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ORIGINAL ARTICLE

Global asymptotic stability of a compartmental model for a pandemic

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Abstract With influenza as a prototype, we propose a compartmental model for a pandemic by taking into account of recruitment. The model has a threshold dynamics. Precisely, when the basic reproduction number $\mathcal{R}_0 \leqslant 1$, the disease free equilibrium is globally asymptotically stable; when $\mathcal{R}_0 > 1$, the disease free equilibrium is unstable and there is a unique endemic equilibrium which globally attracts all solutions except the trivial one (the disease free equilibrium). These results are established by applying the LaSalle's invariance principle.

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1. Introduction

Influenza is one of the most common contagious respiratory illnesses caused by viruses related to negative-sense RNA orthomyxovirade family [1]. The virus can spread from person to person through air by coughs, sneezes or from infected surfaces, and by the direct contact of infected persons. It is also able to shift from species to species and to change its form rapidly. This highly spreadable disease causes about three to five million cases of acute respiratory infections and 250,000–500,000 deaths every year worldwide [2,3]. Even in the developed countries such as USA, Europe, and Canada, the morbidity and the mortality are very high. As an example, in USA more than 200,000 people are hospitalized from flu complication that results in an average 23,600 (approximately) annual deaths [4].

Anyone infected by flu may have symptoms of fever, sore throat, muscle pains, headache, coughing and fatigue. Individuals incubate the virus for nearly 1–3 days before becoming infectious. The infectious period is generally 3–6 days, and the duration of the disease is typically 2–7 days [5].

Epidemic models are important to study the transmission dynamics of infectious diseases and their future risks to human population, and to seek the optimum prevention and control strategies. They provide us with useful information, such as disease transmission, spread of disease agent, epidemiological trends, and preparedness for the disease outbreak.

Arino et al. [6] argued that "as a general policy in preparing for an outbreak of a disease whose parameters are not yet known, it would be better to use a general compartmental model involving relatively few parameters and not depending critically on the particular as yet unknown setting." As a

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result, they proposed a compartmental SLIAR epidemic model with influenza being a prototype. This model was built on the assumption that a significant fraction of the infected individuals never develop symptoms (called asymptomatic cases). The people with asymptomatic infection are able to transmit the disease although they do not have any sign of the disease. Therefore, infectious population is divided into two compartments according to whether or not they develop the symptoms after being infected. They calculated the basic reproduction number and obtained the final size relation. In their study, they neglected the important factor of recruitment.

The purpose of this paper is to study the effect of recruitment. It turns out that the dynamics is quite different from that in [6]. The remaining of this paper is organized as follows. First we formulate the model in Section 2. Then, in Sections 3 and 4, we study the stability of the disease free equilibrium and the endemic equilibrium, respectively. The paper concludes with a brief discussion.

2. Model formulation

The total population $N(t)$ is divided into five classes: susceptible ($S(t)$), latent ($L(t)$), symptomatically infective ($I(t)$), asymptomatically infective ($A(t)$), and recovered ($R(t)$). It is assumed that there is an incubation period between infection and development of disease before an infected person is being infectious. Thus after being infected the susceptible individuals first move to latent class, then to infectious class (either $I(t)$ or $A(t)$), and finally progress to recovered class.

To build a concrete model, we make the following assumptions.

- There is a constant recruitment rate Λ into the susceptible class and the natural death rate is μ.
- The transmission coefficient of the symptomatic infective is β, whereas the infectiousness due to asymptomatic individuals is reduced by a factor δ.
- The rate of having infectiousness is k while the probability being symptomatic infective is p.
- The recovered rates for symptomatic and asymptomatic classes are r_1 and r_2, respectively, and the death rates due to symptomatic and asymptomatic infection are d_1 and d_2, respectively.

Based on the above assumptions, we can sketch the transmission diagram in Fig. 1. These assumptions lead to the model

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \lambda_s(t)S - \mu S,\\
\frac{dL}{dt} &= \lambda_s(t)S - kL - \mu L,\\
\frac{dI}{dt} &= kpL - r_1 I - (\mu + d_1)I,\\
\frac{dA}{dt} &= k(1-p)L - r_2 A - (\mu + d_2)A,\\
\frac{dR}{dt} &= r_1 I + r_2 A - \mu R,
\end{aligned} \tag{2.1}$$

where $\lambda_s(t) = \beta(I + \delta A)$. Since the fifth equation in (2.1) is decoupled from the other four equations, we only focus on the first four equations of (2.1) in the sequel, namely,

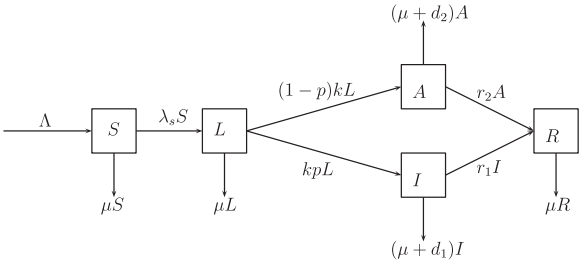

Fig. 1 The transmission diagram for an *SLIAR* model of influenza.

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \lambda_s(t)S - \mu S,\\
\frac{dL}{dt} &= \lambda_s(t)S - kL - \mu L,\\
\frac{dI}{dt} &= kpL - r_1 I - (\mu + d_1)I,\\
\frac{dA}{dt} &= k(1-p)L - r_2 A - (\mu + d_2)A.
\end{aligned} \tag{2.2}$$

It is not difficult to show that the feasible region of (2.2)

$$\Gamma = \left\{(S, L, I, A) \in \mathbb{R}^4_+ : S + L + I + A \leqslant \frac{\Lambda}{\mu}\right\}$$

is a positively invariant and attracting set that attracts all solutions of (2.2) with nonnegative initial conditions. For the long term behavior of (2.2), we only consider solutions in Γ. In the following two sections, we study the stability of the disease free equilibrium and the endemic equilibrium.

3. The global asymptotic stability of the disease free equilibrium

It is easy to see that (2.2) has a unique disease free equilibrium $E^0 = (S_0, 0, 0, 0)$, where $S_0 = \Lambda/\mu$. We first study the local stability of E^0 by linearization.

Let

$$\mathcal{R}_0 = \frac{\beta S_0 kp}{(k+\mu)(r_1+\mu+d_1)} + \frac{\beta\delta S_0 k(1-p)}{(k+\mu)(r_2+\mu+d_2)}.$$

Note that $\mathcal{R}_0$ is called the *basic reproduction number* and it can be calculated by the next generation matrix method [7].

Theorem 3.1. *The disease free equilibrium E^0 of (2.2) is locally exponentially stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix of (2.2) at E^0 is

$$J(E^0) = \begin{bmatrix} -\mu & 0 & -\beta S_0 & -\beta\delta S_0 \\ 0 & -(k+\mu) & \beta S_0 & \beta\delta S_0 \\ 0 & kp & -(r_1+\mu+d_1) & 0 \\ 0 & k(1-p) & 0 & -(r_2+\mu+d_2) \end{bmatrix}.$$

Denote

$$A_{22} = \begin{bmatrix} -(k+\mu) & \beta S_0 & \beta\delta S_0 \\ kp & -(r_1+\mu+d_1) & 0 \\ k(1-p) & 0 & -(r_2+\mu+d_2) \end{bmatrix}.$$

Clearly, one eigenvalue of $J(E^0)$ is $\lambda_1 = -\mu$ (which is negative) and the other three ones are those of A_{22}. Since

$$\begin{aligned}\det(A_{22}) &= -(k+\mu)(r_1+\mu+d_1)(r_2+\mu+d_2)\\ &\quad + \beta S_0 kp(r_2+\mu+d_2) + \beta\delta S_0 k(1-p)(r_1+\mu+d_1)\\ &= (k+\mu)(r_1+\mu+d_1)(r_2+\mu+d_2)(\mathcal{R}_0-1).\end{aligned}$$

If $\mathcal{R}_0 > 1$ then $\det(A_{22}) > 0$. It follows that A_{22} and hence $J(E^0)$ has a positive eigenvalue. Therefore, E^0 is unstable if $\mathcal{R}_0 > 1$. Now, assume that $\mathcal{R}_0 < 1$. In this case, $\det(A_{22}) < 0$ and obviously $\text{trace}(A_{22}) < 0$. Moreover, the second additive compound matrix [8] of A_{22} is

$$A_{22}^{[2]} = \begin{bmatrix} -(l+m) & 0 & -\beta\delta S_0 \\ 0 & -(l+n) & \beta S_0 \\ -k(1-p) & kp & -(m+n) \end{bmatrix},$$

where $l = k+\mu, m = r_1+\mu+d_1$, and $n = r_2+\mu+d_2$. It follows that

$$\begin{aligned}\det(A_{22}^{[2]}) &= (l+m)(l+n)(m+n)\\ &\quad \times \left[\frac{\beta S_0 kp}{(l+n)(m+n)} + \frac{\delta\beta S_0 k(1-p)}{(l+m)(m+n)} - 1\right] < 0\end{aligned}$$

since $\frac{\beta S_0 kp}{(l+n)(m+n)} + \frac{\delta\beta S_0 k(1-p)}{(l+m)(m+n)} < \mathcal{R}_0 < 1$. By Lemma 3 [9], all eigenvalues of A_{22} have negative real parts and this is also true for $J(E^0)$. Therefore, E^0 is locally exponentially stable if $\mathcal{R}_0 < 1$. This completes the proof. □

In fact, E^0 is globally asymptotically stable if $\mathcal{R}_0 \leqslant 1$.

Theorem 3.2. *The disease free equilibrium E^0 of (2.2) is globally asymptotically stable if $\mathcal{R}_0 \leqslant 1$.*

Proof. Note that, for any solution of (2.2), $S(t)$ will be eventually positive and the set

$$\Omega = \left\{(S,L,I,A) \in \mathbb{R}_+^4 : S > 0, S+L+I+A \leqslant \frac{\Lambda}{\mu}\right\}$$

is a positively invariant subset of (2.2). So we only need to show that E^0 is globally asymptotically stable in Ω. To this purpose, we consider the Lyapunov function $V : \Omega \to \mathbb{R}$ defined by

$$V(S,L,I,A) = \left(S - S_0 - S_0 \ln\frac{S}{S_0}\right) + L + a_1 I + a_2 A,$$

where $a_1 = \beta S_0/(r_1+\mu+d_1)$ and $a_2 = \beta\delta S_0/(r_2+\mu+d_2)$. Obviously, V is C^1, $V(E^0) = 0$ and V attains the global minimum 0 in Ω only at E^0.

Now, we calculate the time derivative of V along the trajectories of (2.2).

$$\begin{aligned}\frac{dV}{dt} &= \left(1-\frac{S_0}{S}\right)\frac{dS}{dt} + \frac{dL}{dt} + a_1\frac{dI}{dt} + a_2\frac{dA}{dt}\\ &= \left(1-\frac{S_0}{S}\right)[\Lambda - \beta(I+\delta A)S - \mu S] + [\beta(I+\delta A)S - (k+\mu)L]\\ &\quad + a_1[kpL - (r_1+\mu+d_1)I] + a_2[k(1-p)L - (r_2+\mu+d_2)A]\\ &= \left(1-\frac{S_0}{S}\right)[\mu S_0 - \beta(I+\delta A)S - \mu S] + [\beta(I+\delta A)S - (k+\mu)L]\\ &\quad + a_1[kpL - (r_1+\mu+d_1)I] + a_2[k(1-p)L - (r_2+\mu+d_2)A]\\ &= -\mu S\left(1-\frac{S_0}{S}\right)^2 - \beta(I+\delta A)S + \beta(I+\delta A)S_0 + \beta(I+\delta A)S\\ &\quad - (k+\mu)L + a_1[kpL - (r_1+\mu+d_1)I]\\ &\quad + a_2[k(1-p)L - (r_2+\mu+d_2)A].\end{aligned}$$

Here, we have used the fact that $\Lambda = \mu S_0$. With the choices of a_1 and a_2 (it should be clear why we made such choices), we see that

$$\begin{aligned}\frac{dV}{dt} &= -\mu S\left(1-\frac{S_0}{S}\right)^2\\ &\quad - \left[(k+\mu) - \left\{\frac{\beta S_0 kp}{(r_1+\mu+d_1)} + \frac{\beta\delta S_0 k(1-p)}{(r_2+\mu+d_2)}\right\}\right]L\\ &= -\mu S\left(1-\frac{S_0}{S}\right)^2 - (k+\mu)(1-\mathcal{R}_0)L.\end{aligned}$$

It follows that $\frac{dV}{dt} \leqslant 0$ when $\mathcal{R}_0 \leqslant 1$. Observe that $\left\{(S,L,I,A) : \frac{dV}{dt} = 0\right\} \subseteq \{(S,L,I,A) : S = S^0\}$. We claim that the only complete orbit in $\{(S,L,I,A) : S = S^0\}$ is the disease free equilibrium. In fact, if $(S(t),L(t),I(t),A(t))$ is such a solution, then

$$0 = \frac{dS(t)}{dt} = \Lambda - \beta(I(t)+\delta A(t))S^0 - \mu S^0,$$

which implies that $I(t)+\delta A(t) = 0$. As both $I(t)$ and $A(t)$ are nonnegative, we get that $I(t) = A(t) = 0$. Then $0 = \frac{dI(t)}{dt} = kpL(t)$ produces $L(t) = 0$. This proves the claim. The observation and the claim combined tell us that the only complete orbit in $\left\{(S,L,I,A) : \frac{dV}{dt} = 0\right\}$ is the disease free equilibrium E^0. Therefore, by LaSalle's invariance principle [10,pp. 127], E^0 is globally asymptotically stable in Ω if $\mathcal{R}_0 \leqslant 1$. This completes the proof. □

4. The global asymptotic stability of the endemic equilibrium

In this section, we consider the endemic equilibria of (2.2). For an endemic equilibrium (S^*, L^*, I^*, A^*), we have at least one of I^* and A^* is nonzero. Moreover,

$$\begin{aligned}&\Lambda - \beta(I^*+\delta A^*)S^* - \mu S^* = 0,\\ &\beta(I^*+\delta A^*)S^* - (k+\mu)L^* = 0,\\ &kpL^* - (r_1+\mu+d_1)I^* = 0,\\ &k(1-p)L^* - (r_2+\mu+d_2)A^* = 0.\end{aligned} \tag{4.1}$$

From the third and fourth equations of (4.1), we have

$$I^* = \frac{kpL^*}{(r_1+\mu+d_1)} \quad \text{and} \quad A^* = \frac{k(1-p)L^*}{(r_2+\mu+d_2)}.$$

Substituting these values into the second equation of (4.1) and noting that $L^* \neq 0$, we get $S^* = S_0/\mathcal{R}_0$. These, combined with the first equation of (4.1), give us

$$L^* = \frac{\mu S_0(\mathcal{R}_0-1)}{(k+\mu)\mathcal{R}_0}.$$

It follows that biologically meaningful endemic equilibrium exists if and only if $\mathcal{R}_0 > 1$. When the endemic equilibria exist, there is only one, denoted by $E^* = (S^*, L^*, I^*, A^*)$, where

$$\begin{aligned}S^* &= \frac{S_0}{\mathcal{R}_0},\\ L^* &= \frac{\mu S_0(\mathcal{R}_0-1)}{(k+\mu)\mathcal{R}_0},\\ I^* &= \frac{kp}{r_1+\mu+d_1}L^*,\\ A^* &= \frac{k(1-p)}{r_2+\mu+d_2}L^*.\end{aligned}$$

Though with given parameter values we can obtain the local stability of E^* by linearization and the Hurwitz criterion, it is difficult to discuss its local stability in general. In the following, we use the LaSalle's invariance principle to establish the globally asymptotic stability of E^*. Before doing it, we need a result on persistence of (2.2).

It is easy to see that a solution $(S(t), L(t), I(t), A(t))$ of (2.2) with $L(0)+I(0)+A(0)>0$ is eventually (componentwise) positive. Hence, for the global stability of the endemic equilibrium E^*, we only need to consider the interior of the feasible region Γ,

$$\overset{\circ}{\Gamma} = \{(S, L, I, A) \in \Gamma : S>0, L>0, I>0, A>0\},$$

since it is also a positively invariant set of (2.2). This is achieved with an algebraic approach by Li *et al.* [11] to construct a suitable Lyapunov function. First we need the permanence of the system. With similar arguments as those in [12], one can easily prove the uniform persistence of system (2.2).

Proposition 4.1. *If $\mathcal{R}_0 > 1$, then (2.2) is uniformly persistent, that is, there exists a positive constant $c>0$ such that*

$$\min\left\{\liminf_{t\to\infty} S(t), \liminf_{t\to\infty} L(t), \liminf_{t\to\infty} I(t), \liminf_{t\to\infty} A(t)\right\} > c.$$

As Γ is attractive, it follows that

$$\max\left\{\limsup_{t\to\infty} S(t), \limsup_{t\to\infty} E(t), \limsup_{t\to\infty} I(t), \limsup_{t\to\infty} A(t)\right\} \leqslant \frac{\Lambda}{\mu}.$$

This, combined with Proposition 4.1, implies that (2.2) is permanent if $\mathcal{R}_0 > 1$. Therefore, system (2.2) has a compact absorbing set $K \subset \overset{\circ}{\Gamma}$ (see, for example, [13]).

Theorem 4.1. *If $\mathcal{R}_0 > 1$, then the unique endemic equilibrium E^* of (2.2) is globally asymptotically stable in $\overset{\circ}{\Gamma}$.*

Proof. We only need to prove that E^* is globally asymptotically stable in a compact absorbing set of $\overset{\circ}{\Gamma}$, which exists as we have argued above. Let K be such a compact absorbing set. Define a Lyapunov function $V: K \to \mathbb{R}$ by

$$\begin{aligned} V(S,L,I,A) = {} & \left(S - S^* - S^* \ln\frac{S}{S^*}\right) + a_1\left(L - L^* - L^* \ln\frac{L}{L^*}\right) \\ & + a_2\left(I - I^* - I^* \ln\frac{I}{I^*}\right) \\ & + a_3\left(A - A^* - A^* \ln\frac{A}{A^*}\right), \end{aligned}$$

where a_1, a_2, and a_3 are nonnegative constants to be determined. Clearly, V is C^1, $V(E^*)=0$ and V is strictly positive at other points in K.

The time derivative of V along the solutions of (2.2) is given by

$$\begin{aligned} \frac{dV}{dt} = {} & \left(1-\frac{S^*}{S}\right)\frac{dS}{dt} + a_1\left(1-\frac{L^*}{L}\right)\frac{dL}{dt} \\ & + a_2\left(1-\frac{I^*}{I}\right)\frac{dI}{dt} + a_3\left(1-\frac{A^*}{A}\right)\frac{dA}{dt} \\ = {} & \left(1-\frac{S^*}{S}\right)[\Lambda - \beta(I+\delta A)S - \mu S] \\ & + a_1\left(1-\frac{L^*}{L}\right)[\beta(I+\delta A)S - (k+\mu)L] \\ & + a_2\left(1-\frac{I^*}{I}\right)[kpL - (r_1+\mu+d_1)I] \\ & + a_3\left(1-\frac{A^*}{A}\right)[k(1-p)L - (r_2+\mu+d_2)A] \\ = {} & [\Lambda - \beta(I+\delta A)S - \mu S] + \left[-\Lambda\frac{S^*}{S} + \beta(I+\delta A)S^* + \mu S^*\right] \\ & + a_1[\beta(I+\delta A)S - (k+\mu)L] + a_1\left[-\beta(I+\delta A)L^*\frac{S}{L} + (k+\mu)L^*\right] \\ & + a_2[kpL - (r_1+\mu+d_1)I] + a_2\left[-kpI^*\frac{L}{I} + (r_1+\mu+d_1)I^*\right] \\ & + a_3[k(1-p)L - (r_2+\mu+d_2)A] \\ & + a_3\left[-k(1-p)A^*\frac{L}{A} + (r_2+\mu+d_2)A^*\right] \\ = {} & [\Lambda + \mu S^* + a_1(k+\mu)L^* + a_2(r_1+\mu+d_1)I^* + a_3(r_2+\mu+d_2)A^*] \\ & - \mu S - (1-a_1)\beta SI - (1-a_1)\beta\delta SA + [\beta S^* - a_2(r_1+\mu+d_1)]I \\ & + [\beta\delta S^* - a_3(r_2+\mu+d_2)]A - [a_1(k+\mu) - a_2kp - a_3k(1-p)]L \\ & - \Lambda\frac{S^*}{S} - a_1\beta L^*\frac{SI}{L} - a_1\beta\delta L^*\frac{SA}{L} - a_2kpI^*\frac{L}{I} - a_3k(1-p)A^*\frac{L}{A}. \end{aligned}$$

For the simplicity of notation, denote $w \equiv \frac{S}{S^*}, x \equiv \frac{L}{L^*}, y \equiv \frac{I}{I^*}$, and $z \equiv \frac{A}{A^*}$. Let $C = \Lambda + \mu S^* + a_1(k+\mu)L^* + a_2(r_1+\mu+d_1)I^* + a_3(r_2+\mu+d_2)A^*$. Then

$$\begin{aligned} \frac{dV}{dt} = {} & C - \mu S^* w - (1-a_1)\beta S^* I^* wy - (1-a_1)\beta\delta S^* A^* wz \\ & - [a_1(k+\mu) - a_2kp - a_3k(1-p)]L^* x - [a_2(r_1 \\ & + \mu + d_1) - \beta S^*]I^* y - [a_3(r_2+\mu+d_2) \\ & - \beta\delta S^*]A^* z - a_2kpL^*\frac{x}{y} - a_3k(1-p)L^*\frac{x}{z} - \Lambda\frac{1}{w} \\ & - a_1\beta S^* I^*\frac{wy}{x} - a_1\beta\delta S^* A^*\frac{wz}{x}. \end{aligned} \tag{4.2}$$

As in [11], define a set D of the above terms by

$$D = \left\{w, x, y, z, wy, wz, \frac{1}{w}, \frac{x}{y}, \frac{x}{z}, \frac{wy}{x}, \frac{wz}{x}\right\}.$$

Then we can consider the arithmetic means and geometric means of the following three sets to determine a_1, a_2, and a_3,

$$\left\{w, \frac{1}{w}\right\}, \quad \left\{\frac{1}{w}, \frac{wy}{x}, \frac{x}{y}\right\}, \quad \text{and} \quad \left\{\frac{1}{w}, \frac{wz}{x}, \frac{x}{z}\right\},$$

while the following relationships among S^*, L^*, I^*, and A^* are useful.

$$\begin{aligned} & \Lambda = \beta(I^* + \delta A^*)S + \mu S^*, \\ & \beta(I^* + \delta A^*)S^* = (k+\mu)L^*, \\ & kpL^* = (r_1+\mu+d_1)I^*, \\ & k(1-p)L^* = (r_2+\mu+d_2)A^*. \end{aligned} \tag{4.3}$$

Suppose the right hand side of (4.2) can be rewritten as

$$\begin{aligned} & b_1\left(2 - w - \frac{1}{w}\right) + b_2\left(3 - \frac{1}{w} - \frac{wy}{x} - \frac{x}{y}\right) \\ & \quad + b_3\left(3 - \frac{1}{w} - \frac{wz}{x} - \frac{x}{z}\right), \end{aligned}$$

where the coefficients b_1, b_2, and b_3 are unknown quantities. Then equating the coefficients of like terms of these two expressions gives us

$$\begin{aligned} & 2b_1 + 3b_2 + 3b_3 = C, \\ & b_1 = \mu S^*, \\ & 1 - a_1 = 0, \end{aligned}$$

$$a_2(r_1+\mu+d_1)=\beta S^*,$$
$$a_3(r_2+\mu+d_2)=\beta\delta S^*,$$
$$b_1+b_2+b_3=\Lambda,$$
$$b_2=a_2kpL^*=a_1\beta S^*I^*,$$
$$b_3=a_1\beta\delta S^*A^*=a_3k(1-p)L^*,$$
$$k+\mu=a_2kp+a_3k(1-p).$$

With (4.3), the above linear system is consistent and we can choose

$$a_1=1,\quad a_2=\frac{\beta S^*}{r_1+\mu+d_1}=\frac{\beta S^*I^*}{kpL^*},$$
$$a_3=\frac{\beta\delta S^*}{r_2+\mu+d_2}=\frac{\beta\delta S^*A^*}{k(1-p)L^*}.$$

By the arithmetic mean-geometric mean inequality, we get

$$\frac{dV}{dt}=\mu S^*\left(2-w-\frac{1}{w}\right)+\beta S^*I^*\left(3-\frac{1}{w}-\frac{wy}{x}-\frac{x}{y}\right)+\beta\delta S^*A^*\left(3-\frac{1}{w}-\frac{wz}{x}-\frac{x}{z}\right)\leqslant 0.$$

Note that $\left\{(S,L,I,A)\in K:\frac{dV}{dt}=0\right\}=\{(S,L,I,A)\in K:S=S^*,\frac{L}{L^*}=\frac{I}{I^*}=\frac{A}{A^*}\}$. We claim that the only complete orbit in $\left\{(S,L,I,A)\in K:\frac{dV}{dt}=0\right\}$ is the endemic equilibrium E^*. In fact, let $(S(t),L(t),I(t),A(t))$ be such a solution. Denote $\frac{L(t)}{L^*}=\frac{I(t)}{I^*}=\frac{A(t)}{A^*}=l(t)$. Then

$$0=\frac{dS(t)}{dt}=\frac{dS^*}{dt}=\Lambda-\beta l(t)(I^*+\delta A^*)S^*-\mu S^*,$$

which implies that

$$l(t)=\frac{\Lambda-\mu S^*}{\beta(I^*+\delta A^*)S^*}=1.$$

Therefore, $(S(t),L(t),I(t),A(t))=E^*$. This proves the claim. By LaSalle's invariance principle, we deduce that E^* is globally asymptotically stable in K and hence in Γ. This completes the proof. □

5. Discussion

In this paper, we modified a compartmental model proposed by Arino et al. [6] with the consideration of recruitment. The new model has different dynamics from the original model. For the original model, every equilibrium is a disease free equilibrium and there are infinitely many equilibria. Every solution tends to an equilibrium where the final size is determined. The modified model has a threshold dynamics. There is a basic reproduction number $\mathcal{R}_0$. If $\mathcal{R}_0\leqslant 1$, then there is only the disease free equilibrium and it is globally asymptotically stable. If $\mathcal{R}_0>1$, then the disease free equilibrium is unstable and there is also a unique endemic equilibrium which attracts all solutions other than the disease free equilibrium. The basic reproduction number is expressed in terms of the model parameters. Therefore, prediction and prevention strategies can be easily made.

Nevertheless, the model is a little simple. There are several factors need to be considered. For example, on the one hand, as mentioned in Introduction, there is an incubation period for an infected to be infectious. Hence, it is more reasonable to build models described with delay differential equations. On the other hand, for influenza, the recovered can be susceptible again. As a result, SLIARS models may be proposed. We want to check whether the threshold dynamics preserves or not. We leave these as future study.

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