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ORIGINAL ARTICLE

Locally α-compact spaces based on continuous valued logic

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Abstract This paper is a continuation of [1]. That is, it considers fuzzifying topologies, a special case of *I*-fuzzy topologies (bifuzzy topologies), introduced by Ying [2]. It investigates topological notions defined by means of α-open sets when these are planted into the framework of Ying's fuzzifying topological spaces (by Łukasiewicz logic in $[0,1]$). Other characterizations of fuzzifying α-compactness are given, including characterizations in terms of nets and α-subbases. Several characterizations of locally α-compactness in the framework of fuzzifying topology are introduced and the mapping theorems are obtained.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 54A40; 54B10; 54D30

1. Introduction and preliminaries

In the last few years fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [3–8]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [4], the kind of topologies defined by Chang [9] and Goguen [10] is called the topologies of fuzzy subsets, and further is naturally called L -topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an L -topological space) is a family τ of fuzzy subsets (resp. L -fuzzy subsets) of nonempty set X, and τ satisfies the basic conditions of classical topologies [11]. On the other hand, Höhle in [12] proposed the terminology L-fuzzy topology to be an L-valued mapping on the traditional powerset $P(X)$ of X. The authors in [6,7,13,14] defined an L-fuzzy topology to be an L-valued mapping on the L-powerset L^X of X.

In 1952, Rosser and Turquette [15] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicates calculus, then what are the detail of such theories ? As an attempt to give a partial answer to this problem in the case of point set topology, Ying in 1991-1993 [2,16,17] used a semantical method of continuous-valued logic to develop systematically fuzzifying topology. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus

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established. There are already more than 100 papers in fuzzifying topology published in the last two decades, I guess. But only a few papers can properly use the semantic method introduced in the original papers of Ying, which I strongly believe, can provide more delicate characterization of fuzzifying topological structure. So far, there has been significant research on fuzzifying topologies [18–24]. For example, Ying [22] introduced the concepts of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [24] the concept of local compactness in fuzzifying topology is introduced and some of its properties are established. In [18] the concepts of fuzzifying α-open set and fuzzifying α-continuity were introduced and studied. Also, Sayed [21] introduced some concepts of fuzzifying α-separation axioms and clarified the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms. Quite recently, Sayed in [1] used the finite intersection property to give a characterization of fuzzifying α-compact spaces. In classical topology , α-compact spaces and locally α-compact spaces have been studied in [25,26]. In this paper, the concepts of α-base and α-subbase of fuzzifying α-topology are introduced. Other characterizations of fuzzifying α-compactness are given, including characterizations in terms of nets and α-subbase. Several characterizations of locally α-compactness in the framework of fuzzifying topology are introduced and the mapping theorems are obtained. Thus we fill a gap in the existing literature on fuzzifying topology. We use the terminologies and notations in [1,2,16–18,21,22,24] without any explanation. We note that the set of truth values is the unit interval and we do often not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying does. We will use the symbol ⊗ instead of the second "AND" operation ∧ as dot is hardly visible. This mean that $[\alpha]\leqslant[\varphi\rightarrow\psi] \iff [\alpha]\otimes[\varphi]\leqslant[\psi]$. All of the contributions in general topology in this paper which are not referenced may be original.

We now give some definitions and results which are useful in the rest of the present paper. The family of all fuzzifying α-open sets [18], denoted by $\tau_\alpha\in\Im(P(X))$, is defined as

$$A\in\tau_\alpha := \forall x(x\in A\rightarrow x\in Int(Cl(Int(A)))),\text{ i. e., }\tau_\alpha(A) = \bigwedge_{x\in A} Int(Cl(Int(A)))(x).$$

The family of all fuzzifying α-closed sets [18], denoted by $\mathsf{F}_\alpha\in\Im(P(X))$, is defined as $A\in\mathsf{F}_\alpha := X-A\in\tau_\alpha$. The fuzzifying α-neighborhood system of a point $x\in X$ [18] is denoted by $N_x^{\alpha^X}$ (or N_x^α) $\in\Im(P(X))$ and defined as $N_x^\alpha(A)=\bigvee_{x\in B\subseteq A}\tau_\alpha(B)$. The fuzzifying α-closure of a set $A\subseteq X$ [18], denoted by $Cl_\alpha\in\Im(X)$, is defined as $Cl_\alpha(A)(x)=1-N_x^\alpha(X-A)$. If (X,τ) is a fuzzifying topological space and $N(X)$ is the class of all nets in X, then the binary fuzzy predicates $\rhd^\alpha,\propto^\alpha\in\Im(N(X)\times X)$ [23] are defined as

$$S\rhd^\alpha x := \forall A\left(A\in N_x^{\alpha^X}\rightarrow S\tilde{\subset}A\right), S\propto^\alpha x := \forall A\left(A\in N_x^{\alpha^X}\rightarrow S\tilde{\sqsubset}A\right),$$

where "$S\rhd^\alpha x$", "$S\propto^\alpha x$" stand for "S α-converges to x" , "x is an α-accumulation point of S", respectively; and "$\tilde{\subset}$", "$\tilde{\sqsubset}$" are the binary crisp predicates "almost in ","often in", respectively. The degree to which x is an α-adherence point of S is $adh_\alpha S(x)=[S\propto^\alpha x]$. If (X,τ) and (Y,σ) are two fuzzifying topological spaces and $f\in Y^X$, the unary fuzzy predicates $C_\alpha, I_\alpha\in\Im(Y^X)$, called fuzzifying α-continuity [18], fuzzifying α-irresoluteness [1], are given as $C_\alpha(f) := \forall B(B\in\sigma\rightarrow f^{-1}(B)\in\tau_\alpha)$, $I_\alpha(f) := \forall B(B\in\sigma_\alpha\rightarrow f^{-1}(B)\in\tau_\alpha)$, respectively. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $T_2^\alpha\in\Im(\Omega)$, called fuzzifying α-Hausdorffness [21], is given as follows:

$$T_2^\alpha(X,\tau)=\forall x\forall y((x\in X\wedge y\in X\wedge x\neq y)\rightarrow\exists B\exists C(B\in N_x^\alpha\wedge C\in N_y^\alpha\wedge B\cap C\equiv\phi)).$$

A unary fuzzy predicate $\Gamma\in\Im(\Omega)$, called fuzzifying compactness [22], is given as follows:

$$\Gamma(X,\tau) := (\forall\mathfrak{R})(K_\circ(\mathfrak{R},X)\rightarrow(\exists\wp)((\wp\leqslant\mathfrak{R})\wedge K(\wp,A)\otimes FF(\wp)))$$

and if $A\subseteq X$, then $\Gamma(A) := \Gamma(A,\tau/A)$. For K, $K_\circ$ (resp. $\leqslant$ and FF) see [16, Definition 4.4] (resp. [16, Theorem 4.3] and [22, Definition 1.1 and Lemma 1.1]). A unary fuzzy predicate $fI\in\Im(\Im(P(X)))$, called fuzzy finite intersection property [22], is given as

$$fI(\mathfrak{R}) := \forall\wp((\wp\leqslant\mathfrak{R})\wedge FF(\wp)\rightarrow\exists x\forall B(B\in\wp\rightarrow x\in B)).$$

A fuzzifying topological space (X,τ) is said to be fuzzifying α-topological space [1] if $\tau_\alpha(A\cap B)\geqslant\tau_\alpha(A)\wedge\tau_\alpha(B)$. A unary fuzzy predicate $LC\in\Im(\Omega)$, called fuzzifying locally compactness [24], is given as follows: $(X,\tau)\in LC := (\forall x)(\exists B)((x\in Int(B)\otimes\Gamma(B,\tau/B))$.

2. Fuzzifying α-base and α-subbase

Definition 2.1. Let (X,τ) be a fuzzifying topological space and $\beta_\alpha\subseteq\tau_\alpha$. Then β_α is called an α-base of τ_α if β_α fulfils the condition:

$$\models A\in N_x^{\alpha^X}\rightarrow\exists B((B\in\beta_\alpha)\wedge(x\in B\subseteq A)).$$

Theorem 2.1. *β_α is an α-base of τ_α if and only if $\tau_\alpha=\beta_\alpha^{(\cup)}$, where*

$$\beta_\alpha^{(\cup)}(A)=\bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda=A}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda).$$

Proof. Suppose that β_α is an α-base of τ_α. If

$$\bigcup_{\lambda\in\Lambda}B_\lambda=A,$$

then from Theorem 3.1 (1) (b) in [18],

$$\tau_\alpha(A)=\tau_\alpha\left(\bigcup_{\lambda\in\Lambda}B_\lambda\right)\geqslant\bigwedge_{\lambda\in\Lambda}\tau_\alpha(B_\lambda)\geqslant\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda).$$

Consequently,

$$\tau_\alpha(A)\geqslant\bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda=A}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda).$$

To prove that

$$\tau_\alpha(A)\leqslant\bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda=A}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda),$$

we first prove

$$\tau_\alpha(A)=\bigwedge_{x\in A}\bigvee_{x\in B\subseteq A}\tau_\alpha(B).$$

(Indeed, assume $\gamma_x = \{B: x \in B \subseteq A\}$. Then for any

$$f \in \prod_{x\in A}\gamma_x,\ \bigcup_{x\in A} f(x) = A,$$

and furthermore

$$\tau_\alpha(A) = \tau_\alpha\left(\bigcup_{x\in A} f(x)\right) \geqslant \bigwedge_{x\in A}\tau_\alpha(f(x)) \geqslant \bigvee_{f\in\prod_{x\in A}\gamma_x}\bigwedge_{x\in A}\tau_\alpha(f(x)) = \bigwedge_{x\in A}\bigvee_{x\in B\subseteq A}\tau_\alpha(B).$$

Also

$$\tau_\alpha(A) \leqslant \bigwedge_{x\in A}\bigvee_{x\in B\subseteq A}\tau_\alpha(B).$$

Therefore

$$\tau_\alpha(A) = \bigwedge_{x\in A}\bigvee_{x\in B\subseteq A}\tau_\alpha(B)).$$

Now, since

$$N_x^{\alpha^X}(A) \leqslant \bigvee_{x\in B\subseteq A}\beta_\alpha(B), \tau_\alpha(A) = \bigwedge_{x\in A}\bigvee_{x\in B\subseteq A}\tau_\alpha(B) = \bigwedge_{x\in A}N_x^{\alpha^X}(A) \leqslant \bigwedge_{x\in A}\bigvee_{x\in B\subseteq A}\beta_\alpha(B) = \bigvee_{f\in\prod_{x\in A}\gamma_x}\bigwedge_{x\in A}\beta_\alpha(f(x)).$$

Then

$$\tau_\alpha(A) \leqslant \bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda = A}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda).$$

Therefore

$$\tau_\alpha(A) = \bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda = A}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda)$$

In the other side, we assume

$$\tau_\alpha(A) = \bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda = A}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda)$$

and we will show that β_α is an α-base of τ_α, i.e., for any

$$A \subseteq X,\ N_x^{\alpha^X}(A) \leqslant \bigvee_{x\in B\subseteq A}\beta_\alpha(B).$$

Indeed, if

$$x \in B \subseteq A,\ \bigcup_{\lambda\in\Lambda}B_\lambda = B,$$

then there exists $\lambda_\circ \in \Lambda$ such that $x \in B_{\lambda_\circ}$ and

$$\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda) \leqslant \beta_\alpha(B_{\lambda_\circ}) \leqslant \bigvee_{x\in B\subseteq A}\beta_\alpha(B).$$

Therefore

$$N_x^{\alpha^X}(A) = \bigvee_{x\in B\subseteq A}\tau_\alpha(B) = \bigvee_{x\in B\subseteq A}\bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda = B}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda) \leqslant \bigvee_{x\in B\subseteq A}\beta_\alpha(B). \quad \square$$

Theorem 2.2. *Let $\beta_\alpha \in \Im(P(X))$. Then β_α is an α-base for some fuzzifying α-topology τ_α if and only if it has the following properties:*

(1) $\beta_\alpha^{(\cup)}(X) = 1$;
(2) $\models (A \in \beta_\alpha) \wedge (B \in \beta_\alpha) \wedge (x \in A \cap B) \rightarrow \exists C((C \in \beta_\alpha) \wedge (x \in C \subseteq A \cap B)$.

Proof. If β_α is an α-base for some fuzzifying α-topology τ_α, then $\tau_\alpha(X) = \beta_\alpha^{(\cup)}(X)$. Clearly, $\beta_\alpha^{(\cup)}(X) = 1$. In addition, if $x \in A \cap B$, then

$$\beta_\alpha(A) \wedge \beta_\alpha(B) \leqslant \tau_\alpha(A) \wedge \tau_\alpha(B) \leqslant \tau_\alpha(A \cap B) \leqslant N_x^{\alpha^X}(A \cap B) \leqslant \bigvee_{x\in C\subseteq A\cap B}\beta_\alpha(C).$$

Conversely, if β_α satisfies (1) and (2), then we have τ_α is a fuzzifying α-topology. In fact, $\tau_\alpha(X) = 1$. For any $\{A_\lambda : \lambda \in \Lambda\} \subseteq P(X)$, we set

$$\gamma_\lambda = \left\{\{B_{\delta_\lambda} : \delta_\lambda \in \Lambda_\lambda\} : \bigcup_{\delta_\lambda\in\Lambda_\lambda}B_{\delta_\lambda} = A_\lambda\right\}.$$

Then for any

$$f \in \prod_{\lambda\in\Lambda}\gamma_\lambda,\ \bigcup_{\lambda\in\Lambda}\bigcup_{B_{\delta_\lambda}\in f(\lambda)}B_{\delta_\lambda} = \bigcup_{\lambda\in\Lambda}A_\lambda.$$

Therefore

$$\tau_\alpha\left(\bigcup_{\lambda\in\Lambda}A_\lambda\right) = \bigvee_{\bigcup_{\delta\in\Lambda}B_\delta = \bigcup_{\lambda\in\Lambda}A_\lambda}\bigwedge_{\delta\in\Lambda}\beta_\alpha(B_\delta) \geqslant \bigvee_{f\in\prod_{\lambda\in\Lambda}\gamma_\lambda}\bigwedge_{\lambda\in\Lambda}\bigwedge_{B_{\delta_\lambda}\in f(\lambda)}\beta_\alpha(B_{\delta_\lambda}) \geqslant \bigwedge_{\lambda\in\Lambda}\bigvee_{\{B_{\delta_\lambda}:\delta_\lambda\in\Lambda_\lambda\}\in\gamma_\lambda}\bigwedge_{\delta_\lambda\in\Lambda_\lambda}\beta_\alpha(B_{\delta_\lambda}) = \bigwedge_{\lambda\in\Lambda}\tau_\alpha(A_\lambda).$$

Finally, we need to prove that $\tau_\alpha(A \cap B) \geqslant \tau_\alpha(A) \wedge \tau_\alpha(B)$. If $\tau_\alpha(A) > t$, $\tau_\alpha(B) > t$, then there exists $\{B_{\lambda_1} : \lambda_1 \in \Lambda_1\}$, $\{B_{\lambda_2} : \lambda_2 \in \Lambda_2\}$ such that

$$\bigcup_{\lambda_1\in\Lambda_1}B_{\lambda_1} = A,\ \bigcup_{\lambda_2\in\Lambda_2}B_{\lambda_2} = B$$

and for any $\lambda_1 \in \Lambda_1$, $\beta_\alpha(B_{\lambda_1}) > t$, for any $\lambda_2 \in \Lambda_2$, $\beta_\alpha(B_{\lambda_2}) > t$. Now, for any $x \in A \cap B$, there exists $\lambda_{1x} \in \Lambda_1$, $\lambda_{2x} \in \Lambda_2$ such that $x \in B_{\lambda_{1x}} \cap B_{\lambda_{2x}}$. From the assumption, we know that

$$t < \beta_\alpha(B_{\lambda_{1x}}) \wedge \beta_\alpha(B_{\lambda_{2x}}) \leqslant \bigvee_{x\in C\subseteq B_{\lambda_{1x}}\cap B_{\lambda_{2x}}}\beta_\alpha(C)$$

and furthermore, there exists C_x such that

$$x \in C_x \subseteq B_{\lambda_{1x}} \cap B_{\lambda_{2x}} \subseteq A \cap B,\ \beta_\alpha(C_x) > t.$$

Since $\bigcup_{x\in A\cap B}C_x = A \cap B$, we have

$$t \leqslant \bigwedge_{x\in A\cap B}\beta_\alpha(C_x) \leqslant \bigvee_{\bigcup_{\lambda\in\Lambda}B_\lambda = A\cap B}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda) = \tau_\alpha(A \cap B).$$

Now, let $\tau_\alpha(A) \wedge \tau_\alpha(B) = k$. For any natural number n, we have $\tau_\alpha(A) > k - \frac{1}{n}$, $\tau_\alpha(B) > k - \frac{1}{n}$ and so $\tau_\alpha(A \cap B) \geqslant k - \frac{1}{n}$. Therefore $\tau_\alpha(A \cap B) \geqslant k = \tau_\alpha(A) \wedge \tau_\alpha(B)$. $\square$

Recall that if (X,τ) is a topological space and τ_α is the collection of all α-open sets in X, then an α-subbase of τ_α is a collection S of α-open sets such that every α-open set is the union of sets that are finite intersections of elements from S. Therefore we have the following definition.

Definition 2.2. $\varphi_\alpha \in \Im(P(X))$ is called an α-subbase of τ_α if $\varphi_\alpha^{\Cap}$ is an α-base of τ_α, where

$$\varphi_\alpha^{\Cap}\left(\bigcap_{\lambda\in\Lambda}B_\lambda\right) = \bigvee_{\bigcap_{\lambda\in\Lambda}B_\lambda=A}\bigwedge_{\lambda\in\Lambda}\varphi_\alpha(B_\lambda),\ \{B_\lambda : \lambda\in\Lambda\}\Subset P(X),$$

with "⋐" standing for "a finite subset of".

Theorem 2.3. *$\varphi_\alpha \in \Im(P(X))$ is an α-subbase of some fuzzifying α-topology if and only if $\varphi_\alpha^{(\cup)}(X) = 1$.*

Proof. We only demonstrate that $\varphi_\alpha^{\Cap}$ satisfies the second condition of Theorem 2.2, and others are obvious. In fact

$$\begin{aligned}\varphi_\alpha^{\Cap}(A)\wedge\varphi_\alpha^{\Cap}(B) &= \left(\bigvee_{\bigcap_{\lambda_1\in\Lambda_1}B_{\lambda_1}=A}\bigwedge_{\lambda_1\in\Lambda_1}\varphi_\alpha(B_{\lambda_1})\right)\wedge\left(\bigvee_{\bigcap_{\lambda_2\in\Lambda_2}B_{\lambda_2}=B}\bigwedge_{\lambda_2\in\Lambda_2}\varphi_\alpha(B_{\lambda_2})\right)\\ &= \bigvee_{\bigcap_{\lambda_1\in\Lambda_1}B_{\lambda_1}=A}\ \bigvee_{\bigcap_{\lambda_2\in\Lambda_2}B_{\lambda_2}=B}\left(\bigwedge_{\lambda_1\in\Lambda_1}\varphi_\alpha(B_{\lambda_1})\right)\wedge\left(\bigwedge_{\lambda_2\in\Lambda_2}\varphi_\alpha(B_{\lambda_2})\right)\\ &\leqslant \bigvee_{\bigcap_{\lambda\in\Lambda}B_\lambda=A\cap B}\left(\bigwedge_{\lambda\in\Lambda}\varphi_\alpha(B_\lambda)\right) = \varphi_\alpha^{\Cap}(A\cap B).\end{aligned}$$

Therefore if $x\in A\cap B$, then

$$\varphi_\alpha^{\Cap}(A)\wedge\varphi_\alpha^{\Cap}(B)\leqslant\varphi_\alpha^{\Cap}(A\cap B)\leqslant\bigvee_{x\in C\subseteq A\cap B}\varphi_\alpha^{\Cap}(C). \qquad\square$$

3. Fuzzifying α-compact spaces

Definition 3.1. A binary fuzzy predicate $K_\alpha\in\Im(\Im(P(X))\times P(X))$, called fuzzifying α-open covering [1], is given as $K_\alpha(\Re,A) := K(\Re,A)\otimes(\Re\subseteq\tau_\alpha)$. A unary fuzzy predicate $\Gamma_\alpha\in\Im(\Omega)$, called fuzzifying α-compactness [1], is given as follows:

$$(X,\tau)\in\Gamma_\alpha := (\forall\Re)(K_\alpha(\Re,X)\longrightarrow(\exists\wp)((\wp\leqslant\Re)\wedge K(\wp,X)\otimes FF(\wp)))$$

and if $A\subseteq X$, then $\Gamma_\alpha(A):=\Gamma_\alpha(A,\tau/A)$. It is obvious that $\Gamma_\alpha(X,\tau):=\Gamma(X,\tau_\alpha)$ and $\models K_\circ(\Re,A)\longrightarrow K_\alpha(\Re,A)$.

Theorem 3.1. *Let (X,τ) be a fuzzifying topological space, φ_α be an α-subbase of τ_α, and*

$\beta_1 := (\forall\Re)(K_{\varphi_\alpha}(\Re,X)\to\exists\wp((\wp\leqslant\Re)\wedge K(\wp,X)\otimes FF(\wp)))$,

where $K_{\varphi_\alpha}(\Re,X) := K(\Re,X)\otimes(\Re\subseteq\varphi_\alpha)$;

$\beta_2 := (\forall S)((S \text{ is a universal net in } X)\to\exists x((x\in X)\wedge(S\rhd^\alpha x))$;

$\beta_3 := (\forall S)((S\in N(X)\to(\exists T)(\exists x)((T<S)\wedge(x\in X)\wedge(T\rhd^\alpha x))$,

where "$T<S$" stands for "T is a subnet of S";

$\beta_4 := (\forall S)((S\in N(X)\to\neg(adh_\alpha S\equiv\phi))$;

$\beta_5 := (\forall\Re)(\Re\in\Im(P(X))\wedge\Re\subseteq F_\alpha\otimes fI(\Re)\to\exists x\forall A(A\in\Re\to x\in A))$.

Then $\models(X,\tau)\in\Gamma_\alpha\leftrightarrow\beta_i$ *$i = 1, 2, \ldots, 5$.*

Proof.

(1) Since $\varphi_\alpha\subseteq\tau_\alpha$, $[\Re\subseteq\varphi_\alpha]\leqslant[\Re\subseteq\tau_\alpha]$ for any $\Re\in\Im(P(X))$. Then $[K_{\varphi_\alpha}(\Re,X)]\leqslant[K_\alpha(\Re,X)]$. Therefore $\Gamma_\alpha(X,\tau)\leqslant[\beta_1]$.

(2) $[\beta_2] = \bigwedge\left\{\bigvee_{x\in X}[S\rhd^\alpha x] : S \text{ is a universal net in } X\right\}$.

(2.1) Assume X is finite. We set $X = \{x_1,\ldots,x_m\}$. For any universal net S in X, there exists $i_\circ\in\{1,\ldots,m\}$ with $S\tilde{\subset}\{x_{i_\circ}\}$. In fact, if not, then for any $i\in\{1,\ldots,m\}$, $S\tilde{\not\subset}\{x_i\}$, $S\tilde{\subset}X-\{x_i\}$ and $S\tilde{\subset}\bigcap_{i=1}^m(X-\{x_i\})=\phi$, a contradiction. Therefore $x_{i_\circ}\notin A$ and $N_{x_{i_\circ}}^\alpha(A)=0$ (see[18], Theorem 4.2 (1)) provided $S\tilde{\not\subset}A$, and furthermore $[S\rhd^\alpha x_{i_\circ}] = \bigwedge_{S\tilde{\not\subset}A}\left(1-N_{x_{i_\circ}}^\alpha(A)\right) = 1$. Therefore $[\beta_2] = 1\geqslant[\beta_1]$.

(2.2) In general, to prove that $[\beta_1]\leqslant[\beta_2]$ we prove that for any $\lambda\in[0,1]$, if $[\beta_2]<\lambda$, then $[\beta_1]<\lambda$. Assume for any $\lambda\in[0,1]$, $[\beta_2]<\lambda$. Then there exists a universal net S in X such that $\bigvee_{x\in X}[S\rhd^\alpha x]<\lambda$ and for any $x\in X$, $[S\rhd^\alpha x] = \bigwedge_{S\tilde{\not\subset}A}\left(1-N_x^\alpha(A)\right)<\lambda$, i.e., there exists $A\subseteq X$ with $S\tilde{\not\subset}A$ and $N_x^\alpha(A)>1-\lambda$. Since φ_α is an α-subbase of τ_α, $\varphi_\alpha^{\Cap}$ is an α-base of τ_α and from Definition 2.1, we have $\bigvee_{x\in B\subseteq A}\varphi_\alpha^{\Cap}(B)\geqslant N_x^\alpha(A)>1-\lambda$, i.e., there exists $B\subseteq A$ such that $x\in B\subseteq A$ and

$$\bigvee\left\{\min_{\lambda\in\Lambda}\varphi_\alpha(B_\lambda) : \bigcap_{\lambda\in\Lambda}B_\lambda = B, B_\lambda\subseteq X, \lambda\in\Lambda\right\} = \varphi_\alpha^{\Cap}(B) > 1-\lambda,$$

where Λ is finite. Therefore there exists a finite set Λ and $B_\lambda\subseteq X(\lambda\in\Lambda)$ such that $\bigcap_{\lambda\in\Lambda}B_\lambda = B$ and for any $\lambda\in\Lambda, \varphi_\alpha(B_\lambda)>1-\lambda$. Since $S\tilde{\not\subset}A$ and Λ is finite, there exists $\lambda(x)\in\Lambda$ such that $S\tilde{\not\subset}B_{\lambda(x)}$.We set $\Re_\circ(B_{\lambda(x)}) = \bigvee_{x\in X}\varphi_\alpha(B_{\lambda(x)})$. If $\wp\leqslant\Re_\circ$, then for any $\delta>0, \wp_\delta\subseteq\{B_{\lambda(x)}:x\in X\}$. Consequently, for any $B\in\wp_\delta, S\tilde{\not\subset}B$ and $S\tilde{\subset}B^c$ because S is a universal net. If $[FF(\wp)] = 1-\inf\{\delta\in[0,1]:F(\wp_\delta)\} = t$, then for any $n\in w$ (the non-negative integer), $\inf\{\delta\in[0,1]:F(\wp_\delta)\}<1-t+\frac{1}{n}$, and there exists $\delta_\circ<1-t+\frac{1}{n}$ such that $F(\wp_{\delta\circ})$. If $\delta_\circ=0$, then $P(X)=\wp_{\delta\circ}$ is finite and it is proved in (2.1). If $\delta_\circ>0$, then for any $B\in\wp_{\delta\circ}, S\tilde{\subset}B^c$. Since $F(\wp_{\delta\circ})$, we have $S\tilde{\subset}\bigcap\{B^c : B\in\wp_{\delta\circ}\}\neq\phi$. i.e., $\cup\wp_{\delta\circ}\neq X$and there exist $x_\circ\in X$ such that for any $B\in\wp_{\delta\circ}, x_\circ\notin B$. Therefore, if $x_\circ\in B$, then $B\notin\wp_{\delta\circ}$, i.e.,

$$\wp(B)<\delta\circ, K(\wp,X) = \bigwedge_{x\in X}\bigvee_{x\in B}\wp(B)\leqslant\bigvee_{x_\circ\in B}\wp(B)\leqslant\delta\circ<1-t+\frac{1}{n}.$$

Let $n\to\infty$. We obtain $K(\wp,X)\leqslant 1-t$ and $[K(\wp,X)\otimes FF(\wp)] = 0$. In addition, $[K_{\varphi_\alpha}(\Re_\circ,X)]\geqslant1-\lambda$. In fact, $[\Re_\circ\subseteq\varphi_\alpha] = 1$ and

$$[K(\Re_\circ, X)] = \bigwedge_{x\in X}\bigvee_{x\in B}\Re_\circ(B) \geqslant \bigwedge_{x\in X}\Re_\circ(B_{\lambda(x)}) \geqslant \bigwedge_{x\in X}\varphi_\alpha(B_{\lambda(x)}) \geqslant 1-\lambda$$

because $x \in B_{\lambda(x)}$. Now, we have

$$\begin{aligned}[\beta_1] &= (\forall\Re)(K_{\varphi_\alpha}(\Re, X) \to \exists\wp((\wp \leqslant \Re) \wedge K(\wp, X) \otimes FF(\wp))) \\ &\leqslant K_{\varphi_\alpha}(\Re_\circ, X) \to \exists\wp((\wp \leqslant \Re_\circ) \wedge K(\wp, X) \otimes FF(\wp)) \\ &= \min(1, 1 - K_{\varphi_\alpha}(\Re_\circ, X) + \bigvee_{\wp\leqslant\Re_\circ}[K(\wp, X) \otimes FF(\wp))] \leqslant \lambda.\end{aligned}$$

By noticing that λ is arbitrary, we have $[\beta_1] \leqslant [\beta_2]$.

(3) It is immediate that $[\beta_2] \leqslant [\beta_3]$.

(4) To prove that $[\beta_3] \leqslant [\beta_4]$, first we prove that $[\exists T((T < S) \wedge (T\rhd^\alpha x))] \leqslant [S \propto {}^\alpha x]$, where $[\exists T\ ((T < S) \wedge (T\rhd^\alpha x))] = \bigvee_{T<S}\bigwedge_{T\tilde{\not\subset} A}(1 - N_x^\alpha(A))$ and $[S\propto^\alpha x] = \bigwedge_{S\tilde{\not\supset} A}(1 - N_x^\alpha(A))$. Indeed, for any $T < S$ one can deduce $\{A : S\tilde{\not\supset}A\} \subseteq \{A : T\tilde{\not\subset}A\}$ as follows. Suppose $T = S\circ K$. If $S\tilde{\not\supset}A$, then there exists $m \in D$ such that $S(n) \notin A$ when $n \geqslant m$, where $\geqslant$ directs the domain D of S. Now, we will show that $T\tilde{\not\subset}A$. If not, then there exists $p \in E$ such that $T(q) \in A$ when $q \geqslant p$, where $\geqslant$ directs the domain E of T. Moreover, there exists $n_1 \in E$ such that $K(n_1) \geqslant m$ because $T < S$, and there exists $n_2 \in E$ such that $n_2 \geqslant n_1, p$ because $(E, \geqslant)$ is directed. So, $K(n_2) \geqslant K(n_1) \geqslant m, S\circ K(n_2) \notin A$ and $S\circ K(n_2) = T(n_2) \in A$. They are contrary. Hence $\{A : S\tilde{\not\supset}A\} \subseteq \{A : T\tilde{\not\subset}A\}$. Therefore

$$\begin{aligned}[\exists T((T < S) \wedge (T\rhd^\alpha x))] &= \bigvee_{T<S}\bigwedge_{T\tilde{\not\subset} A}(1 - N_x^\alpha(A)) \\ &= \bigvee_{T<S}\bigwedge_{\{A:T\tilde{\not\subset} A\}}(1 - N_x^\alpha(A)) \leqslant \bigwedge_{\{A:S\tilde{\not\supset} A\}}(1 - N_x^\alpha(A)) \\ &= \bigwedge_{S\tilde{\not\supset} A}(1 - N_x^\alpha(A)) = [S\propto^\alpha x].\end{aligned}$$

Therefore for any $x \in X$ and $S \in N(X)$ we have

$$\begin{aligned}[\beta_3] &= \bigwedge_{S\in N(X)}\bigvee_{x\in X}[\exists T((T < S) \wedge (T\rhd^\alpha x))] \leqslant \bigwedge_{S\in N(X)}\bigvee_{x\in X}[S\propto^\alpha x] \\ &= \bigwedge_{S\in N(X)}\neg\left(\bigwedge_{x\in X}(1 - [S\propto^\alpha x])\right) = \bigwedge_{S\in N(X)}[\neg(adh_\alpha S \equiv \phi)] \\ &= [\beta_4].\end{aligned}$$

(5) We want to show that $[\beta_4] \leqslant [\beta_5]$. For any $\Re \in \Im(P(X))$, assume $[fI(\Re)] = \lambda$. Then for any $\delta > 1 - \lambda$, if $A_1, \ldots, A_n \in \Re_\delta$, $A_1 \cap A_2 \cap \ldots \cap A_n \neq \phi$. In fact, we set $\wp(A_i) = \bigvee_{i=1}^n \Re(A_i)$. Then $\wp \leqslant \Re$ and $FF(\wp) = 1$. By putting $\varepsilon = \lambda + \delta - 1 > 0$, we obtain

$$\begin{aligned}\lambda - \varepsilon < \lambda &\leqslant [FF(\wp) \to (\exists x)(\forall B)(B \in \wp \to x \in B)] \\ &= \bigvee_{x\in X}\bigwedge_{x\notin B}(1 - \wp(B)).\end{aligned}$$

There exists $x_\circ \in X$ such that $\lambda - \varepsilon < \bigwedge_{x_\circ\notin B}(1 - \wp(B)), x_\circ \notin B$ implies $\wp(B) < 1 - \lambda + \varepsilon = \delta$ and $x_\circ \in \cap\wp_\delta = A_1 \cap A_2 \cap \ldots \cap A_n$. Now, we set $\vartheta_\delta = \{A_1 \cap A_2 \cap \ldots \cap A_n : n \in N, A_1, \ldots, A_n \in \Re_\delta\}$ and $S{:}\vartheta_\delta \to X, B \mapsto x_B \in B, B \in \vartheta_\delta$ and know that $(\vartheta_\delta, \subseteq)$ is a directed set and S is a net in X. Therefore

$$[\beta_4] \leqslant [\neg(adh_\alpha S \equiv \phi)] = \bigvee_{x\in X}\bigwedge_{S\tilde{\not\supset} A}(1 - N_x^\alpha(A)).$$

Assume $[\Re \subseteq F_\alpha] = \mu$. Then for any $B \in P(X)$, $\Re(B) \leqslant 1 + F_\alpha(B) - \mu$, and

$$\begin{aligned}[\Re \subseteq F_\alpha \otimes fI(\Re) &\to (\exists x)(\forall A)((A \in \Re) \to x \in A)] \\ &= \min(1, 2 - \mu - \lambda + \bigvee_{x\in X}\bigwedge_{x\notin A}(1 - \Re(A))).\end{aligned}$$

Therefore, it suffices to show that for any

$$x \in X,\ \bigwedge_{S\tilde{\not\supset} A}(1 - N_x^\alpha(A)) \leqslant 2 - \mu - \lambda + \bigwedge_{x\notin A}(1 - \Re(A)),$$

i.e.,

$$\bigvee_{x\notin A}\Re(A) \leqslant 2 - \mu - \lambda + \bigvee_{S\tilde{\not\supset} A}N_x^\alpha(A)$$

for some $\delta > 1 - \lambda$. For any $t \in [0,1]$, if $\bigvee_{x\notin A}\Re(A) > t$, then there exists $A_\circ$ such that $x_\circ \notin A_\circ$ and $\Re(A_\circ) > t$.

Case 1. $t \leqslant 1 - \lambda$, then $t \leqslant 2 - \mu - \lambda + \bigvee_{S\tilde{\not\supset} A}N_x^\alpha(A)$.

Case 2. $t > 1 - \lambda$. Here we set $\delta = \frac{1}{2}(t + 1 - \lambda)$ and have $A_\circ \in \Re_\delta, A_\circ \in \vartheta_\delta$. In addition,

$$t < \Re(A_\circ) \leqslant 1 + F_\alpha(A_\circ) - \mu, t + \mu - 1 \leqslant F_\alpha(A_\circ) = \tau_\alpha(A_\circ^c).$$

Since $A_\circ \in \vartheta_{\delta_\sim}$ we know that $S_B \in A_\circ$, *i.e.*, $S_B \notin A_\circ^c$ when $B \subseteq A_\circ$ and $S \not\subset A_\circ^c$. Therefore,

$$\begin{aligned}2 - \mu - \lambda + \bigvee_{S\tilde{\not\supset} A}N_x^\alpha(A) &\geqslant 2 - \mu - \lambda + N_x^\alpha(A_\circ^c) \geqslant 2 - \mu - \lambda \\ &+ \tau_\alpha(A_\circ^c) \geqslant t + (1 - \lambda) \geqslant t.\end{aligned}$$

By noticing that t is arbitrary, we have completed the proof.

(6) To prove that $[\beta_5] = [(X, \tau) \in \Gamma_\alpha]$ see [1] Theorem 3.3. □

The above theorem is a generalization of the following corollary.

Corollary 3.1. *The following are equivalent for a topological space* (X, τ).

(a) *X is an* α*-compact space.*
(b) *Every cover of X by members of an* α*-subbase of* τ_α *has a finite subcover.*
(c) *Every universal net in X* α*-converges to a point in X.*
(d) *Each net in X has a subnet that* α*-converges to some point in X.*
(e) *Each net in X has an* α*-adherent point.*
(f) *Each family of* α*-closed sets in X that has the finite intersection property has a non-void intersection.*

Definition 3.2. Let $\{(X_s, \tau_s){:}s \in S\}$ be a family of fuzzifying topological spaces, $\prod_{s\in S}X_s$ be the cartesian product of $\{X_s{:}s \in S\}$ and $\varphi = \{p_s^{-1}(U_s) : s \in S, U_s \in P(X_s)\}$, where $p_t : \prod_{s\in S}X_s \to X_t (t \in S)$ is a projection. For $\Phi \subseteq \varphi$, $S(\Phi)$ stands for the set of indices of elements in Φ. The α-base $\beta_\alpha \in \Im(\prod_{s\in S}X_s)$ of $\prod_{s\in S}(\tau_\alpha)_s$ is defined as

$V \in \beta_\alpha := (\exists \Phi)\left(\Phi \Subset \varphi \wedge \left(\bigcap \Phi = V\right)\right) \to \forall s(s \in S(\Phi) \to V_s \in (\tau_\alpha)_s)$, *i.e.*, $\beta_\alpha(V) = \bigvee_{\Phi \Subset \varphi, \bigcap \Phi = V} \bigwedge_{s \in S(\Phi)} (\tau_\alpha)_s(V_s)$.

Definition 3.3. Let (X,τ) and (Y,σ) be two fuzzifying topological space. A unary fuzzy predicate $O_\alpha \in \Im(Y^X)$, is called fuzzifying α-openness, is given as follows: $O_\alpha(f) := \forall U(U \in \tau_\alpha \to f(U) \in \sigma_\alpha)$. Intuitively, the degree to which f is α-open is $[O_\alpha(f)] = \bigwedge_{U \subseteq X} \min(1, 1 - \tau_\alpha(U) + \sigma_\alpha(f(U)))$.

Lemma 3.1. *Let (X,τ) and (Y,σ) be two fuzzifying topological space. For any $f \in Y^X$, $O_\alpha(f) := \forall B(B \in \beta_\alpha^X \to f(B) \in \sigma_\alpha)$, where β_α^X is an α-base of τ_α.*

Proof. Clearly, $[O_\alpha(f)] \leqslant \left[\forall U\left(U \in \beta_\alpha^X \to f(U) \in \sigma_\alpha\right)\right]$. Conversely, for any $U \subseteq X$, we are going to prove

$$\min(1, 1 - \tau_\alpha(U) + \sigma_\alpha(f(U))) \geqslant \left[\forall V\left(V \in \beta_\alpha^X \to f(V) \in \sigma_\alpha\right)\right].$$

If $\tau_\alpha(U) \leqslant \sigma_\alpha(f(U))$, it is hold clearly. Now assume $\tau_\alpha(U) > \sigma_\alpha(f(U))$. If $\Re \subseteq P(X)$ with $\bigcup \Re = U$, then $\bigcup_{V \in \Re} f(V) = f(\bigcup \Re) = f(U)$. Therefore

$$\begin{aligned}\tau_\alpha(U) - \sigma_\alpha(f(U)) &= \bigvee_{\Re \subseteq P(X), \bigcup \Re = U} \bigwedge_{V \in \Re} \beta_\alpha^X(V) - \bigvee_{\wp \subseteq P(Y), \bigcup \wp = f(U)} \bigwedge_{W \in \wp} \sigma_\alpha(W) \\ &\leqslant \bigvee_{\Re \subseteq P(X), \bigcup \Re = U} \bigwedge_{V \in \Re} \beta_\alpha^X(V) - \bigvee_{\Re \subseteq P(X), \bigcup \Re = U} \bigwedge_{V \in \Re} \sigma_\alpha(f(V)) \\ &\leqslant \bigvee_{\Re \subseteq P(X), \bigcup \Re = U} \bigwedge_{V \in \Re} \left(\beta_\alpha^X(V) - \sigma_\alpha(f(V))\right),\end{aligned}$$

$$\min(1, 1 - \tau_\alpha(U) + \sigma_\alpha(f(U))) \geqslant \bigvee_{\Re \subseteq P(X), \bigcup \Re = U} \bigwedge_{V \in \Re} \min\left(1, 1 - \beta_\alpha^X(V) + \sigma_\alpha(f(V))\right) \geqslant \left[\forall V\left(V \in \beta_\alpha^X \to f(V) \in \sigma_\alpha\right)\right]. \quad \square$$

Lemma 3.2. *For any family $\{(X_s, \tau_s) : s \in S\}$ of fuzzifying topological spaces. (1) $\models (\forall s)(s \in S \to p_s \in O_\alpha)$; and (2) $\models (\forall s)(s \in S \to p_s \in C_\alpha)$.*

Proof.

(1) For any $t \in S$, we have

$$O_\alpha(p_t) = \bigwedge_{U \in P\left(\prod_{s \in S} X_s\right)} \min\left(1, 1 - \left(\prod_{s \in S} (\tau_\alpha)_s\right)(U) + (\tau_\alpha)_t(p_t(U))\right).$$

Then it suffices to show that for any $U \in P\left(\prod_{s \in S} X_s\right)$, we have

$$(\tau_\alpha)_t(p_t(U)) \geqslant \left(\prod_{s \in S} (\tau_\alpha)_s\right)(U).$$

Assume

$$\left(\prod_{s \in S} (\tau_\alpha)_s\right)(U) = \bigvee_{\cup_{\lambda \in \Lambda} B_\lambda = U} \bigwedge_{\lambda \in \Lambda} \bigvee_{\Phi_\lambda \Subset \varphi, \cap \Phi_\lambda = B_\lambda} \bigwedge_{s \in S(\Phi_\lambda)} (\tau_\alpha)_s(V_s) > \mu,$$

where $\Phi_\lambda = \{p_s^{-1}(V_s) : s \in S(\Phi_\lambda)\} (\lambda \in \Lambda)$. Hence there exists $\{B_\lambda : \lambda \in \Lambda\} \subseteq P\left(\prod_{s \in S} X_s\right)$ such that $\bigcup_{\lambda \in \Lambda} B_\lambda = U$ and furthermore, for any $\lambda \in \Lambda$, there exists $\Phi_\lambda \Subset \varphi$ such that $\cap \Phi_\lambda = B_\lambda$ and $\bigcap_{s \in S(\Phi_\lambda)} p_s^{-1}(V_s) = B_\lambda$, where for any $s \in S(\Phi_\lambda)$ we have $(\tau_\alpha)_s(V_s) > \mu$. Thus $p_t(U) = p_t\left(\bigcup_{\lambda \in \Lambda} \bigcap_{s \in S(\Phi_\lambda)} p_s^{-1}(V_s)\right)$.

(1) If for any $\lambda \in \Lambda$, $\bigcap_{s \in S(\Phi_\lambda)} p_s^{-1}(V_s) = \phi$, then $U = \phi$, $p_t(U) = \phi$ and $(\tau_\alpha)_t(p_t(U)) = 1$. Therefore $(\tau_\alpha)_t(p_t(U)) \geqslant \left(\prod_{s \in S} (\tau_\alpha)_s\right)(U)$.

(2) If there exists $\lambda_\circ \in \Lambda$, such that $\phi \neq \bigcap_{s \in S(\Phi_\lambda)} p_s^{-1}(V_s) = B_{\lambda_\circ}$,
   - (i) If $t \notin S(\Phi_{\lambda_\circ})$, i.e., $t \in S - S(\Phi_{\lambda_\circ})$, $p_t(B_{\lambda_\circ}) = X_t$. Therefore $(\tau_\alpha)_t(p_t(B_{\lambda_\circ})) = (\tau_\alpha)_t(X_t) = 1$.
   - (ii) If $t \in S(\Phi_{\lambda_\circ})$, then $p_t(B_{\lambda_\circ}) = V_t \subseteq X_t$. Thus

$$\begin{aligned}p_t(U) &= p_t\left(\left(\bigcup_{t \in S(\Phi_{\lambda_\circ})} B_{\lambda_\circ}\right) \cup \left(\bigcup_{t \notin S(\Phi_{\lambda_\circ})} B_{\lambda_\circ}\right)\right) \\ &= \left(\bigcup_{t \in S(\Phi_{\lambda_\circ})} p_t(B_{\lambda_\circ})\right) \cup \left(\bigcup_{t \notin S(\Phi_{\lambda_\circ})} p_t(B_{\lambda_\circ})\right) = V_t \cup X_t = X_t.\end{aligned}$$

Hence $(\tau_\alpha)_t(p_t(U)) = (\tau_\alpha)_t(X_t) = 1$ or $(\tau_\alpha)_t(p_t(U)) = (\tau_\alpha)_t(V_t) > \mu$.
Therefore $(\tau_\alpha)_t(p_t(U)) \geqslant \left(\prod_{s \in S} (\tau_\alpha)_s\right)(U)$. Thus $O_\alpha(p_t) = 1$.

(2) From Lemma 3.1 in [17] we have $\models (\forall s)(s \in S \to p_s \in C)$. Furthermore, for any two fuzzifying topological spaces (X,τ) and (Y,σ) and $f \in Y^X$, we have $C(f) \leqslant C_\alpha(f)$ (Theorem 6.3 (3) in [18]). Therefore $\models (\forall s)(s \in S \to p_s \in C_\alpha)$. $\square$

Theorem 3.2. *Let $\{(X_s, \tau_s) : s \in S\}$ be the family of fuzzifying topological spaces, then*

$$\models \exists U\left(U \subseteq \prod_{s \in S} X_s \wedge \Gamma_\alpha(U, \tau/U) \wedge \exists x(x \in Int_\alpha(U))\right) \to \exists T(T \Subset S \wedge \forall t(t \in S - T \wedge \Gamma_\alpha(X_t, \tau_t))).$$

Proof. It suffices to show that

$$\bigvee_{U \in P\left(\prod_{s \in S} X_s\right)} \left(\Gamma_\alpha(U, \tau/U) \wedge \bigvee_{x \in \prod_{s \in S} X_s} N_x^\alpha(U)\right) \leqslant \bigvee_{T \Subset S} \bigwedge_{t \in S - T} \Gamma_\alpha(X_t, \tau_t).$$

Indeed, if

$$\bigvee_{U \in P\left(\prod_{s \in S} X_s\right)} \left(\Gamma_\alpha(U, \tau/U) \wedge \bigvee_{x \in \prod_{s \in S} X_s} N_x^\alpha(U)\right) > \mu > 0,$$

then there exists $U \in P\left(\prod_{s \in S} X_s\right)$ such that $\Gamma_\alpha(U, \tau/U) > \mu$ and $\bigvee_{x \in \prod_{s \in S} X_s} N_x^\alpha(U) > \mu$, where $N_x^\alpha(U) = \bigvee_{x \in V \subseteq U} \left(\prod_{s \in S} (\tau_\alpha)_s\right)(V)$. Furthermore, there exists V such that $x \in V \subseteq U$ and $\left(\prod_{s \in S} (\tau_\alpha)_s\right)(V) > \mu$. Since β_α is an α-base of $\prod_{s \in S} (\tau_\alpha)_s$,

$$\left(\prod_{s\in S}(\tau_\alpha)_s\right)(V)=\bigvee_{\cup_{\lambda\in\Lambda}B_\lambda=V}\bigwedge_{\lambda\in\Lambda}\beta_\alpha(B_\lambda)=\bigvee_{\cup_{\lambda\in\Lambda}B_\lambda=V}\bigwedge_{\lambda\in\Lambda}\bigvee_{\Phi_\lambda\Subset\varphi,\cap\Phi_\lambda=B_\lambda}\bigwedge_{s\in S(\Phi_\lambda)}(\tau_\alpha)_s(V_s)>\mu,$$

where

$$\Phi_\lambda=\left\{p_s^{-1}(V_s):s\in S(\Phi_\lambda)\right\}(\lambda\in\Lambda).$$

Hence there exists $\{B_\lambda:\lambda\in\Lambda\}\subseteq P\left(\prod_{s\in S}X_s\right)$ such that $\cup_{\lambda\in\Lambda}B_\lambda=V$. Furthermore, for any $\lambda\in\Lambda$, there exists $\Phi_\lambda\Subset\varphi$ such that $\cap\Phi_\lambda=B_\lambda$ and for any $s\in S(\Phi_\lambda)$, we have $(\tau_\alpha)_s(V_s)>\mu$. Since $x\in V$, there exists B_{λ_x} such that $x\in B_{\lambda_x}\subseteq V\subseteq U$. Hence there exists $\Phi_{\lambda_x}\Subset\varphi$ such that $\cap\Phi_{\lambda_x}=B_{\lambda_x}$ and

$$\bigcap_{s\in S(\Phi_\lambda)}p_s^{-1}(V_s)=B_{\lambda_x}\subseteq\prod_{s\in S}X_s$$

and for any $s\in S(\Phi_\lambda)$, we have $(\tau_\alpha)_s(V_s)>1-\mu$. By

$$\bigcap_{s\in S(\Phi_\lambda)}p_s^{-1}(V_s)=B_{\lambda_x},$$

we have $P_\delta(B_{\lambda_x})=V_\delta\subseteq X_\delta$, if $\delta\in S(\Phi_{\lambda_x})$; $P_\delta(B_{\lambda_x})=X_\delta$, if $\delta\in S-S(\Phi_{\lambda_x})$. Since $B_{\lambda_x}\subseteq U$, for any $\delta\in S-S(\Phi_{\lambda_x})$, we have $P_\delta(U)\supseteq P_\delta(B_{\lambda_x})=X_\delta$ and $P_\delta(U)=X_\delta$. On the other hand, since for any $s\in S$ and

$$U_s\in P(X_s),\left(\prod_{t\in S}(\tau_\alpha)_t\right)\left(p_s^{-1}(U_s)\right)\geqslant(\tau_\alpha)_s(U_s),$$

we have, for any

$$s\in S,\ I_\alpha(p_s)=\bigwedge_{U_s\in P(X_s)}\min\left(1,1-(\tau_\alpha)_s(U_s)+\prod_{t\in S}(\tau_\alpha)_t\left(p_s^{-1}(U_s)\right)\right)=1.$$

Furthermore, since by Theorem 4.4 [1], we have $\vDash\Gamma_\alpha(X,\tau)\otimes I_\alpha(f)\to\Gamma_\alpha(f(X))$, then $\Gamma_\alpha(U,\tau/U)=\Gamma_\alpha(U,\tau/U)\otimes I_\alpha(p_\delta)\leqslant\Gamma_\alpha(P_\delta(U),\tau_\delta)=\Gamma_\alpha(X_\delta,\tau_\delta)$ for each $\delta\in S-S(\Phi_\lambda)$. Therefore,

$$\bigvee_{T\Subset S}\bigwedge_{t\in S-T}\Gamma_\alpha(X_t,\tau_t)\geqslant\bigwedge_{\delta\in S-S(\Phi_\lambda)}\Gamma_\alpha(X_\delta,\tau_\delta)\geqslant\Gamma_\alpha(U,\tau/U)>\mu.\quad\square$$

The above theorem is a generalization of the following corollary.

Corollary 3.2. *If there exists a coordinate α-neighborhood α-compact subset U of some point $x\in X$ of the product space, then all except a finite number of coordinate spaces are α-compact.*

Lemma 3.3. *For any fuzzifying topological space (X,τ), $A\subseteq X$, $\models T_2^\alpha(X,\tau)\to T_2^\alpha(A,\tau/A)$.*

Proof.

$$\begin{aligned}\left[T_2^\alpha(X,\tau)\right]&=\bigwedge_{x,y\in X,x\neq y}\bigvee_{U,V\in P(X),U\cap V=\phi}\left(N_x^\alpha(U),N_y^\alpha(V)\right)\\&\leqslant\bigwedge_{x,y\in X,x\neq y}\bigvee_{(U\cap A)\cap(V\cap A)=\phi}\left(N_x^{\alpha^A}(U\cap A),N_y^{\alpha^A}(V\cap A)\right)\\&\leqslant\bigwedge_{x,y\in A,x\neq y}\bigvee_{U'\cap V'=\phi,U',V'\in P(A)}\left(N_x^{\alpha^A}(U'),N_y^{\alpha^A}(V')\right)\\&=T_2^\alpha(A,\tau_{/A}),\end{aligned}$$

where $N_x^{\alpha^A}(U)=\bigvee_{x\in C\subseteq U}\tau_\alpha/A(C)$ and $\tau_\alpha/A(B)=\bigvee_{B=V\cap A}\tau_\alpha(V)$. $\square$

Lemma 3.4. *For any fuzzifying α-topological space $(X,\tau),\models T_2^\alpha(X,\tau)\otimes\Gamma_\alpha(X,\tau)\to T_4^\alpha(X,\tau)$.*

For the definition of $T_4^\alpha(X,\tau)$ see [21].

Proof. If $\left[T_2^\alpha(X,\tau)\otimes\Gamma_\alpha(X,\tau)\right]=0$, then the result holds. Now, suppose that $\left[T_2^\alpha(X,\tau)\otimes\Gamma_\alpha(X,\tau)\right]>\lambda>0$. Then $T_2^\alpha(X,\tau)+\Gamma_\alpha(X,\tau)-1>\lambda>0$. Therefore from Theorem 4.6 [1],

$$\begin{aligned}&T_2^\alpha(X,\tau)\otimes(\Gamma_\alpha(A)\wedge\Gamma_\alpha(B))\wedge(A\cap B=\phi)\models^{ws}T_2^\alpha(X,\tau)\\&\quad\to(\exists U)(\exists V)((U\in\tau_\alpha)\wedge(V\\&\quad\in\tau_\alpha)\wedge(A\subseteq U)\wedge(B\subseteq V)\wedge(A\cap B=\phi)).\end{aligned}$$

Then for any $A,B\subseteq X,A\cap B=\phi$,

$$T_2^\alpha(X,\tau)\otimes(\Gamma_\alpha(A)\wedge\Gamma_\alpha(B))\leqslant\bigvee_{U\cap V=\phi,A\subseteq U,B\subseteq V}\min(\tau_\alpha(U),\tau_\alpha(V))$$

or equivalently

$$T_2^\alpha(X,\tau)\leqslant\Gamma_\alpha(A)\wedge\Gamma_\alpha(B)\to\bigvee_{U\cap V=\phi,A\subseteq U,B\subseteq V}\min(\tau_\alpha(U),\tau_\alpha(V)).$$

Hence for any

$$\begin{aligned}&A,\ B\subseteq X,\ A\cap B=\phi,\ 1-[\Gamma_\alpha(A)\wedge\Gamma_\alpha(B)]\\&\quad+\bigvee_{U\cap V=\phi,A\subseteq U,B\subseteq V}\min(\tau_\alpha(U),\tau_\alpha(V))+\Gamma_\alpha(X,\tau)-1>\lambda.\end{aligned}$$

From Theorem 4.1(1) in [1] we have $\models\Gamma_\alpha(X,\tau)\otimes A\in\mathsf{F}_\alpha\to\Gamma_\alpha(A)$. Then

$$\begin{aligned}\Gamma_\alpha(X,\tau)+[\tau_\alpha(A^c)\wedge\tau_\alpha(B^c)]-1&=(\Gamma_\alpha(X,\tau)+\tau_\alpha(A^c)-1)\\&\quad\wedge(\Gamma_\alpha(X,\tau)+\tau_\alpha(B^c)-1)\\&\leqslant(\Gamma_\alpha(X,\tau)\otimes\tau_\alpha(A^c))\\&\quad\wedge(\Gamma_\alpha(X,\tau)\otimes\tau_\alpha(B^c))\\&\leqslant[\Gamma_\alpha(A)\wedge\Gamma_\alpha(B)].\end{aligned}$$

Thus $\Gamma_\alpha(X,\tau)-[\Gamma_\alpha(A)\wedge\Gamma_\alpha(B)]-1\leqslant-[\tau_\alpha(A^c)\wedge\tau_\alpha(B^c)]$. So, $1-[\tau_\alpha(A^c)\wedge\tau_\alpha(B^c)]+\bigvee_{U\cap V=\phi,A\subseteq U,B\subseteq V}\min(\tau_\alpha(U),\tau_\alpha(V))>\lambda$, i.e.,

$$\begin{aligned}T_4^\alpha(X,\tau)=&\bigwedge_{A\cap B=\phi}\min(1,1-[\tau_\alpha(A^c)\wedge\tau_\alpha(B^c)]\\&+\bigvee_{U\cap V=\phi,A\subseteq U,B\subseteq V}\min(\tau_\alpha(U),\tau_\alpha(V)))>\lambda.\quad\square\end{aligned}$$

The above lemma is a generalization of the following corollary.

Corollary 3.3. *Every α-compact α-Hausdorff topological space is α-normal.*

Lemma 3.5. *For any fuzzifying α-topological space (X,τ), $\models T_2^\alpha(X,\tau)\otimes\Gamma_\alpha(X,\tau)\to T_3^\alpha(X,\tau)$. For the definition of $T_3^\alpha(X,\tau)$ see* 21, Definition 2.2].

Proof. Immediate, set $A=\{x\}$ in the above lemma. $\square$

The above lemma is a generalization of the following corollary.

Corollary 3.4. *Every α-compact α-Hausdorff topological space is α-regular.*

Theorem 3.3. *For any fuzzifying topological space (X,τ) and $A\subseteq X$, $\models T_2^\alpha(X,\tau)\otimes\Gamma_\alpha(A)\to A\in \mathsf{F}_\alpha$.*

Proof. For any $\{x\}\subset A^c$, we have $\{x\}\cap A=\phi$ and $\Gamma_\alpha(\{x\})=1$. By Theorem 4.6 [1],

$$\left[T_2^\alpha(X,\tau)\otimes(\Gamma_\alpha(A)\wedge\Gamma_\alpha(\{x\}))\right]\leqslant\bigvee_{G\cap H_x=\phi,\ A\subseteq G,\ x\in H_x}\min(\tau_\alpha(G),\tau_\alpha(H_x))\Big).$$

Assume

$$\beta_x=\{H_x: A\cap H_x=\phi, x\in H_x\},\quad\bigcup_{x\in X\setminus A}f(x)\supseteq A^c$$

and

$$\bigcup_{x\in A^c}f(x)\cap A=\bigcup_{x\in A^c}(f(x)\cap A)=\phi.$$

So, $\bigcup_{x\in A^c}f(x)=A^c$.

Therefore

$$\begin{aligned}[T_2^\alpha(X,\tau)\otimes\Gamma_\alpha(A)]&\leqslant\bigvee_{G\cap H_x=\phi,\ A\subseteq G,\ x\in H_x}\tau_\alpha(H_x)\\&\leqslant\bigwedge_{x\in A^c}\bigvee_{A\cap H_x=\phi,x\in H_x}\tau_\alpha(H_x)\\&=\bigvee_{f\in\prod_{x\in A^c}\beta_x}\bigwedge_{x\in A^c}\tau_\alpha(f(x))\\&\leqslant\bigvee_{f\in\prod_{x\in A^c}\beta_x}\tau_\alpha\left(\bigcup_{x\in A^c}f(x)\right)=\bigvee_{f\in\prod_{x\in X\setminus A}\beta_x}\tau_\alpha(A^c)\\&=\mathsf{F}_\alpha(A).\qquad\square\end{aligned}$$

The above theorem is a generalization of the following corollary.

Corollary 3.5. *α-compact subspace of an α-Hausdorff topological space is α-closed.*

4. Fuzzifying locally α-compactness

Definition 4.1. Let Ω be a class of fuzzifying topological spaces. A unary fuzzy predicate $L_\alpha C\in\Im(\Omega)$, called fuzzifying locally α-compactness, is given as follows:

$(X,\tau)\in L_\alpha C:=(\forall x)(\exists B)((x\in Int_\alpha(B)\otimes\Gamma_\alpha(B,\tau/B))$. Since $[x\in Int_\alpha(X)]=N_x^\alpha(X)=1$, then $L_\alpha C(X,\tau)\geqslant\Gamma_\alpha(X,\tau)$. Therefore, $\models(X,\tau)\in\Gamma_\alpha\to(X,\tau)\in L_\alpha C$.

Also, since $\models(X,\tau)\in\Gamma\to(X,\tau)\in LC$ [24] and $\models(X,\tau)\in\Gamma_\alpha\to(X,\tau)\in\Gamma$[1], $\models(X,\tau)\in\Gamma_\alpha\to(X,\tau)\in LC$, which is a generalization of Corollary 4.4 [26].

Theorem 4.1. *For any fuzzifying topological space (X,τ) and $A\subseteq X$, $\models(X,\tau)\in L_\alpha C\otimes A\in\mathsf{F}_\alpha\to(A,\tau/A)\in L_\alpha C$.*

Proof. We have

$$L_\alpha C(X,\tau)=\bigwedge_{x\in X}\bigvee_{B\subseteq X}\max\left(0,N_x^{\alpha^X}(B)+\Gamma_\alpha(B,\tau/B)-1\right)$$

and

$$L_\alpha C(A,\tau/A)=\bigwedge_{x\in A}\bigvee_{G\subseteq A}\max\left(0,N_x^{\alpha^A}(G)+\Gamma_\alpha(G,(\tau/A)/G)-1\right).$$

Now, suppose that $[(X,\tau)\in L_\alpha C\otimes A\in\mathsf{F}_\alpha]>\lambda>0$. Then for any $x\in A$, there exists $B\subseteq X$ such that

$$N_x^{\alpha^X}(B)+\Gamma_\alpha(B,\tau/B)+\tau_\alpha(X-A)-2>\lambda.\qquad(*)$$

Set $E=A\cap B\in P(A)$. Then

$$N_x^{\alpha^A}(E)=\bigvee_{E=C\cap B}N_x^{\alpha^X}(C)\geqslant N_x^{\alpha^X}(B)$$

and for any $U\in P(E)$, we have

$$\begin{aligned}(\tau_\alpha/A)_\alpha/E(U)&=\bigvee_{U=C\cap E}\tau_\alpha/A(C)=\bigvee_{U=C\cap E}\bigvee_{C=D\cap A}\tau_\alpha(D)\\&=\bigvee_{U=D\cap A\cap E}\tau_\alpha(D)=\bigvee_{U=D\cap E}\tau_\alpha(D).\end{aligned}$$

Similarly,

$$(\tau_\alpha/B)_\alpha/E(U)=\bigvee_{U=D\cap E}\tau_\alpha(D).$$

Thus, $(\tau_\alpha/B)_\alpha/E=(\tau_\alpha/A)_\alpha/E$ and $\Gamma_\alpha(E,(\tau/A)/E)=\Gamma_\alpha(E,(\tau/B)/E)$. Furthermore,

$$\begin{aligned}[E\in\mathsf{F}_\alpha/B]&=\tau_\alpha/B(B-E)=\tau_\alpha/B(B\cap E^c)=\bigvee_{B\cap E^c=B\cap D}\tau_\alpha(D)\\&\geqslant\tau_\alpha(X-A)=\mathsf{F}_\alpha(A).\end{aligned}$$

Since $\models(X,\tau)\in\Gamma_\alpha\otimes A\in\mathsf{F}_\alpha\to(A,\tau/A)\in\Gamma_\alpha$ (see [1], Theorem 4.1 (1)], from (*) we have for any $x\in A$ that

$$\begin{aligned}&\bigvee_{G\subseteq A}\max\left(0,N_x^{\alpha^A}(G)+\Gamma_\alpha(G,(\tau/A)/G)-1\right)\\&\geqslant N_x^{\alpha^A}(E)+\Gamma_\alpha(E,(\tau/A)/E)-1\\&=N_x^{\alpha^A}(E)+\Gamma_\alpha(E,(\tau/B)/E)-1\\&\geqslant N_x^{\alpha^X}(B)+[\Gamma_\alpha(B,\tau/B)\otimes E\in\mathsf{F}_\alpha/B]-1\\&\geqslant N_x^{\alpha^X}(B)+\Gamma_\alpha(B,\tau/B)+[E\in\mathsf{F}_\alpha/B]-2\\&\geqslant N_x^{\alpha^X}(B)+\Gamma_\alpha(B,\tau/B)+[A\in\mathsf{F}_\alpha]-2>\lambda.\end{aligned}$$

Therefore

$$\begin{aligned}L_\alpha C(A,\tau/A)&=\bigwedge_{x\in A}\bigvee_{G\subseteq A}\max\left(0,N_x^{\alpha^A}(G)+\Gamma_\alpha(G,(\tau/A)/G)-1\right)\\&>\lambda.\end{aligned}$$

Hence $[(X,\tau)\in L_\alpha C\otimes A\in\mathsf{F}_\alpha]\leqslant L_\alpha C(A,\tau/A)$. $\square$

As a crisp result of the above theorem we have the following corollary.

Corollary 4.1. *Let A be an α-closed subset of locally α-compact space (X,τ). Then A with the relative topology τ/A is locally α-compact.*

The following theorem is a generalization of the statement "If X is an α-Hausdorff topological space and A is an α-dense α-locally compact subspace, then A is α-open", where A is an α-dense in a topological space X if and only if the α-closure of A is X.

Theorem 4.2. *For any fuzzifying α-topological space* (X, τ) *and* $A \subseteq X$,

$$\models T_2^\alpha(X,\tau) \otimes L_\alpha C(A,\tau/A) \otimes (Cl_\alpha(A) \equiv X) \to A \in \tau_\alpha.$$

Proof. Assume

$$[T_2^\alpha(X,\tau) \otimes L_\alpha C(A,\tau/A) \otimes (Cl_\alpha(A) \equiv X)] > \lambda > 0.$$

Then

$$L_\alpha C(A,\tau/A) > \lambda - \left[T_2^\alpha(X,\tau) \otimes (Cl_\alpha(A) \equiv X)\right] + 1 = \lambda' > \lambda,$$

i.e.,

$$\bigwedge_{x\in A}\bigvee_{B\subseteq A} \max\left(0, N_x^{\alpha^A}(B) + \Gamma_\alpha(B,(\tau/A)/B) - 1\right) > \lambda'.$$

Thus for any $x \in A$, there exists $B_x \subseteq A$ such that

$$N_x^{\alpha^A}(B_x) + \Gamma_\alpha(B_x,(\tau/A)/B_x) - 1 > \lambda'.$$

i.e.,

$$\bigvee_{H\cap A=B_x}\bigvee_{x\in K\subseteq H} \tau_\alpha(K) + \Gamma_\alpha(B_x,(\tau/A)/B_x) - 1 > \lambda'.$$

Hence there exists K_x such that $K_x \cap A = B_x, \tau_\alpha(K_x) + \Gamma_\alpha(B_x,(\tau/A)/B_x) - 1 > \lambda'$. Therefore $\tau_\alpha(K_x) > \lambda'$.

(1) If for any $x \in A$ there exists K_x such that $x \in K_x \subseteq B_x \subseteq A$, then $\bigcup_{x\in A} K_x = A$ and $\tau_\alpha(A) = \tau_\alpha\left(\bigcup_{x\in A} K_x\right) \geqslant \bigwedge_{x\in A} \tau_\alpha(K_x) \geqslant \lambda' > \lambda$.

(2) If there exists $x_\circ \in A$ such that

$$K_{x_\circ} \cap \left(B_{x_\circ}^c\right) \neq \phi,\ \tau_\alpha(K_{x_\circ}) + \Gamma_\alpha(B_{x_\circ},(\tau/A)/B_{x_\circ}) - 1 > \lambda'.$$

From the hypothesis

$$\left[T_2^\alpha(X,\tau) \otimes L_\alpha C(A,\tau/A) \otimes (Cl_\alpha(A) \equiv X)\right] > \lambda > 0,$$

we have $\left[T_2^\alpha(X,\tau) \otimes (Cl_\alpha(A) \equiv X)\right] \neq 0$. So

$$\tau_\alpha(K_{x_\circ}) + \Gamma_\alpha(B_{x_\circ},(\tau/A)/B_{x_\circ}) - 1 + [T_2^\alpha(X,\tau) \otimes (Cl_\alpha(A) \equiv X)] - 1 > \lambda.$$

Therefore

$$\tau_\alpha(K_{x_\circ}) + \Gamma_\alpha(B_{x_\circ},(\tau/A)/B_{x_\circ}) - 1 + T_2^\alpha(X,\tau) + [(Cl_\alpha(A) \equiv X)] - 1 - 1 > \lambda.$$

Since

$$\begin{aligned}(\tau_\alpha/A)_\alpha/B_{x_\circ}(U) &= \bigvee_{U=C\cap B_{x_\circ}} \tau_\alpha/A(C)\\ &= \bigvee_{U=C\cap B_{x_\circ}}\bigvee_{C=D\cap A} \tau_\alpha(D)\\ &= \bigvee_{U=D\cap B_{x_\circ}} \tau_\alpha(D) = \tau_\alpha/B_{x_\circ}(U),\end{aligned}$$

$$\Gamma_\alpha(B_{x_\circ},(\tau/A)/B_{x_\circ}) = \Gamma_\alpha(B_{x_\circ},\tau/B_{x_\circ}).$$

From Theorem 3.3 we have

$$\begin{aligned}\tau_\alpha\left(B_{x_\circ}^c\right) &\geqslant T_2^\alpha(X,\tau) \otimes \Gamma_\alpha(B_{x_\circ},\tau/B_{x_\circ})\\ &\geqslant T_2^\alpha(X,\tau) + \Gamma_\alpha(B_{x_\circ},\tau/B_{x_\circ}) - 1.\end{aligned}$$

Hence

$$\tau_\alpha(K_{x_\circ}) + \tau_\alpha\left(B_{x_\circ}^c\right) + [Cl_\alpha(A) \equiv X] - 2 > \lambda.$$

Now, for any $y \in A^c$ we have

$$[Cl_\alpha(A) \equiv X] = \bigwedge_{x\in X}\left(1 - N_x^{\alpha^X}(A^c)\right) \leqslant 1 - N_y^{\alpha^X}(A^c).$$

Since (X, τ) is a fuzzifying α-topological space,

$$\begin{aligned}\tau_\alpha(K_{x_\circ}) + \tau_\alpha\left(B_{x_\circ}^c\right) - 1 &\leqslant \tau_\alpha(K_{x_\circ}) \otimes \tau_\alpha\left(B_{x_\circ}^c\right) \leqslant \tau_\alpha(K_{x_\circ}) \wedge \tau_\alpha\left(B_{x_\circ}^c\right)\\ &\leqslant \tau_\alpha\left(K_{x_\circ} \cap B_{x_\circ}^c\right) \leqslant N_y^{\alpha^X}\left(K_{x_\circ} \cap B_{x_\circ}^c\right)\\ &\leqslant N_y^{\alpha^X}(A^c),\end{aligned}$$

where

$$\begin{aligned}y \in K_{x_\circ} \cap B_{x_\circ}^c &\subseteq H_{x_\circ} \cap (H_{x_\circ} \cap A)^c = H_{x_\circ} \cap \left(H_{x_\circ}^c \cup A^c\right)\\ &= H_{x_\circ} \cap A^c \subseteq A^c.\end{aligned}$$

Therefore

$$\begin{aligned}0 < \lambda &< \tau_\alpha(K_{x_\circ}) + \tau_\alpha\left(B_{x_\circ}^c\right) + [Cl_\alpha(A) \equiv X] - 2\\ &= \tau_\alpha(K_{x_\circ}) + \tau_\alpha\left(B_{x_\circ}^c\right) - 1 + [Cl_\alpha(A) \equiv X] - 1\\ &\leqslant N_y^{\alpha^X}(A^c) + 1 - N_y^{\alpha^X}(A^c) - 1 = 0,\end{aligned}$$

a contradiction. So, case (2) does not hold. We complete the proof. □

Theorem 4.3. *For any fuzzifying α-topological space* (X, τ),

$$\models T_2^\alpha(X,\tau) \otimes (L_\alpha C(X,\tau))^2 \to \forall x \forall U\left(U \in N_x^{\alpha^X} \to \exists V\left(V \in N_x^{\alpha^X} \wedge Cl_\alpha(V) \subseteq U \wedge \Gamma_\alpha(V)\right)\right),$$

where $(L_\alpha C(X,\tau))^2 := L_\alpha C(X,\tau) \otimes L_\alpha C(X,\tau)$.

Proof. We need to show that for any x and U, $x \in U$,

$$T_2^\alpha(X,\tau) \otimes (L_\alpha C(X,\tau))^2 \otimes N_x^{\alpha^X}(U) \leqslant \bigvee_{V\subseteq X}\left(N_x^{\alpha^X}(V) \wedge \bigwedge_{y\in U^c} N_x^{\alpha^X}(V^c) \wedge \Gamma_\alpha(V,\tau/V)\right).$$

Assume that $T_2^\alpha(X,\tau) \otimes (L_\alpha C(X,\tau))^2 \otimes N_x^{\alpha^X}(U) > \lambda > 0$. Then for any $x \in X$ there exists C such that

$$T_2^\alpha(X,\tau) + N_x^{\alpha^X}(C) + (L_\alpha C(X,\tau))^2 + N_x^{\alpha^X}(U) - 3 > \lambda. \qquad (*)$$

Since (X, τ) is fuzzifying α-topological space,

$$\begin{aligned}N_x^{\alpha^X}(C) + N_x^{\alpha^X}(U) - 1 &\leqslant N_x^{\alpha^X}(C) \otimes N_x^{\alpha^X}(U) \leqslant N_x^{\alpha^X}(C) \wedge N_x^{\alpha^X}(U)\\ &\leqslant N_x^{\alpha^X}(C \cap U) = \bigvee_{x\in W\subseteq C\cap U} \tau_\alpha(W).\end{aligned}$$

Therefore there exists W such that $x \in W \subseteq C \cap U$, and $T_2^\alpha(X,\tau) + (L_\alpha C(X,\tau))^2 + \tau_\alpha(W) - 2 > \lambda$. By Lemmas 3.3 and 3.5 we have $T_2^\alpha(X,\tau) \leqslant T_2^\alpha(C,\tau/C)$ and

$$\begin{aligned}T_2^\alpha(C,\tau/C) + \Gamma_\alpha(C,\tau/C) - 1 &\leqslant T_2^\alpha(C,\tau/C) \otimes \Gamma_\alpha(C,\tau/C)\\ &\leqslant T_3^\alpha(C,\tau/C).\end{aligned}$$

Thus $T_3^\alpha(X,\tau) + \Gamma_\alpha(C,\tau/C) + \tau_\alpha(W) - 2 > \lambda$. Since for any $x \in W \subseteq U$, we have

$$T_3^\alpha(C,\tau/C) \leqslant 1-\tau_\alpha/C(W) + \bigvee_{G\subseteq C}\left(\left(N_x^{\alpha^C}(G)\wedge\bigwedge_{y\in C-W}N_y^{\alpha^C}(C-G)\right)\right),$$

so there exists $G, x\in G\subseteq W$ such that

$$\left(\left(N_x^{\alpha^C}(G)\wedge\bigwedge_{y\in C-W}N_y^{\alpha^C}(C-G)\right)\right) \geqslant T_3^\alpha(C,\tau/C)+\tau_\alpha/C(W)-1 \geqslant T_3^\alpha(C,\tau/C)+\tau_\alpha(W)-1$$

and

$$\left(\left(N_x^{\alpha^C}(G)\wedge\bigwedge_{y\in C-W}N_y^{\alpha^C}(C-G)\right)\right)+\Gamma_\alpha(C,\tau/C)-1>\lambda.$$

Thus

$$N_x^{\alpha^C}(G)=\bigvee_{D\cap C=G}N_x^{\alpha^X}(D)=N_x^{\alpha^X}(G\cup C^c)>\lambda' = \lambda+1-\Gamma_\alpha(C,\tau/C)\geqslant\lambda.$$

Furthermore, for any

$$y\in C-W, N_y^{\alpha^C}(C-G)=\bigvee_{D\cap C=C\cap G^c}N_y^{\alpha^X}(G^c\cup C^c)=N_y^{\alpha^X}(G^c)>\lambda'$$

and

$$N_x^{\alpha^X}(G)=N_x^{\alpha^X}((G\cup C^c)\cap C)\geqslant N_x^{\alpha^X}(G\cup C^c)\wedge N_x^{\alpha^X}(C)>\lambda'.$$

Since $N_y^{\alpha^X}(G^c)=\bigvee_{x\in B^c\subseteq G^c}\tau_\alpha(B^c)>\lambda'$, for any $y\in C-W$, there exists B_y^c such that $y\in B_y^c\subseteq G^c$ and $\tau_\alpha(B_y^c)>\lambda'$. Set $B^c=\bigcup_{y\in C-W}B_y^c$. Then $C-W\subseteq B^c\subseteq G^c$ and $\tau_\alpha(B^c)\geqslant\bigwedge_{y\in C-W}\tau_\alpha(B_y^c)\geqslant\lambda'$. Again, set $V=B\cap C$, then $V\subseteq(C-W)^c\cap C=(C^c\cup W)\cap C=C\cap W=W\subseteq U\cap C$ and $V^c=B^c\cup C^c$.Since (X,τ) is fuzzifying α-topological space,

$$N_x^{\alpha^X}(V)=N_x^{\alpha^X}(B\cap C)\geqslant N_x^{\alpha^X}(B)\wedge N_x^{\alpha^X}(C) \geqslant N_x^{\alpha^X}(G)\wedge N_x^{\alpha^X}(C)>\lambda. \quad (1)$$

By $(*)$ and Theorem3.3, $\tau_\alpha(C^c)\geqslant T_2^\alpha(X,\tau)\otimes\Gamma_\alpha(C,\tau/C) \geqslant T_2^\alpha(X,\tau)+\Gamma_\alpha(C,\tau/C)-1\geqslant\lambda'$. So $\tau_\alpha(V^c) = \tau_\alpha(B^c\cup C^c)\geqslant\tau_\alpha(B^c)\wedge\tau_\alpha(C^c) \geqslant\lambda'$, i.e., $\tau_\alpha(V^c)+\Gamma_\alpha(C,\tau/C)-1 \geqslant\lambda$ and $\Gamma_\alpha(V,\tau/V)=\Gamma_\alpha(V,(\tau/C)/V) \geqslant\tau_\alpha/C(C-V)+\Gamma_\alpha(C,\tau/C)-1 \geqslant\tau_\alpha(V^c)+\Gamma_\alpha(C,\tau/C)-1\geqslant\lambda \quad (2)$

Finally,

$$\bigwedge_{y\in U^c}N_y^{\alpha^X}(V^c)\geqslant\bigwedge_{y\in V^c}N_y^{\alpha^X}(V^c)=\tau_\alpha(V^c)\geqslant\lambda \quad (3)$$

Thus by (1)–(3), for any $x\in U$, there exists $V\subseteq U$ such that $N_x^{\alpha^X}(V)>\lambda$, $\bigwedge_{y\in U^c}N_y^{\alpha^X}(V^c)\geqslant\lambda$ and $\Gamma_\alpha(V,\tau/V)\geqslant\lambda$. So

$$\bigvee_{V\subseteq X}\left(N_x^{\alpha^X}(V)\wedge\bigwedge_{y\in U^c}N_y^{\alpha^X}(V^c)\wedge\Gamma_\alpha(V,\tau/V)\right)\geqslant\lambda. \quad \square$$

Theorem 4.4. *For any fuzzifying α-topological space (X,τ),* $\models T_2^\alpha(X,\tau)\otimes(L_\alpha C(X,\tau))^2\to T_3^\alpha(X,\tau)$

Proof. By Theorem 4.3, for any $x\in U$, we have

$$\bigvee_{x\in V\subseteq U}\left(N_x^{\alpha^X}(V)\wedge\bigwedge_{y\in U^c}N_y^{\alpha^X}(V^c) \geqslant\left[T_2^\alpha(X,\tau)\otimes(\Gamma_\alpha(C,\tau/C))^2\otimes N_x^{\alpha^X}(U)\right]\right).$$

Thus

$$1-N_x^{\alpha^X}(U)+\bigvee_{x\in V\subseteq U}\left(N_x^{\alpha^X}(V)\wedge\bigwedge_{y\in U^c}N_y^{\alpha^X}(V^c)\geqslant\left[T_2^\alpha(X,\tau)\otimes(\Gamma_\alpha(C,\tau/C))^2\right]\right),$$

i.e., $\left[T_3^\alpha(X,\tau)\right]\geqslant\left[T_2^\alpha(X,\tau)\otimes(\Gamma_\alpha(C,\tau/C))^2\right]$. $\square$

Theorem 4.5. *For any fuzzifying α-topological space (X,τ),*

$$\models T_3^\alpha(X,\tau)\otimes L_\alpha C(X,\tau)\to\forall A\forall U\left(U\in N_A^{\alpha^X}\otimes\Gamma_\alpha(A,\tau/A) \to\exists V\left(V\subseteq U\wedge U\in N_A^{\alpha^X}\wedge\tau_\alpha(V^c)\wedge\Gamma_\alpha(V,\tau/V)\right)\right),$$

where $U\in N_A^{\alpha^X}:=(\forall x)\left(x\in A\wedge U\in N_x^{\alpha^X}\right)$.

Proof. We only need to show that for any A, $U\in P(X)$,

$$\left[T_3^\alpha(X,\tau)\otimes L_\alpha C(X,\tau)\otimes\Gamma_\alpha(A,\tau/A)\otimes N_A^{\alpha^X}(U)\right] \leqslant\bigvee_{V\subseteq U}\left(N_A^{\alpha^X}(V)\wedge\tau_\alpha(V^c)\wedge\Gamma_\alpha(V,\tau/V)\right).$$

Indeed, if

$$\left[T_3^\alpha(X,\tau)\otimes L_\alpha C(X,\tau)\otimes\Gamma_\alpha(A,\tau/A)\otimes N_A^{\alpha^X}(U)\right]>\lambda>0,$$

then for any $x\in A$, there exists $C\in P(X)$ such that

$$\left[T_3^\alpha(X,\tau)\otimes N_x^{\alpha^X}(C)\otimes\Gamma_\alpha(C,\tau/C)\otimes\Gamma_\alpha(A,\tau/A)\otimes N_A^{\alpha^X}(U)\right]>\lambda.$$

Since (X,τ) is fuzzifying α-topological space,

$$\bigvee_{x\in W\subseteq C\cap U}\tau_\alpha(W)=N_x^{\alpha^X}(C\cap U)\geqslant N_x^{\alpha^X}(C)\wedge N_x^{\alpha^X}(U) \geqslant N_x^{\alpha^X}(C)\wedge N_A^{\alpha^X}(U)\geqslant N_x^{\alpha^X}(C)\otimes N_A^{\alpha^X}(U).$$

Then there exists W such that $x\in W\subseteq C\cap U$, and $\left[T_3^\alpha(X,\tau)\otimes\tau_\alpha(W)\otimes\Gamma_\alpha(C,\tau/C)\otimes\Gamma_\alpha(A,\tau/A)\right]>\lambda$.

Therefore

$$\left[T_3^\alpha(X,\tau)\right]+\tau_\alpha(W)-1>\lambda+2-\Gamma_\alpha(C,\tau/C)-\Gamma_\alpha(A,\tau/A)] =\lambda'\geqslant\lambda. \quad (*)$$

Since for any

$$x\in W,\ \left[T_3^\alpha(X,\tau)\right] \leqslant 1-\tau_\alpha(W)+\bigvee_{B\subseteq W}\left(N_x^{\alpha^X}(B)\wedge\bigwedge_{y\in W^c}N_y^{\alpha^X}(B^c)\right),$$

we have

$$\bigvee_{B\subseteq W}\left(N_x^{\alpha^X}(B)\wedge\bigwedge_{y\in W^c}N_y^{\alpha^X}(B^c)\right)>\lambda'.$$

Thus there exists B_x such that $x\in B_x\subseteq W\subseteq C\cap U$ and for any $y\in W^c$, we have

$$N_y^{\alpha^X}(B_x^c)>\lambda',\ N_x^{\alpha^X}(B_x)>\lambda'.$$

Since

$$N_y^{\alpha^X}(B_x^c)=\bigvee_{x\in G^c\subseteq B_x^c}\tau_\alpha(G^c)>\lambda',$$

then for any $y\in W^c$, there exists G_{xy} such that $x\in G_{xy}^c\subseteq B_x^c$ and $\tau_\alpha\left(G_{xy}^c\right)>\lambda'$. Set $G_x^c=\bigcup_{y\in W^c}G_{xy}^c$, then $W^c\subseteq G_{xy}^c\subseteq B_x^c$ and $\tau_\alpha(G_x^c)\geqslant\bigwedge_{y\in W^c}\tau_\alpha\left(G_{xy}^c\right)\geqslant\lambda'$. Since $G_x\supseteq B_x$, $N_x^{\alpha^X}(G_x)\geqslant N_x^{\alpha^X}(B_x)>\lambda'$, i.e., $\bigvee_{x\in H\subseteq G_x}\tau_\alpha(H)>\lambda'$. Thus there exists H_x such that $x\in H_x\subseteq G_x$ and $\tau_\alpha(H_x)>\lambda'$. Hence for any $x\in A$, there exists H_x and G_x such that $x\in H_x\subseteq G_x\subseteq U,\tau_\alpha(H_x)>\lambda'$ and $W\supseteq\bigcup_{x\in A}G_x\supseteq\bigcup_{x\in A}H_x\supseteq A$. We define $\mathfrak{R}\in\mathfrak{I}(P(A))$ as follows:

$$\mathfrak{R}(D)=\begin{cases}\bigvee_{H_x\cap A=D}\tau_\alpha(H_x), & \text{there exists } H_x \text{ such that } H_x\cap A=D,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\Gamma_\alpha(A,\tau/A)=\mu>\mu-\epsilon(\epsilon>0)$. Then $1-K_\alpha(\mathfrak{R},A)+\bigvee_{\wp\leqslant\mathfrak{R}}[K(\wp,A)\otimes FF(\wp)]>\mu-\epsilon$, where

$$[K(\mathfrak{R},A)]=\bigwedge_{x\in A}\bigvee_{x\in B}\mathfrak{R}(B)=\bigwedge_{x\in A}\bigvee_{x\in D}\mathfrak{R}(D)=\bigwedge_{x\in A}\bigvee_{x\in D}\bigvee_{H_{x'}\cap A=D}\tau_\alpha(H_{x'})\geqslant\lambda'$$

and

$$[\mathfrak{R}\subseteq\tau_\alpha/A]=\bigwedge_{B\subseteq X}\min(1,1-\mathfrak{R}(B)+\tau_\alpha/A(B))$$
$$=\bigwedge_{B\subseteq X}\min\left(1,1-\bigvee_{H_x\cap A=B}\tau_\alpha(H_x)+\bigvee_{H\cap A=B}\tau_\alpha(H)\right)=1.$$

So, $K_\alpha(\mathfrak{R},A)=[K(\mathfrak{R},A)]\geqslant\lambda'$. By (*),

$$[K(\wp,A)\otimes FF(\wp)]>\mu-\epsilon-1+K_\alpha(\mathfrak{R},A)\geqslant\mu-\epsilon-1+\lambda'\geqslant\lambda-\epsilon.$$

Thus

$$\bigwedge_{x\in A}\bigvee_{x\in E}\wp(E)+1-\bigwedge\{\delta:F(\wp_\delta)\}-1>\lambda-\epsilon,\ \text{and}\ \bigwedge_{x\in A}\bigvee_{x\in E}\wp(E)>\lambda-\epsilon+\bigwedge\{\delta:F(\wp_\delta)\}.$$

Hence there exists $\beta>0$ such that $F(\wp_\beta)$ and

$$\bigwedge_{x\in A}\bigvee_{x\in D}\wp(D)>\lambda-\epsilon+\beta.$$

Therefore for any $x\in A$, there exists $D_x\subseteq A$ such that $\wp(D_x)>\lambda-\epsilon+\beta$ and

$$\bigcup_{x\in A}D_x\subseteq A.$$

Suitably choose ϵ such that $\lambda-\epsilon>0$, then $\wp(D_x)>\beta>0$. Since

$$\mathfrak{R}(D_x)\geqslant\wp(D_x)>0,\ D_x=H_{x'}\cap A,$$

i.e., $H_{x'}\cap A\in\wp_\beta$. By $F(\wp_\beta)$, so there exists finite $H_{x'_1}, H_{x'_2},\ldots,H_{x'_n}$ such that

$$\bigcup_{i=1}^n H_{x'_i}\supseteq A$$

and

$$\bigcup_{i=1}^n H_{x'_i}\subseteq\bigcup_{i=1}^n G_{x'_i}.$$

Set $V=\bigcup_{i=1}^n G_{x'_i}$, and $V^c=\bigcap_{i=1}^n G_{x'_i}^c$, $A\subseteq V\subseteq U$, and $\tau_\alpha(V^c)\geqslant\bigwedge_{1\leqslant i\leqslant n}\tau_\alpha\left(G_{x'_i}^c\right)\geqslant\lambda'>\lambda$. Since for any $x\in A, G_x\subseteq W\subseteq C\cap U\subseteq C$, we have $V=\bigcup_{i=1}^n G_{x'_i}\subseteq W\subseteq C$. Because $\tau_\alpha/C(C-V)=\bigvee_{D\cap C=C\cap V^c}\tau_\alpha(D)\geqslant\tau_\alpha(V^c)\geqslant\lambda'$. Thus by (*), $\tau_\alpha/C(C-V)+\Gamma_\alpha(C,\tau/C)-1>\lambda$, and by Theorem 4.1 [1], $\Gamma_\alpha(V,\tau/V)=\Gamma_\alpha(V,\tau/C/V)\geqslant[\Gamma_\alpha(C,\tau/C)\otimes\tau_\alpha/C(C-V)]>\lambda$.

Finally, we have for any $x\in A$,

$$N_x^{\alpha^X}(V)=N_x^{\alpha^X}\left(\bigcup_{i=1}^n G_{x'_i}\right)\geqslant N_x^{\alpha^X}\left(\bigcup_{i=1}^n H_{x'_i}\right)\geqslant\tau_\alpha\left(\bigcup_{i=1}^n H_{x'_i}\right)\geqslant\bigwedge_{1\leqslant i\leqslant n}\tau_\alpha\left(H_{x'_i}\right)\geqslant\lambda'>\lambda.$$

So, $N_A^{\alpha^X}(V)=\bigwedge_{x\in A}N_x^{\alpha^X}(V)\geqslant\lambda$. Therefore $N_A^{\alpha^X}(V)\wedge\tau_\alpha(V^c)\wedge\Gamma_\alpha(V,\tau/V)\geqslant\lambda$.

Thus

$$\bigvee_{V\subseteq U}\left(N_A^{\alpha^X}(V)\wedge\tau_\alpha(V^c)\wedge\Gamma_\alpha(V,\tau/V)\right)\geqslant\lambda.\qquad\square$$

Theorem 4.6. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and $f\in Y^X$ be surjective. Then $\models L_\alpha C(X,\tau)\otimes C_\alpha(f)\otimes O(f)\to LC(Y,\sigma)$. For the definition of $O(f)$, see [17].*

Proof. If $[L_\alpha C(X,\tau)\otimes C_\alpha(f)\otimes O(f)]>\lambda>0$, then for any $x\in X$, there exists $U\subseteq X$, such that $[N_x^{\alpha^X}(U)\otimes\Gamma_\alpha(U,\tau/U)\otimes C_\alpha(f)\otimes O(f)]>\lambda$. Since $N_x^{\alpha^X}(U)=\bigvee_{x\in V\subseteq U}\tau_\alpha(V)$, so there exists $V'\subseteq X$ such that $x\in V'\subseteq U$ and $[\tau_\alpha(V')\otimes\Gamma_\alpha(U,\tau/U)\otimes C_\alpha(f)\otimes O(f)]>\lambda$. By Theorem 4.3 in [1], $[\Gamma_\alpha(U,\tau/U)\otimes C_\alpha(f)]\leqslant[\Gamma(f(U),\sigma/f(U))]$ and

$$[\tau(V')\otimes O(f)]=\max(0,\tau(V')+O(f)-1)=\max(0,\tau(V')+\bigwedge_{V\subseteq X}\min(1,1-\tau(V')+\sigma(f(V)))-1)$$
$$\leqslant\max(0,\tau(V')+1-\tau(V')+\sigma(f(V))-1)$$
$$=\sigma(f(V))\leqslant N_{f(x)}^Y(f(V'))\leqslant N_{f(x)}^Y(f(U)).$$

Since f is surjective,

$$LC(Y,\sigma)=LC(f(X),\sigma)=\bigwedge_{y\in f(x)\subseteq f(X)}\bigvee_{U'=f(U)\subseteq f(X)}\left[N_y^Y(U')\otimes[\Gamma(U',\sigma/U')]\right]$$
$$\geqslant\bigwedge_{y\in f(x)\subseteq f(X)}\left[N_{f(x)}^Y(f(U))\otimes[\Gamma(f(U),\sigma/f(U))]\right]$$
$$\geqslant\bigwedge_{y\in f(x)\subseteq f(X)}[\tau(V')\otimes O(f)\otimes\Gamma_\alpha(U,\tau/U)\otimes C_\alpha(f)]\geqslant\lambda.\qquad\square$$

Theorem 4.7. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and $f\in Y^X$ be surjective. Then* $\models L_\alpha C(X,\tau)\otimes I_\alpha(f)\otimes O_\alpha(f)\to L_\alpha C(Y,\sigma)$.

Proof. By Theorem 4.3 in [1], the proof is similar to the proof of Theorem 4.6. □

Theorems 4.6 and 4.7 are a generalization of the following corollary.

Corollary 4.2. *Let (X,τ) and (Y,σ) be two topological spaces and $f:(X,\tau)\to(Y,\sigma)$ be surjective mapping. If f is an α-continuous (resp. α-irresolute), open (resp. α-open) and X is locally α-compact, then Y is locally compact (resp. locally α-compact) space.*

Theorem 4.8. *Let $\{(X_s,\tau_s):s\in S\}$ be a family of fuzzifying topological spaces, then*

$$\models L_\alpha C\left(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\alpha)_s\right)\to\forall s(s\in S\wedge L_\alpha C(X_s,(\tau_\alpha)_s)\wedge\exists T(T\Subset S\wedge\forall t(t\in S-T\wedge\Gamma_\alpha(X_t,\tau_t)))).$$

Proof. It suffices to show that

$$L_\alpha C\left(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\alpha)_s\right)\leqslant\bigwedge_{s\in S}\left[L_\alpha C(X_s,(\tau_\alpha)_s)\wedge\bigvee_{T\Subset S}\bigwedge_{t\in S-T}\Gamma_\alpha(X_t,\tau_t)\right].$$

From Theorem 4.7 and Lemma 3.1 we have for any $t\in S$,

$$L_\alpha C\left(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\alpha)_s\right)=\left[L_\alpha C\left(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\alpha)_s\right)\otimes C_\alpha(p_t)\otimes O_\alpha(p_t)\right]\leqslant L_\alpha C(X_t,\tau_t).$$

So,

$$\bigwedge_{t\in S}L_\alpha C(X_t,\tau_t)\geqslant L_\alpha C\left(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\alpha)_s\right).$$

By Theorem 3.2 we have

$$\begin{aligned}\bigvee_{T\Subset S}\bigwedge_{t\in S-T}\Gamma_\alpha(X_t,\tau_t)&\geqslant\left[\bigvee_{U\subseteq\prod_{s\in S}X_s}\Gamma_\alpha\left(U,\prod_{s\in S}(\tau_\alpha)_s/U\right)\otimes\bigvee_{X\subseteq\prod_{s\in S}X_s}N_x^{\alpha^X}(U))\right]\\&\geqslant\bigvee_{U\subseteq\prod_{s\in S}X_s}\bigvee_{X\subseteq\prod_{s\in S}X_s}\left[\Gamma_\alpha\left(U,\prod_{s\in S}(\tau_\alpha)_s/U\right)\otimes N_x^{\alpha^X}(U)\right)\right]\\&\geqslant\bigwedge_{X\subseteq\prod_{s\in S}X_s}\bigvee_{U\subseteq\prod_{s\in S}X_s}\left[\Gamma_\alpha\left(U,\prod_{s\in S}(\tau_\alpha)_s/U\right)\otimes N_x^{\alpha^X}(U)\right)\right]\\&=L_\alpha C\left(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\alpha)_s\right).\end{aligned}$$

Therefore

$$L_\alpha C\left(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\alpha)_s\right)\leqslant\left[\bigwedge_{t\in S}L_\alpha C(X_t,\tau_t)\wedge\bigvee_{T\Subset S}\bigwedge_{t\in S-T}\Gamma_\alpha(X_t,\tau_t)\right].\quad\square$$

We can obtain the following corollary in crisp setting.

Corollary 4.3. *Let $\{X_\lambda:\lambda\in\Lambda\}$ be a family of nonempty topological spaces. If $\prod_{\lambda\in\Lambda}X_\lambda$ is locally α-compact, then each X_λ is locally α-compact and all but finitely many X_λ are α-compact*

5. Conclusion

The present paper investigates topological notions when these are planted into the framework of Ying's fuzzifying topological spaces (in semantic method of continuous valued-logic). It continue various investigations into fuzzy topology in a legitimate way and extend some fundamental results in general topology to fuzzifying topology. An important virtue of our approach (in which we follow Ying) is that we define topological notions as fuzzy predicates (by formulae of Łukasiewicz fuzzy logic) and prove the validity of fuzzy implications (or equivalences). Unlike the (more wide-spread) style of defining notions in fuzzy mathematics as crisp predicates of fuzzy sets, fuzzy predicates of fuzzy sets provide a more genuine fuzzification; furthermore the theorems in the form of valid fuzzy implications are more general than the corresponding theorems on crisp predicates of fuzzy sets. The main contributions of the present paper are to give characterizations of fuzzifying α-compactness. Also, we define the concept of locally α-compactness of fuzzifying topological spaces and obtain some basic properties of such spaces. There are some problems for further study:

(1) One obvious problem is: our results are derived in the Łukasiewicz continuous logic. It is possible to generalize them to more general logic setting, like residuated lattice-valued logic considered in [27,28].
(2) What is the justification for fuzzifying locally α-compactness in the setting of $(2,L)$ topologies.
(3) Obviously, fuzzifying topological spaces in [14] form a fuzzy category. Perhaps, this will become a motivation for further study of the fuzzy category.
(4) What is the justification for fuzzifying locally α-compactness in (M,L)-topologies etc.

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