_{k=0}^n \frac{\varphi(x, \lambda_k^-)}{a_k^-} \int_0^x \alpha(y) \sin \eta_k^- y dy. \end{aligned} \quad (2.14)$$

We prove by using (2.10), (2.11) that the left hand side of (2.14) tends to zero as $n \rightarrow \infty$

$$\begin{aligned} & \left| \frac{1}{2\pi i} \oint_{L_n} \left\{ r(x, 0, \lambda_n) \int_0^x \alpha(y) \sin \eta y dy \right\} \right| \leq \frac{1}{2\pi i} \oint_{L_n} |r(x, 0, \lambda)| \\ & \quad \left| \int_0^x \alpha(y) \sin \eta y dy \right| d\lambda \leq \frac{BB_1}{\pi} \oint_{\Gamma_n^+} \frac{e^{-|\tau|(x-\delta)}}{|\eta|} d\eta \\ & \leq \frac{2BB_1}{\pi} \left\{ \int_0^{\frac{\pi}{2a}(n-\frac{1}{4}) + \frac{\pi}{2(a-\sigma)}} \frac{e^{-|\tau|(a+\delta-x)}}{|\tau|} d\tau \right. \\ & \quad \left. + \int_0^{\frac{\pi}{2a}(n-\frac{1}{4}) + \frac{\pi}{2a}} \frac{e^{-|\tau|(a+\delta-x)}}{|\tau|} d\sigma \right\} \leq \frac{2BB_1}{\pi \left(\frac{\pi}{2a}(n-\frac{1}{4}) + \frac{\pi}{2a} \right)} \\ & \quad \left\{ \frac{1 - e^{-\left[\frac{\pi}{2a}(n-\frac{1}{4}) + \frac{\pi}{2(a-\sigma)} \right] (a+\delta-x)}}{a+\delta-x} + \left[\frac{\pi}{2a}(n-\frac{1}{4}) + \frac{\pi}{2a} \right] e^{-\left[\frac{\pi}{2a}(n-\frac{1}{4}) + \frac{\pi}{2(a-\sigma)} \right] (a+\delta-x)} \right\}, \end{aligned} \quad (2.15)$$

which tends to zero as $n \rightarrow \infty$.

By the help of asymptotic formulas of $\varphi(x, \lambda_n^\pm)$, a_n^\pm [16, (2.15) and (3.29)] and the inequality (2.11), the absolute convergence is followed, which completes the proof. \square

Lemma 2.2. *The sequence of functions*

$$\begin{aligned} T_n(x, y) &= \sum_{k=0}^n \left(\frac{1}{a_k^+} \sin \eta_k^+ x \sin \eta_k^+ y - \frac{1}{a_k^{\circ+}} \sin \eta_k^{\circ+} x \sin \eta_k^{\circ+} y \right), \\ & \quad + \sum_{k=0}^n \left(\frac{1}{a_k^-} \sin \eta_k^- x \sin \eta_k^- y - \frac{1}{a_k^{\circ-}} \sin \eta_k^{\circ-} x \sin \eta_k^{\circ-} y \right), \end{aligned} \quad (2.16)$$

converges in metric $L_2(0, a)$ to some function $T(x, y)$ as a function of y ($x < a$ is fixed) where a_k^\pm and η_k^\pm are the spectral characters of problem (1.1)–(1.2) when $q(x) \equiv 0$

N.B: It should be noted here that, the statement of this lemma still true with respect to x while y is fixed y because $T_n(x, y) = T_n(y, x)$ and consequently $T(x, y) = T(y, x)$

Proof. Let

$$T_n^\pm(x, y) = \sum_{k=0}^n \left(\frac{1}{a_k^\pm} \sin \eta_k^\pm x \sin \eta_k^\pm y - \frac{1}{a_k^{\circ\pm}} \sin \eta_k^{\circ\pm} x \sin \eta_k^{\circ\pm} y \right).$$

For simplicity we introduce the notations $\Phi_n^\pm(t), T_n^\pm(x, y), T_n^-(x, y)$ such that $T_n^\pm(x, y) = \frac{1}{2} [\Phi_n^\pm(x-y) - \Phi_n^\pm(x+y)]$, where $T_n(x, y) = T_n^+(x, y) + T_n^-(x, y)$ and

$$\Phi_n^\pm(t) = \sum_{k=0}^n \left(\frac{1}{a_k^\pm} \sin \eta_k^\pm t - \frac{1}{a_k^{\circ\pm}} \sin \eta_k^{\circ\pm} t \right), \quad (2.17)$$

so that, the convergence of $\Phi_n^\pm(t)$ implies the convergence of $T_n^\pm(t)$

From [16], $a_n^- = -\frac{(\pi-a)^3}{\pi^2 n^2} e^{2\eta_n^- a} [1 + \mathcal{O}(\frac{1}{n})]$ and $\eta_n^- = \frac{\pi-a}{n - \frac{1}{4}} + \mathcal{O}(\frac{1}{n})$, from which and a similar formula for $a_n^{\circ-}$ and $\eta_n^{\circ-}$, we have

$$\begin{aligned} \left| \frac{\sin \eta_k^- t}{a_k^-} - \frac{\sin \eta_k^{\circ-} t}{a_k^{\circ-}} \right| &\leq \frac{\sinh \eta_k^- t}{a_k^-} + \frac{\sinh \eta_k^{\circ-} t}{a_k^{\circ-}} \\ &\leq C [e^{\eta_k^- (t-2a)} + e^{\eta_k^{\circ-} (t-2a)}], \end{aligned} \quad (2.18)$$

where C is some constant. From inequality (2.18) and (2.17) we see that Φ_n^- converges as $n \rightarrow \infty, t \in [0, 2a - \varepsilon], \varepsilon > 0$.

We prove now the convergence of

$$\Phi_n^+ = \sum_{k=0}^n \left(\frac{1}{a_k^+} \sin \eta_k^+ t - \frac{1}{a_k^{\circ+}} \sin \eta_k^{\circ+} t \right). \quad (2.19)$$

Again, from [16], we have

$$\begin{aligned} a_k^+ &= \frac{b_1}{k^2} + \frac{b_2}{k^3} + \mathcal{O}\left(\frac{1}{k^4}\right) = a_k^{\circ+} + \mathcal{O}\left(\frac{1}{k^4}\right), \text{ where } a_k^{\circ+} \\ &= \frac{b_1}{k^2} + \mathcal{O}\left(\frac{1}{k}\right), b_1 = \frac{a^3}{2\pi^2}, \end{aligned}$$

and

$$\eta_k^+ = \eta_k^{\circ+} + \mathcal{O}\left(\frac{1}{k}\right), \text{ where } \eta_k^{\circ+} = \frac{\pi}{a} \left(n - \frac{1}{4} \right) + \mathcal{O}\left(\frac{1}{k}\right),$$

so that

$$\begin{aligned} \eta_k^+ &= \eta_k^{\circ+} + \frac{\varepsilon_k}{k}, \text{ and } \eta_k^{\circ+} = \frac{\pi}{a} \left(n - \frac{1}{4} \right) + \frac{\varepsilon'_k}{k}, \\ & \text{where } \text{Sup}|\varepsilon_k| < \infty, \text{ Sup}|\varepsilon'_k| < \infty, \end{aligned}$$

and hence we write

$$\frac{\sin(\eta_k^+)}{a_k^+} = \frac{\sin(\eta_k^{\circ+})}{a_k^{\circ+}} - \frac{t\varepsilon_k}{ka_k^{\circ+}} \cos(\eta_k^{\circ+} t) + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (2.20)$$

From (2.19) and (2.20) we have

$$\Phi_n^+(t) = \sum_{k=0}^n \left[-\frac{t\varepsilon_k}{ka_k^{\circ+}} \sin(\eta_k^{\circ+} t) + \mathcal{O}\left(\frac{1}{k^2}\right) \right], \quad \eta_k^{\circ+} = \frac{\pi}{a} \left(k - \frac{1}{4} \right). \quad (2.21)$$

$-\frac{t\varepsilon_k}{ka_k^{\circ+}} \sin(\eta_k^{\circ+} t)$ can be written as

$$\begin{aligned} \frac{t\varepsilon_k}{ka_k^{\circ+}} \cos(\eta_k^{\circ+} t) &= \frac{\varepsilon_k}{ka_k^{\circ+}} \cos\left(\frac{k\pi t}{a}\right) t \cos\left(\frac{\pi t}{4a}\right) - \frac{\varepsilon_k}{ka_k^{\circ+}} \\ & \quad \times \sin\left(\frac{k\pi t}{a}\right) t \sin\left(\frac{\pi t}{4a}\right). \quad \square \end{aligned} \quad (2.22)$$

From (2.21) and (2.22) together with the orthogonality of $\left\{ \sqrt{\frac{1}{a_k^{\sigma+}}} \sin \frac{k\pi t}{a} \right\}$ and $\left\{ \sqrt{\frac{1}{a_k^{\sigma-}}} \cos \frac{k\pi t}{a} \right\}$, we deduce that $\Phi_n^+(t)$ is convergent for $n \rightarrow \infty$ in the metric $L_2(0, a)$, which completes the proof. The function $T(x, y)$ can be, then, written in the form

$$T(x, y) = \sum_{k=0}^{\infty} \left(\frac{1}{a_k^+} \sin \eta_k^+ x \sin \eta_k^+ y - \frac{1}{a_k^{\sigma+}} \sin \eta_k^{\sigma+} x \sin \eta_k^{\sigma+} y \right) + \sum_{k=0}^{\infty} \left(\frac{1}{a_k^-} \sin \eta_k^- x \sin \eta_k^- y - \frac{1}{a_k^{\sigma-}} \sin \eta_k^{\sigma-} x \sin \eta_k^{\sigma-} y \right). \tag{2.23}$$

It is well known from [17] that the solution $\varphi(x, \lambda)$ of problem (1.1) subject to the initial conditions $\varphi(0, \lambda) = 0, \varphi'(0, \lambda) = 1$ has the representation, for $0 \leq x \leq a$,

$$\varphi(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_0^x A(x, t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt, \tag{2.24}$$

where, $\frac{\partial A(x, t)}{\partial x}, \frac{\partial A(x, t)}{\partial t}, (0 \leq t \leq x \leq a)$ are local integrable and

$$A(x, x) = \int_0^x q(t) dt, \left. \frac{\partial A(x, t)}{\partial t} \right|_{t=0} = 0 (0 \leq x \leq a).$$

In the following theorem we introduce, by the help of Lemma 2.1 and Lemma 2.2, the main integral equation of the inverse problem.

Theorem 2.1. *The function $T(x, y)$ which is defined by (2.22) satisfies the integral equation*

$$T(x, y) + A(x, y) + \int_0^x A(x, t) T(t, y) dy = 0, (0 \leq y \leq x \leq a), \tag{2.25}$$

where $A(x, t)$, as in (2.24), is the kernel of the representation of $\varphi(x, \lambda)$. This equation is called the Gelfand-Levitan integral equation.

Proof. By substituting from (2.24) into (1.4), for $0 \leq y \leq x \leq a$ we have

$$\begin{aligned} Y_n(x, y) &= \sum_{k=0}^n \frac{\sin \eta_k^+ y}{a_k^+} \left[\sin \eta_k^+ x + \int_0^x A(x, t) \sin \eta_k^+ t dt \right] \\ &+ \sum_{k=0}^n \frac{\sin \eta_k^- y}{a_k^-} \left[\sin \eta_k^- x + \int_0^x A(x, t) \sin \eta_k^- t dt \right] \\ &= \sum_{k=0}^n \left(\frac{1}{a_k^+} \sin \eta_k^+ x \sin \eta_k^+ y - \frac{1}{a_k^{\sigma+}} \sin \eta_k^{\sigma+} x \sin \eta_k^{\sigma+} y \right) \\ &+ \sum_{k=0}^n \left(\frac{1}{a_k^-} \sin \eta_k^- x \sin \eta_k^- y - \frac{1}{a_k^{\sigma-}} \sin \eta_k^{\sigma-} x \sin \eta_k^{\sigma-} y \right) \\ &+ \sum_{k=0}^n \frac{1}{a_k^{\sigma+}} \sin \eta_k^{\sigma+} x \sin \eta_k^{\sigma+} y + \sum_{k=0}^n \frac{1}{a_k^{\sigma-}} \sin \eta_k^{\sigma-} x \sin \eta_k^{\sigma-} y \\ &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^+} \sin \eta_k^+ t \sin \eta_k^+ x - \frac{1}{a_k^{\sigma+}} \sin \eta_k^{\sigma+} t \sin \eta_k^{\sigma+} x \right) \right] dt \\ &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^-} \sin \eta_k^- t \sin \eta_k^- x - \frac{1}{a_k^{\sigma-}} \sin \eta_k^{\sigma-} t \sin \eta_k^{\sigma-} x \right) \right] dt \\ &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^{\sigma+}} \sin \eta_k^{\sigma+} t \sin \eta_k^{\sigma+} x \right) \right] dt \end{aligned}$$

$$\begin{aligned} &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^{\sigma-}} \sin \eta_k^{\sigma-} t \sin \eta_k^{\sigma-} x \right) \right] dt, \\ &= T_n(x, y) + \int_0^x A(x, t) T_n(t, y) dt \\ &+ \sum_{k=0}^n \frac{1}{a_k^{\sigma+}} \sin \eta_k^{\sigma+} x \sin \eta_k^{\sigma+} y + \sum_{k=0}^n \frac{1}{a_k^{\sigma-}} \sin \eta_k^{\sigma-} x \sin \eta_k^{\sigma-} y \\ &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^+} \sin \eta_k^+ t \sin \eta_k^+ x \right) \right] dt \\ &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^-} \sin \eta_k^- t \sin \eta_k^- x \right) \right] dt. \tag{2.26} \end{aligned}$$

Multiplying both sides of (2.26) by $\alpha(y)$, where $\alpha(y)$ is an arbitrary function satisfying the condition of lemma (2.1), and integrating with respect to y on the interval $(0, x)$ we get

$$\begin{aligned} \int_0^x Y_n(x, y) \alpha(y) dy &= \int_0^x T_n(x, y) \alpha(y) dy \\ &+ \int_0^x A(x, t) \left[\int_0^x T_n(t, y) \alpha(y) dy \right] dt \\ &+ \sum_{k=0}^n \frac{1}{a_k^{\sigma+}} \sin \eta_k^{\sigma+} x \int_0^x \alpha(y) \sin \eta_k^{\sigma+} y dy \\ &+ \sum_{k=0}^n \frac{1}{a_k^{\sigma-}} \sin \eta_k^{\sigma-} x \int_0^x \alpha(y) \sin \eta_k^{\sigma-} y dy \\ &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^+} \sin \eta_k^+ t \int_0^x \alpha(y) \sin \eta_k^+ y dy \right) \right] dt \\ &+ \int_0^x A(x, t) \left[\sum_{k=0}^n \left(\frac{1}{a_k^-} \sin \eta_k^- t \int_0^x \alpha(y) \sin \eta_k^- y dy \right) \right] dt. \tag{2.27} \end{aligned}$$

Passing, here, to the limit as $n \rightarrow \infty$ and using lemma (2.1), lemma (2.2) and the eigenfunction expansion formula [14] for $q = 0$ and keeping in mind that $\alpha(x) = 0$, we obtain

$$\begin{aligned} 0 &= \int_0^x T(x, y) \alpha(y) dy + \int_0^x A(x, t) \left[\int_0^x T(t, y) \alpha(y) dy \right] dt \\ &+ \int_0^x A(x, t) \alpha(t) dt. \end{aligned}$$

From here and by virtue of arbitrariness of $\alpha(y)$, Eq. (2.25) followed and the theorem is proved. \square

2.2. The uniqueness of the solutions of the main integral equation

Now we prove the uniqueness of the solution of the integral Eq. (2.25) with respect to $A(x, t)$. Because of the introduction of the turning point $\rho(x)$ as in (1.3), the proof of the uniqueness theorem is carried out in two steps, (i) for $x \leq a$ (ii) for $x \geq a$

Case (i) for $x \leq a$

Theorem 2.2. *For every fixed $x \leq a$ the solution $\alpha(y) \in L_2(0, x)$ of the integral equation*

$$f(y) + \alpha(y) + \int_0^x \alpha(t) T(t, y) dt = 0, (0 \leq y \leq x),$$

is unique

Proof. We prove that the homogenous integral equation

$$\alpha(y) + \int_0^x \alpha(t)T(t, y)dt = 0, \quad (0 \leq y \leq x \leq a), \quad (2.28)$$

has only the zero solution in the space $L_2(0, x)$. In addition to the boundary value problem (1.1)–(1.2) we consider the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq \pi, \quad (2.29)$$

$$y(0) = 0, \quad y(\pi) = 0, \quad (2.30)$$

with the same function $q(x)$ as in (1.1)–(1.2), we denote by $\tilde{\varphi}(x, \lambda)$ the solution of the Eq. (2.29) satisfying

$$\tilde{\varphi}(0, \lambda) = 0, \quad \tilde{\varphi}'(0, \lambda) = 1, \quad (2.31)$$

$\tilde{\varphi}(x, \lambda)$ has the representation

$$\tilde{\varphi}(x, \lambda) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \tilde{A}(x, t) \frac{\sin \sqrt{\lambda}t}{\sqrt{\lambda}} dt. \quad (2.32)$$

Since the two Eqs. (1.1) and (2.29) are identical for $0 \leq x \leq a$ and the solutions $\tilde{\varphi}(x, \lambda)$, $\varphi(x, \lambda)$ satisfy the same initial condition, it follows that

$$\tilde{\varphi}(x, \lambda) \equiv \varphi(x, \lambda), \quad 0 \leq x \leq a. \quad (2.33)$$

From which and by virtue of (2.32) it follows that

$$A(x, t) \equiv \tilde{A}(x, t), \quad 0 \leq t \leq x \leq a. \quad (2.34)$$

Let $\tilde{\eta}_k$ and \tilde{a}_k be the eigen values and the normalization numbers of the problem (2.29)–(2.30), where $\tilde{a}_k = \int_0^\pi \tilde{\varphi}^2(x, \tilde{\eta}_k) dx$, and $\tilde{\eta}_k^o$ and \tilde{a}_k^o be the eigen values and the normalization numbers of the same problem with $q(x) = 0$. So that the following Gelfand–Levitan integral equation takes place

$$\tilde{T}(x, y) + \tilde{A}(x, y) + \int_0^x \tilde{A}(x, t) \tilde{T}(t, y) dt = 0, \quad 0 \leq y \leq x \leq \pi, \quad (2.35)$$

where

$$\tilde{T}(x, y) = \sum_{k=0}^{\infty} \left(\frac{1}{\tilde{a}_k} \sin \tilde{\eta}_k x \sin \tilde{\eta}_k y - \frac{1}{\tilde{a}_k^o} \sin \tilde{\eta}_k^o x \sin \tilde{\eta}_k^o y \right). \quad (2.36)$$

From (2.35), (2.28) and (2.34) we get

$$T(x, y) - \tilde{T}(x, y) + \int_0^x A(x, t)[T(t, y) - \tilde{T}(t, y)] dt = 0, \quad 0 \leq y \leq x \leq a, \quad (2.37)$$

from which, in particular, for $y = 0$ we have

$$T(x, 0) - \tilde{T}(x, 0) + \int_0^x A(x, t)[T(t, 0) - \tilde{T}(t, 0)] dt = 0, \quad 0 \leq x \leq a. \quad (2.38)$$

The integral Eq. (2.38) is a homogenous integral equation of Volterra type with respect to $T(x, 0) - \tilde{T}(t, 0)$ so that,

$$T(x, 0) = \tilde{T}(x, 0), \quad 0 \leq x \leq a. \quad (2.39)$$

It is clear that

$$T(x, y) = \frac{1}{2}[T(x+y, 0) - T(x-y, 0)], \quad 0 \leq y \leq x \leq a, \quad (2.40)$$

$$\tilde{T}(x, y) = \frac{1}{2}[\tilde{T}(x+y, 0) - \tilde{T}(x-y, 0)], \quad 0 \leq y \leq x \leq \pi. \quad (2.41)$$

From (2.40), (2.41) and by virtue of (2.38) we have

$$T(x, y) = \tilde{T}(x, y), \quad 0 \leq y \leq x \leq \frac{a}{2}. \quad (2.42)$$

From (2.37), for $y = \frac{a}{2}$, we have

$$T\left(x, \frac{a}{2}\right) - \tilde{T}\left(x, \frac{a}{2}\right) + \int_0^{\frac{a}{2}} A(x, t) \left[T\left(t, \frac{a}{2}\right) - \tilde{T}\left(t, \frac{a}{2}\right) \right] dt + \int_{\frac{a}{2}}^x A(x, t) \left[T\left(t, \frac{a}{2}\right) - \tilde{T}\left(t, \frac{a}{2}\right) \right] dt = 0, \quad (2.43)$$

$\frac{a}{2} \leq x \leq a$. By the aid of (2.42), the last integral equation becomes

$$T\left(x, \frac{a}{2}\right) - \tilde{T}\left(x, \frac{a}{2}\right) + \int_{\frac{a}{2}}^x A(x, t) \left[T\left(t, \frac{a}{2}\right) - \tilde{T}\left(t, \frac{a}{2}\right) \right] dt = 0, \quad \frac{a}{2} \leq x \leq a. \quad (2.44)$$

From which and by virtue of the uniqueness of the solution of Volterra integral equation, we get

$$T\left(x, \frac{a}{2}\right) = \tilde{T}\left(x, \frac{a}{2}\right) \quad \frac{a}{2} \leq x \leq a. \quad (2.45)$$

By putting $y = \frac{a}{2}$ into (2.40) and (2.41), we have from (2.39), (2.43)

$$T(x, 0) = \tilde{T}(x, 0) \quad 0 \leq x \leq \frac{3a}{2}. \quad (2.46)$$

From (2.40), (2.41), keeping in mind (2.46) we get

$$T(x, y) = \tilde{T}(x, y) \quad 0 \leq y \leq x \leq \frac{3a}{4}. \quad (2.47)$$

From (2.37), for $y = \frac{3a}{4}$, we have

$$T\left(x, \frac{3a}{4}\right) - \tilde{T}\left(x, \frac{3a}{4}\right) + \int_0^{\frac{3a}{4}} A(x, t) \left[T\left(t, \frac{3a}{4}\right) - \tilde{T}\left(t, \frac{3a}{4}\right) \right] dt + \int_{\frac{3a}{4}}^x A(x, t) \left[T\left(t, \frac{3a}{4}\right) - \tilde{T}\left(t, \frac{3a}{4}\right) \right] dt = 0, \quad \frac{3a}{4} \leq x \leq a. \quad (2.48)$$

The first integral is zero by (2.47) and hence as we did before we have $T\left(x, \frac{3a}{4}\right) = \tilde{T}\left(x, \frac{3a}{4}\right)$, $\frac{3a}{4} \leq x \leq a$, $T(x, 0) = \tilde{T}(x, 0)$, $0 \leq x \leq \frac{7a}{8}$, and

$$T(x, y) = \tilde{T}(x, y) \quad 0 \leq y \leq x \leq \frac{7a}{8}. \quad (2.49)$$

Continuing this process k-times we reach to

$$T(x, y) = \tilde{T}(x, y) \quad 0 \leq y \leq x \leq \frac{2^{k-1} - 1}{2^k} a. \quad (2.50)$$

Passing to the limit as $k \rightarrow \infty$ in (2.50) we obtain

$$T(x, y) = \tilde{T}(x, y) \quad 0 \leq y \leq x \leq a. \quad (2.51)$$

Eq. (2.28), By the aid of (2.51), can be written in the form

$$\alpha(y) + \int_0^x \alpha(t) \tilde{T}(t, y) dt = 0, \quad (0 \leq y \leq x \leq a). \quad (2.52)$$

From which as in [18, p. 27] $\alpha(t) = 0$ and the theorem is proved

Case (ii) for $x \geq a$

The solution $\psi(x, \lambda)$ of the Eq. (1.1) subject to the conditions $\psi(\pi, \lambda) = 0, \psi'(\pi, \lambda) = 1$ has the representation

$$\psi(x, \lambda) = \frac{\sinh \lambda(x - \pi)}{\lambda} + \int_x^\pi B(x, t) \frac{\sinh \lambda(t - \pi)}{\lambda} dt, \quad (a \leq x \leq \pi), \tag{2.53}$$

where the kernel $B(x, t)$ has integrable first derivative $\frac{\partial B}{\partial x}, \frac{\partial B}{\partial t}$ ($a < x \leq t \leq \pi$) and has the properties

$$B(x, x) = \frac{1}{2} \int_x^\pi q(t) dt, \quad B(\pi, \pi) = 0, \quad \frac{\partial B}{\partial t} \Big|_{t=0} = 0. \tag{2.54}$$

From [14, p. 12], we have $c_n^\pm = \frac{\psi(x, \lambda_n^\pm)}{\varphi(x, \lambda_n^\pm)}$, $0 \leq x \leq \pi$ by the help of which the formulas (2.12) and (2.13) can be written in the form

$$r^+(x, \pi, \lambda) = \frac{-1}{\lambda - \lambda_k^+} \frac{\varphi(x, \lambda_k^+)}{(c_k^+)^2 a_k^+} + r_1^+(x, \lambda), \tag{2.55}$$

$$r^-(x, \pi, \lambda) = \frac{-1}{\lambda - \lambda_k^-} \frac{\varphi(x, \lambda_k^-)}{(c_k^-)^2 a_k^-} + r_1^-(x, \lambda), \tag{2.56}$$

where $r_1^+(x, \lambda)$ is regular in the neighborhood of $\lambda = \lambda_k^+, r_1^-(x, \lambda)$ is regular in the neighborhood of $\lambda = \lambda_k^-$ and $a_k^\pm = \int_0^\pi \rho(\xi) \varphi^2(x, \lambda_k^\pm) dx$. Let $g(y), y \in (x, \pi)$ be a smooth function tending to zero in some neighborhood of the point x . By using (2.55), (2.56) and arguing as in the proof of Lemma 2.1, we obtain the equation

$$\sum_{k=0}^\infty \frac{\psi(x, \lambda_k^+)}{(c_k^+)^2 a_k^+} \int_x^\pi g(y) \sinh \lambda_k^+(y - \pi) dy + \sum_{k=0}^\infty \frac{\psi(x, \lambda_k^-)}{(c_k^-)^2 a_k^-} \int_x^\pi g(y) \sinh \lambda_k^-(y - \pi) dy = 0, \quad a < x \leq \pi. \tag{2.57}$$

From (2.57) and the representation (2.53) and By using the same proof technique of Theorem 2.1 we get the integral equation

$$T_1(x, y) + B(x, y) + \int_x^\pi B(x, t) T_1(t, y) dt = 0, \quad 0 < x \leq y \leq \pi, \tag{2.58}$$

where

$$T_1(x, y) = \sum_{k=0}^\infty \left[\frac{\sinh \eta_k^+(x - \pi) \sinh \eta_k^+(y - \pi)}{(c_k^+)^2 a_k^+} - \frac{\sinh \eta_k^{o+}(x - \pi) \sinh \eta_k^{o+}(y - \pi)}{(c_k^+)^2 a_k^+} \right] + \sum_{k=0}^\infty \left[\frac{\sinh \eta_k^-(x - \pi) \sinh \eta_k^-(y - \pi)}{(c_k^-)^2 a_k^-} - \frac{\sinh \eta_k^{o-}(x - \pi) \sinh \eta_k^{o-}(y - \pi)}{(c_k^-)^2 a_k^-} \right], \tag{2.59}$$

where η_k^{\pm}, c_k^{\pm} , and a_k^{\pm} are the numbers η_k^\pm, c_k^\pm , and a_k^\pm of the problem (1.1)–(1.2) for $q(x) \equiv 0$ Using the same way and technique used in the proof of Theorem 2.1, it can be proved that the integral Eq. (2.58) has only the zero solution with respect to $B(x, t)$. The Eqs. (2.25) and (2.58) are called the main integral equation of the inverse problem of the problem (1.1)–(1.2) □

3. The uniqueness theorem of the inverse problem

Now we come to the last of our goals of this paper which is the uniqueness theorem of the inverse problem by spectral data $\{\lambda_n^\pm\}_{n=0}^\infty, \{a_n^\pm\}_{n=0}^\infty$.

Theorem 3.3. *The spectral data $\{\lambda_n^\pm\}_{n=0}^\infty, \{a_n^\pm\}_{n=0}^\infty$ of the boundary value problem (1.1)–(1.2) uniquely define the coefficients $q(x)$ and $\rho(x)$ of the Eqs. (1.1) and (1.3)*

Proof. We introduce, from (1.4) and (1.5), the spectral data $\{\lambda_n^\pm\}_{n=0}^\infty, \{a_n^\pm\}_{n=0}^\infty$ of the boundary value problem (1.1)–(1.2). By virtue of (1.4) the number a is uniquely defined by the formula

$$a = \lim_{k \rightarrow \infty} \frac{\pi k}{\sqrt{\lambda_k^+}}, \tag{3.60}$$

and consequently the function $\rho(x)$, which is defined by (1.3), is uniquely defined. By virtue of Theorem 2.2 the solution $A(x, y)$ of the integral Eq. (2.25) is uniquely defined for ($0 \leq x \leq a$), so that $q(x) = 2 \frac{dA(x, x)}{dx}$ is uniquely defined for. The uniqueness of the potential $q(x), a \leq x \leq \pi$ is followed from the uniqueness of the solution $B(x, y)$ of the integral Eq. (2.58) because according to (2.54) $q(x) = -2 \frac{dB(x, x)}{dx}$ is unique, which complete the proof. □

4. Conclusion and comments

- 1– This problem is one of the suggested inverse problems in [19, page 6];
- 2– The author had studied, in Ph.D., the inverse problem of the boundary value problem (1.1) subject to $y'(0) + hy(0) = 0, y'(\pi) + Hy(\pi) = 0,$ (4.61)
- 3– The present problem, with $y(0) = 0, y(\pi) = 0$ cannot considered as a spacial case of (4.61)
- 4– Due to the absence of the numbers H and h the inverse problem by two specters cannot be studied in the present work
- 5– Besides the direct spectral investigation of the of the present problem as in [14,16], the author had studied the eigenfunction expansion, equiconvergence of the eigenfunction expansion, and the regularized trace formula, [20,21]

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