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ORIGINAL ARTICLE

Numerical study for systems of fractional differential equations via Laplace transform

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Abstract In this paper, we propose a numerical algorithm for solving system of fractional differential equations by using the homotopy analysis transform method. The homotopy analysis transform method is the combined form of the homotopy analysis method and Laplace transform method. The solutions of our modeled equations are calculated in the form of convergent power series with easily computable components. The numerical results shows that the approach is easy to implement and accurate when applied to various fractional differential equations.

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1. Introduction

Fractional order ordinary and partial differential equations, as generalization of classical integer order differential equations, are increasingly used to model problems in fluid mechanics, viscoelasticity, biology, physics and engineering, and others applications [1–4]. The most important advantage of making use of fractional differential equations in mathematical modelling is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular in scientific and technological fields [5–11]. The homotopy analysis method is introduced by Liao [12] for solving linear and non-linear differential and integral equations. Different from perturbation technique, the HAM does not need any small or large parameters in the equations. The HAM was successfully applied to solve many nonlinear problems [13–17]. In recent years various techniques have been applied to handle various physical problems [18–24]. The Laplace transform [25] is a powerful technique for solving various linear partial differential equations having considerable significance in various fields such as engineering and applied sciences. Coupling of semi-analytical methods with Laplace transform giving time-consuming consequences and less C.P.U time (Processor 2.65 GHz or more and RAM-1 GB or more) for solving non-linear problems. Many authors

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have paid attention for using these techniques in literature [26–28]. On the other hand, homotopy analysis method (HAM) is also combined with well defined Laplace transform to produce highly effective technique, namely the homotopy analysis transform method (HATM) for non-linear problems [29].

In this paper we consider the system of fractional differential equations of the type:

$$\begin{aligned} D_*^{\alpha_1}x &= a_1x^n + b_1y^n + c_1z^n,\\ D_*^{\alpha_2}y &= a_2x^n + b_2y^n + c_2z^n,\\ D_*^{\alpha_3}z &= a_3x^n + b_3y^n + c_3z^n, \end{aligned} \tag{1}$$

with the initial conditions:

$$D^{n-1}x(0) = A_{n-1}, D^{n-1}y(0) = B_{n-1} \text{ and } D^{n-1}z(0) = C_{n-1}, \tag{2}$$

where $D \equiv \frac{d}{dt}, a_i, b_i, c_i, A_{n-1}, B_{n-1}$ and C_{n-1} are constant. For $n = 1$ the system of Eq. (1) combined with initial conditions is said to be linear fractional differential equations while for $n \geqslant 2$ the system is non-linear. Momani and Odibat [30] used Adomain decomposition method (ADM) and homotopy perturbation method (HPM) for solving these types of differential equations. Zurigat et al. [31] applied homotopy analysis method (HAM) for solving systems of differential Eq. (1).

In this paper we implement the homotopy analysis transform method (HATM) for solving systems of differential equations. The HATM is an elegant combination of the Laplace transform method and HAM. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for non-linear equations.

2. Basic definitions of fractional calculus and Laplace transform

In this section, we mention the following basic definitions of fractional calculus and Laplace transform.

Definition 1.1. The Laplace transform of the function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty e^{-st}f(t)dt. \tag{3}$$

Definition 1.2. The Laplace transform $L[u(x,t)]$ of the Riemann–Liouville fractional integral is defined by [7]:

$$L[I_t^\alpha u(x,t)] = s^{-\alpha}L[u(x,t)]. \tag{4}$$

Definition 1.3. The Laplace transform $L[u(x,t)]$ of the Caputo fractional derivative is given as [7]:

$$L\left[D_t^\alpha u(x,t)\right] = s^\alpha L[u(x,t)] - \sum_{k=0}^{n-1} s^{(\alpha-k-1)}u^{(k)}(x,0), \quad n-1 < \alpha \leqslant n. \tag{5}$$

3. Basic idea of homotopy analysis transform method (HATM)

To illustrate the basic idea, let us consider the following fractional differential equation

$$D_*^\alpha v(t) = F(t, v(t).v'(t)), \qquad t \geqslant 0, \quad 0 < \alpha \leqslant 2. \tag{6}$$

subject to the initial conditions:

$$v(0) = a. \tag{7}$$

Applying Laplace transform on both sides of Eq. (6), we get

$$\mathcal{L}[v(t)] = \frac{a}{s} + \frac{1}{s^\alpha}\mathcal{L}[F(t, v(t), v'(t))]. \tag{8}$$

The zero-order deformation equation of the Laplace Eq. (8) has the form

$$(1-q)[\phi(s;q) - \bar{v}_0(s)] = q\hbar[\phi(s;q) - \frac{a}{s} - \frac{1}{s^\alpha}\mathcal{L}[F(t, \phi(s;q), \phi'(s;q))], \tag{9}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, we have $\phi(s;0) = \bar{v}_0(s)$ and $\phi(s;1) = \bar{v}(s)$. Thus, as q increases from 0 to 1, the solution $\phi(s;q)$ varies from the initial guess $\bar{v}_0(s)$ to the solution $\bar{v}(s)$. Expanding $\phi(s;q)$ by Taylor series with respect to q, we get

$$\phi(s;q) = \bar{v}_0(s) + \sum_{m=1}^\infty \bar{v}_m(s)q^m, \tag{10}$$

where

$$\bar{v}_m(s) = \frac{1}{m!}\frac{\partial^m\phi(s;q)}{\partial q^m}|q=0. \tag{11}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, the series (10) converges at $q = 1$, then we have

$$\bar{v}(s) = \bar{v}_0(s) + \sum_{m=1}^\infty \bar{v}_m(s), \tag{12}$$

which must be one of the solutions of the original non-linear equation. The governing equation can be deduced from the zero-order deformation Eq. (9). Define the vector

$$\vec{v}_n = \{v_0(s), v_1(s), \ldots v_n(s)\}. \tag{13}$$

Differentiating Eq. (9) m-times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation.

$$\bar{v}_m(s) = \chi_m\bar{v}_{m-1} - R_m(\bar{v}_{m-1}(s)), \tag{14}$$

where

$$\begin{aligned} R_m(\bar{v}_{m-1}(s)) = \bar{v}_{m-1}(s) &- \frac{1}{s^\alpha}\left(\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[L(F(t), \phi(t;q), \phi'(t;q))]|q=0\right)\\ &- \frac{a}{s}(1-\chi_m), \end{aligned} \tag{15}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1,\\ 1, & m > 1. \end{cases} \tag{16}$$

Operating inverse Laplace transform of Eq. (14), we get power series solution of Eq. (6) with convergence control parameter.

4. Numerical implementations

In order to assess both the accuracy and convergence of the HATM presented in this paper for fractional system of differential equations, we have applied it to the following three examples.

Example 4.1. Consider the following linear system of fractional differential equations:

$$D_*^{\alpha_1} x = x + y,$$
$$D_*^{\alpha_2} y = -x + y, \quad 0 < \alpha_1, \alpha_2 \leqslant 1, \tag{17}$$

subject to the initial condition:

$$x(0) = 0, \quad y(0) = 1. \tag{18}$$

The exact solution of this system, when $\alpha_1 = \alpha_2 = 1$, is given by

$$x(t) = e^t \sin t, \qquad y(t) = e^t \cos t.$$

Taking Laplace transform both of sides, we get

$$s^\alpha \mathcal{L}[x(t)] - s^{\alpha-1}x(0) = \mathcal{L}[x(t) + y(t)],$$
$$s^\alpha \mathcal{L}[y(t)] - s^{\alpha-1}y(0) = \mathcal{L}[-x(t) + y(t)].$$

By applying initial conditions, we get

$$\mathcal{L}[x(t)] = \frac{1}{s^\alpha}\mathcal{L}[x(t) + y(t)], \tag{19}$$
$$\mathcal{L}[y(t)] = \frac{1}{s} + \frac{1}{s^\alpha}\mathcal{L}[-x(t) + y(t)]. \tag{20}$$

We now define the non-linear operators as

$$N_1(\phi_1(t;q), \phi_2(t;q)) = \mathcal{L}[\phi_1(t;q)] - \frac{1}{s^\alpha}\mathcal{L}[\phi_1(t;q) + \phi_2(t;q)], \tag{21}$$
$$N_2(\phi_1(t;q), \phi_2(t;q)) = \mathcal{L}[\phi_2(t;q)] - \frac{1}{s} - \frac{1}{s^\alpha}\mathcal{L}[-\phi_1(t;q) + \phi_2(t;q)]. \tag{22}$$

Using the above definition, we construct the zero-order deformation equation

$$(1-q)\mathcal{L}[\phi_i(t;q) - x_{i,0}(t)] = q\hbar_i H_i(t) N_i[\phi_1(t;q)\phi_2(t;q)]. \tag{23}$$

Obviously when $q = 0$ and $q = 1$

$$\phi_1(t;0) = x_{1,0} = x_0(t), \qquad \phi_1(t;1) = x(t),$$
$$\phi_2(t;0) = x_{2,0} = y_0(t), \qquad \phi_2(t;1) = y(t). \tag{24}$$

We now define the mth-order deformation equation is given by

$$\mathcal{L}[x_m(t) - \chi_m x_{i(m-1)}(t)] = \hbar_i H_i(t) R_{im}(\vec{x}_{(m-1)}(t), \vec{y}_{(m-1)}t),$$
$$i = 1, 2, 3 \ldots n. \tag{25}$$

where $R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1}) = \mathcal{L}[x_{m-1}(t)] - \frac{1}{s^\alpha}\mathcal{L}[x_{m-1}(t) + y_{m-1}(t)]$

$$R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1}) = \mathcal{L}[y_{m-1}(t)] - \frac{1}{s}(1 - \chi_m) - \frac{1}{s^\alpha}\mathcal{L}[-x_{m-1}(t) + y_{m-1}(t)]. \tag{26}$$

Applying inverse Laplace transform, we get

$$x_m(t) = \chi_m x_{(m-1)}(t) + h_1 \ \mathcal{L}^{-1}[R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1})], \tag{27}$$
$$y_m(t) = \chi_m y_{(m-1)}(t) + h_2 \ \mathcal{L}^{-1}[R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1})]. \tag{28}$$

for simplicity let us take $\hbar_1 = \hbar_2 = \hbar$ and $H_1(t) = H_2(t) = 1$.

$$x_1(t) = -\hbar t,$$
$$y_1(t) = -\hbar t,$$
$$x_2(t) = -\hbar t + \hbar^2 t^2 - \frac{\hbar^2 t^{(2-\alpha_1)}}{(2-\alpha_1)\Gamma(2-\alpha_1)},$$
$$y_2(t) = -\hbar t - \frac{\hbar^2 t^{(2-\alpha_2)}}{(2-\alpha_2)\Gamma(2-\alpha_2)}, \tag{29}$$
$$\vdots$$

and so on.

We now define the mth-order deformation equation is given by

$$\mathcal{L}[x_m(t) - \chi_m x_{(m-1)}(t)] = \hbar_1 J^{\alpha_1} R_{1m}(\vec{x}_{(m-1)}(t), \vec{y}_{(m-1)}t),$$
$$\mathcal{L}[y_m(t) - \chi_m y_{(m-1)}(t)] = \hbar_2 J^{\alpha_2} R_{2m}(\vec{x}_{(m-1)}(t), \vec{y}_{(m-1)}t).$$

Taking inverse Laplace transform, we get

$$x_m(t) = \chi_m x_{(m-1)}(t) + \hbar\mathcal{L}^{-1}[J^{\alpha_1} R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1})], \tag{30}$$
$$y_m(t) = \chi_m y_{(m-1)}(t) + \hbar\mathcal{L}^{-1}[J^{\alpha_2} R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1})]. \tag{31}$$

Then we obtain the following components as follows:

$$x_1(t) = -\frac{\hbar t^{\alpha_1}}{\Gamma(1+\alpha_1)},$$
$$y_1(t) = -\frac{\hbar t^{\alpha_2}}{\Gamma(1+\alpha_2)},$$
$$x_2(t) = -\frac{\hbar(1+\hbar)t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\hbar^2 t^{(\alpha_1+\alpha_2)}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{\hbar^2 t^{2\alpha_1}}{\Gamma(1+2\alpha_1)}, \tag{32}$$
$$y_2(t) = -\frac{\hbar(1+\hbar)t^{\alpha_2}}{\Gamma(1+\alpha_2)} - \frac{\hbar^2 t^{(\alpha_1+\alpha_2)}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{\hbar^2 t^{2\alpha_2}}{\Gamma(1+2\alpha_2)},$$
$$\vdots$$

Setting $\hbar = -1$ in Eqs. (29) and (32), the above expressions are exactly the same as given by the HPM and ADM [30] and HAM [31].

Example 4.2. Consider the following non-linear fractional system

$$D_*^{\alpha_1} x = \frac{1}{2}x,$$
$$D_*^{\alpha_2} y = x^2 + y, \quad 0 < \alpha_1, \alpha_2 \leqslant 1. \tag{33}$$

The exact solution of this system, when $\alpha_1 = \alpha_2 = 1$, is given by

$$x(t) = e^{t/2}, \qquad y(t) = te^t.$$

To solve above system using HATM, we select the initial guesses as $x_0(t) = 1$, $y_0(t) = 0$

$$R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1}) = \mathcal{L}[x_{m-1}(t)] - \frac{1}{s}(1-\chi_m) - \frac{1}{2s^\alpha}\mathcal{L}[x_{m-1}(t)],$$
$$R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1}) = \mathcal{L}[y_{m-1}(t)] - \frac{1}{s^\alpha}\mathcal{L}\left[\sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t) + y_{m-1}\right]. \tag{34}$$

Applying inverse Laplace transform, we get

$$x_m(t) = \chi_m x_{(m-1)}(t) + h_1 \quad \mathcal{L}^{-1}[R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1})],$$
$$y_m(t) = \chi_m y_{(m-1)}(t) + h_2 \quad \mathcal{L}^{-1}[R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1})] \tag{35}$$

for simplicity let us take $\hbar_1 = \hbar_2 = \hbar$ and $H_1(t) = H_2(t) = 1$. Then we obtain the following components as follows:

$$\begin{aligned}
x_1(t) &= -\frac{\hbar t}{2},\\
y_1(t) &= -\hbar t,\\
x_2(t) &= -\frac{\hbar t}{2} + \frac{\hbar^2 t^2}{8} - \frac{\hbar^2 t^{(2-\alpha_1)}}{2(2-\alpha_1)\Gamma(2-\alpha_1)},\\
y_2(t) &= -\hbar t + \hbar^2 t^2 - \frac{\hbar^2 t^{(2-\alpha_2)}}{2(2-\alpha_2)\Gamma(2-\alpha_2)},\\
&\vdots
\end{aligned} \tag{36}$$

and so on.

we now define the mth-order deformation equation is given by

$$\mathcal{L}[y_m(t) - \chi_m y_{(m-1)}(t)] = \hbar_1 J^{\alpha_1} R_{1m}(\vec{x}_{(m-1)}(t), \vec{y}_{(m-1)} t),$$
$$\mathcal{L}[y_m(t) - \chi_m y_{(m-1)}(t)] = \hbar_2 J^{\alpha_2} R_{2m}(\vec{x}_{(m-1)}(t), \vec{y}_{(m-1)} t).$$

Taking inverse Laplace transform, we get

$$x_m(t) = \chi_m x_{(m-1)}(t) + \hbar \mathcal{L}^{-1}[J^{\alpha_1} R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1})], \tag{37}$$
$$y_m(t) = \chi_m y_{(m-1)}(t) + \hbar \mathcal{L}^{-1}[J^{\alpha_2} R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1})]. \tag{38}$$

Then we obtain the following components as follows:

$$\begin{aligned}
x_1(t) &= -\frac{\hbar t^{\alpha_1}}{2\Gamma(1+\alpha_1)},\\
y_1(t) &= -\frac{\hbar t^{\alpha_2}}{\Gamma(1+\alpha_2)},\\
x_2(t) &= -\frac{\hbar(1+\hbar) t^{\alpha_1}}{2\Gamma(1+\alpha_1)} + \frac{\hbar^2 t^{2\alpha_1}}{4\Gamma(1+2\alpha_1)},\\
y_2(t) &= -\frac{\hbar(1+\hbar) t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{\hbar^2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{\hbar^2 t^{2\alpha_2}}{\Gamma(1+2\alpha_2)},\\
&\vdots
\end{aligned} \tag{39}$$

and so on.

Setting $\hbar = -1$ in Eqs. (36) and (39), the above expressions are exactly the same as given by the HPM and ADM [30] and HAM [31].

Example 4.3. Consider the following non-linear fractional system

$$\begin{aligned}
D_*^{\alpha_1} x &= x,\\
D_*^{\alpha_2} y &= 2x^2,\\
D_*^{\alpha_3} z &= 3xy,
\end{aligned} \tag{40}$$

with the initial conditions:

$$x(0) = 1, y(0) = 1, z(0) = 0. \tag{41}$$

To solve above system using HATM, we select the initial guesses as $x_0(t) = 1, y_0(t) = 1$ and $z_0(t) = 0$.

$$\begin{aligned}
R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}) &= \mathcal{L}[x_{m-1}(t)] - \frac{1}{s}(1-\chi_m) - \frac{1}{s^\alpha}\mathcal{L}[x_{m-1}(t)],\\
R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}) &= \mathcal{L}[y_{m-1}(t)] - \frac{1}{s}(1-\chi_m)\\
&\quad - \frac{1}{s^\alpha}\mathcal{L}\left[2\sum_{i=0}^{m-1} x_i(t) x_{m-1-i}(t)\right],\\
R_{3,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}) &= \mathcal{L}[z_{m-1}(t)] - \frac{1}{s^\alpha}\mathcal{L}\left[3\sum_{i=0}^{m-1} x_i(t) y_{m-1-i}(t)\right].
\end{aligned} \tag{42}$$

Applying inverse Laplace transform, we get

$$\begin{aligned}
x_m(t) &= \chi_m x_{(m-1)}(t) + h_1\ \mathcal{L}^{-1}[R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1})],\\
y_m(t) &= \chi_m y_{(m-1)}(t) + h_2\ \mathcal{L}^{-1}[R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1})],\\
z_m(t) &= \chi_m z_{(m-1)}(t) + h_3\ \mathcal{L}^{-1}[R_{3,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1})]
\end{aligned} \tag{43}$$

for simplicity let us take $\hbar_1 = \hbar_2 = \hbar$ and $H_1(t) = H_2(t) = 1$. Then we obtain the following components as follows:

$$\begin{aligned}
x_1(t) &= -\hbar t,\\
y_1(t) &= -2\hbar t,\\
z_1(t) &= -3\hbar t,\\
x_2(t) &= -\hbar t + \frac{\hbar^2 t^2}{2} - \frac{\hbar^2 t^{2-\alpha_1}}{(2-\alpha_1)\Gamma(2-\alpha_1)},\\
y_2(t) &= -2\hbar t + 2\hbar^2 t^2 - \frac{2\hbar^2 t^{2-\alpha_2}}{(2-\alpha_2)\Gamma(2-\alpha_2)},\\
z_2(t) &= -3\hbar t + \frac{9}{2}\hbar^2 t^2 - \frac{3\hbar^2 t^{2-\alpha_3}}{(2-\alpha_3)\Gamma(2-\alpha_3)},\\
&\vdots
\end{aligned} \tag{44}$$

and so on.

We now define the mth-order deformation equation is given by

$$\begin{aligned}
\mathcal{L}[x_m(t) - \chi_m x_{m-1}(t)] &= \hbar_1 J^{\alpha_1} R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}),\\
\mathcal{L}[y_m(t) - \chi_m y_{m-1}(t)] &= \hbar_2 J^{\alpha_2} R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}),\\
\mathcal{L}[z_m(t) - \chi_m z_{m-1}(t)] &= \hbar_3 J^{\alpha_3} R_{3,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1}).
\end{aligned} \tag{45}$$

Taking inverse Laplace transform, we get

$$\begin{aligned}
x_m(t) &= \chi_m x_{(m-1)}(t) + \hbar \quad \mathcal{L}^{-1}[J^{\alpha_1} R_{1,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1})],\\
y_m(t) &= \chi_m y_{(m-1)}(t) + \hbar \quad \mathcal{L}^{-1}[J^{\alpha_2} R_{2,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1})],\\
z_m(t) &= \chi_m z_{(m-1)}(t) + \hbar \quad \mathcal{L}^{-1}[J^{\alpha_3} R_{3,m}(\vec{x}_{m-1}, \vec{y}_{m-1}, \vec{z}_{m-1})].
\end{aligned} \tag{46}$$

Then we obtain the following components as follows:

$$\begin{aligned}
x_1(t) &= -\frac{\hbar t^{\alpha_1}}{\Gamma(1+\alpha_1)},\\
y_1(t) &= -\frac{2\hbar t^{\alpha_2}}{\Gamma(1+\alpha_2)},\\
z_1(t) &= -\frac{3\hbar t^{\alpha_3}}{\Gamma(1+\alpha_3)},\\
x_2(t) &= -\frac{\hbar(1+\hbar) t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\hbar^2 t^{2\alpha_1}}{\Gamma(1+2\alpha_1)},\\
y_2(t) &= -\frac{2\hbar(1+\hbar) t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{4\hbar^2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \frac{2\hbar^2 t^{2\alpha_2}}{\Gamma(1+2\alpha_2)},\\
z_2(t) &= -\frac{3\hbar(1+\hbar) t^{\alpha_3}}{\Gamma(1+\alpha_3)} + \frac{6\hbar^2 t^{\alpha_2+\alpha_3}}{\Gamma(1+\alpha_2+\alpha_3)} + \frac{3\hbar^2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)},\\
&\vdots
\end{aligned} \tag{47}$$

and so on

Setting $\hbar = -1$ in Eqs. (44) and (47), the above expressions are exactly the same as given by the HPM and ADM [30] and HAM [31].

5. Discussions

Setting $\hbar = -1$ in the above three examples of two and three respective systems of equations, it can be seen that the above

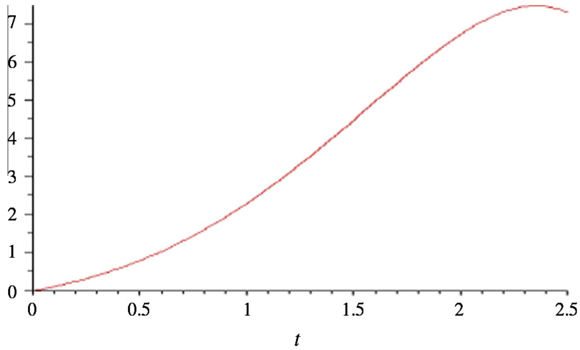

Fig. 1 Exact solution graph of $x[t]$ of Example 4.1 for $t \in [0,2.5)$, when $\alpha_1 = \alpha_2 = 1$.

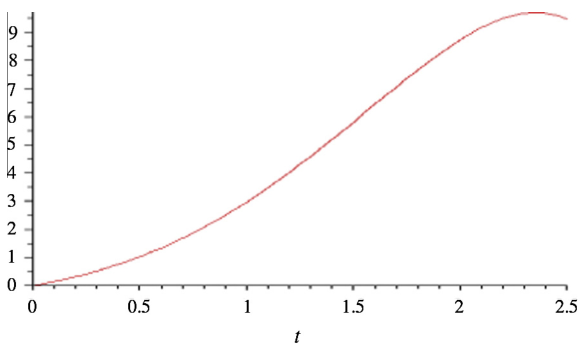

Fig. 2 Approximate solution graph of $x[t]$ of Example 4.1 by HATM for fourth-order approximations for $t \in [0,2.5)$, when $\alpha_1 = \alpha_2 = 1$ and $\hbar = -1$.

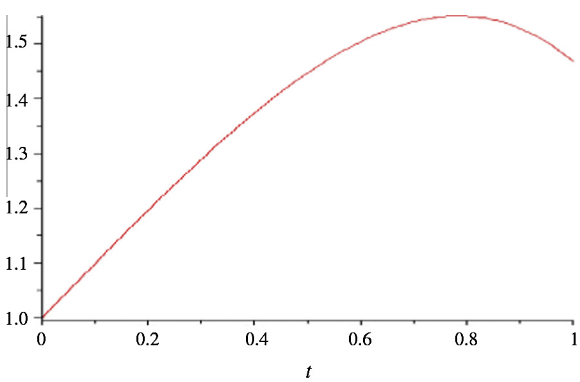

Fig. 3 Exact solution graph of $y[t]$, of Example 4.1 for $t \in [0,1)$, when $\alpha_1 = \alpha_2 = 1$.

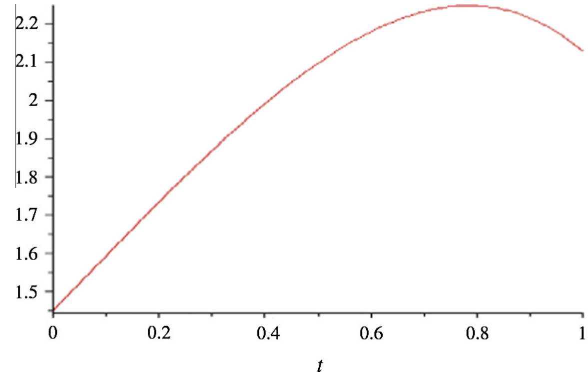

Fig. 4 Approximate solution graph of $y[t]$ of Example 4.1 by HATM for fourth-order approximations for $t \in [0,1)$, when $\alpha_1 = \alpha_2 = 1$ and $\hbar = -1$.

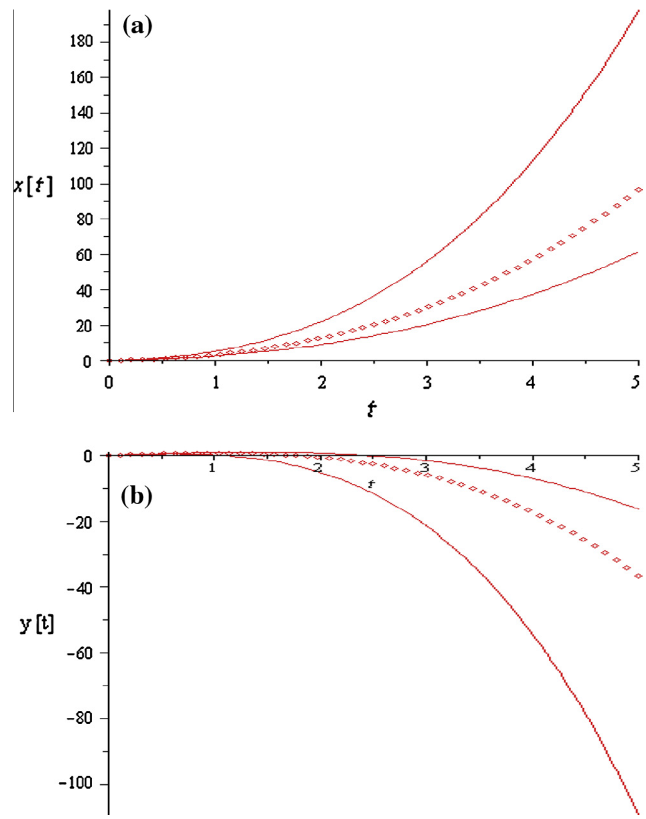

Fig. 5 Plot of system (17) for (a) and (b) when $\alpha_1 = \alpha_2 = 1$; dotted line: exact solution; upper and lower solid lines denote approximate solution at: $\hbar = -1.4$ and $\hbar = -0.8$ respectively for Example 4.1

expression are exactly same as those by HPM and ADM [30] respectively. This illustrates that the two methods are indeed special cases of HAM [31] and HATM. However, the results given by the ADM and HPM converge to the corresponding numerical solutions in a rather small region and calculations of complicated Adomain's polynomials. But, different from those two methods, the HATM provides us with a simple way to adjust and control the convergence region of series solution by choosing proper values for the auxiliary parameter $\hbar$, by using the suitable auxiliary linear operators $L_i = \frac{d}{dt}$, and $L_i = D_*^{\alpha_i}$. It is to be noted that only four terms of the HATM series solution were used in evaluating the approximate solutions as shown in the Figs. 1–10. It manifests that the efficiency of this approach can be significantly improved by computing further terms of $x(t)$, $y(t)$ and $z(t)$.

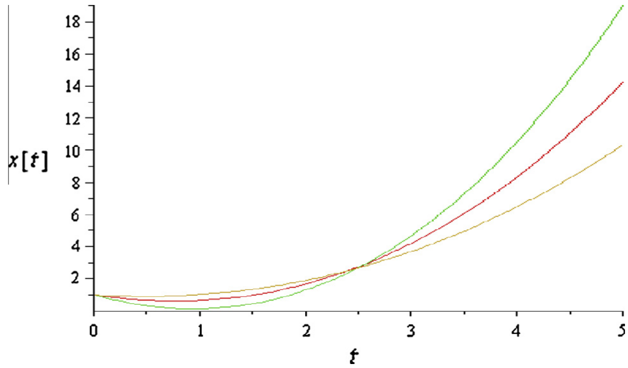

Fig. 6 plot of system (46) for $x[t]$ when $\alpha_1 = \alpha_2 = 1$; red line: exact solution; green and yellow solid lines denote approximate solution at: $\hbar = -1.4$ and $\hbar = -0.8$ respectively for Example 4.2

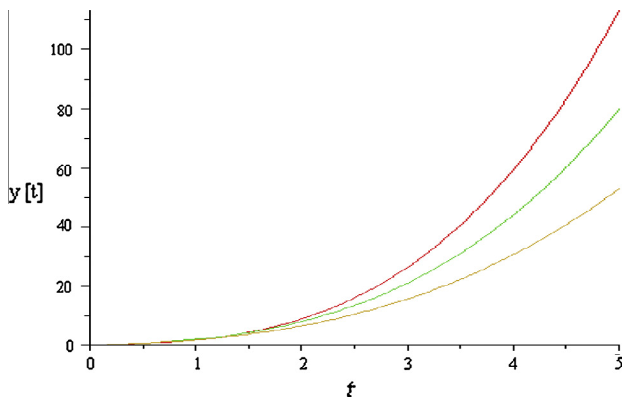

Fig. 7 Plot of system (46) for $y[t]$ when $\alpha_1 = \alpha_2 = 1$; red line: exact solution; green and yellow solid lines denote approximate solution at: $\hbar = -1.4$ and $\hbar = -0.8$ respectively for Example 4.2.

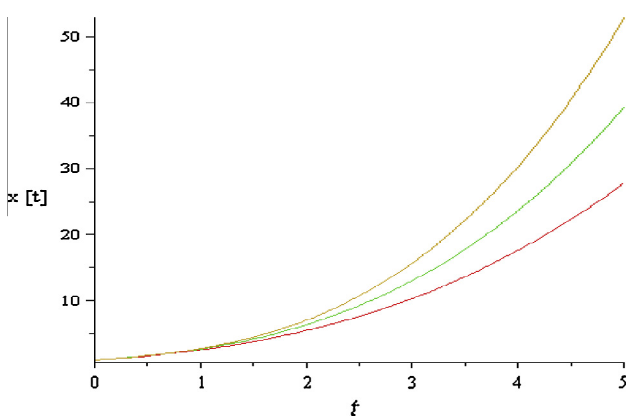

Fig. 8 plot of system (40) for $x[t]$ when $\alpha_1 = \alpha_2 = 1$; red line: exact solution; green and yellow solid lines denote approximate solution at: $\hbar = -1.4$ and $\hbar = -0.8$ respectively for Example 4.3

6. Concluding remarks

In the present paper the HATM has been successfully applied to calculate the approximate solutions for systems of fractional differential equations. Different from all other analytic methods, it provides us with a convenient way to adjust and control the convergence region of solution series by choosing proper values for auxiliary parameter $\hbar$and auxiliary linear operator. In conclusion, the HATM may be considered as a nice refinement in existing numerical techniques and might find the wide applications in science and engineering.

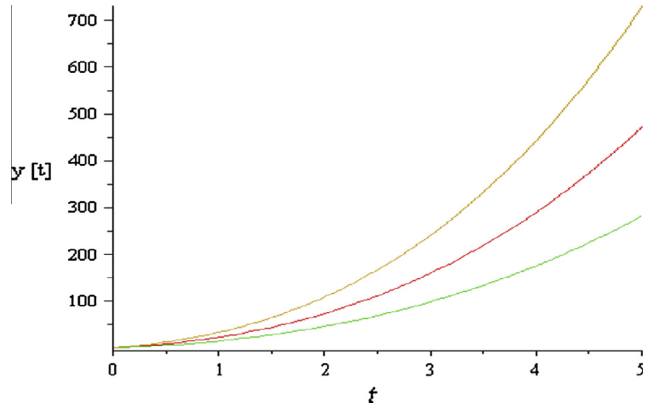

Fig. 9 Plot of system (40) for $y[t]$ when $\alpha_1 = \alpha_2 = 1$; red line: exact solution; green and yellow solid lines denote approximate solution at: $\hbar = -1.4$ and $\hbar = -0.8$ respectively for Example 4.3

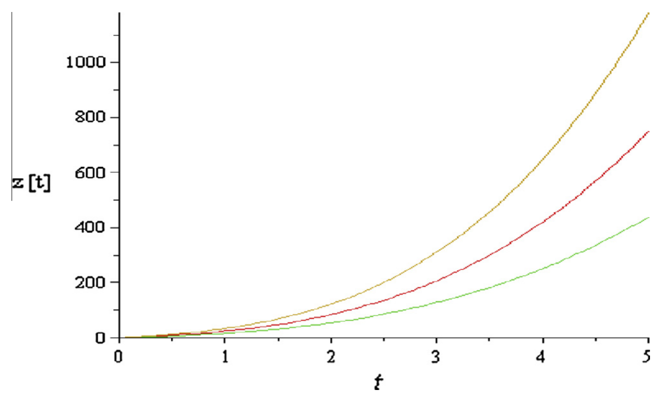

Fig. 10 Plot of system (40) for $z[t]$ when $\alpha_1 = \alpha_2 = 1$; red line: exact solution; green and yellow solid lines denote approximate solution at: $\hbar = -1.4$ and $\hbar = -0.8$ respectively for Example 4.3.

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