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ORIGINAL ARTICLE

Fixed point theorems in fuzzy metric spaces ☆

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Abstract In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).

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1. Introduction

In 1965, the theory of fuzzy sets was investigated by Zadeh [2]. In 1981, Heilpern [3] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [4]. Therefore, several fixed point theorems for types of fuzzy contractive mappings have appeared (see, for instance [1,5–9]).

In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results (see [1]).

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2. Basic preliminaries

The definitions and terminologies for further discussions are taken from Heilpern [3]. Let (X,d) be a metric linear space. A **fuzzy set** in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the **grade of membership** of x in A. The collection of all fuzzy sets in X is denoted by $\Im(X)$.

Let $A \in \Im(X)$ and $\alpha \in [0,1]$. The **α-level set** of A, denoted by A_α, is defined by

$$A_\alpha = \{x : A(x) \geqslant \alpha\} \quad \text{if} \quad \alpha \in (0,1], \quad A_0 = \overline{\{x : A(x) > 0\}},$$

whenever $\overline{B}$ is the closure of set (nonfuzzy) B.

Definition 2.1. A fuzzy set A in X is an **approximate quantity** iff its α-level set is a nonempty compact convex subset (nonfuzzy) of X for each $\alpha \in [0,1]$ and $sup_{x \in X} A(x) = 1$.

The set of all approximate quantities, denoted by $W(X)$, is a subcollection of $\Im(X)$.

Definition 2.2. Let $A, B \in W(X)$, $\alpha \in [0,1]$ and $CP(X)$ be the set of all nonempty compact subsets of X. Then

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$$p_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y), \quad \delta_\alpha(A,B) = \sup_{x\in A_\alpha, y\in B_\alpha} d(x,y) \quad \text{and}$$
$$D_\alpha(A,B) = H(A_\alpha, B_\alpha),$$

where H is the **Hausdorff metric** between two sets in the collection $CP(X)$. We define the following functions

$$p(A,B) = \sup_\alpha p_\alpha(A,B), \quad \delta(A,B) = \sup_\alpha \delta_\alpha(A,B) \quad \text{and}$$
$$D(A,B) = \sup_\alpha D_\alpha(A,B).$$

It is noted that p_α is nondecreasing function of α.

Definition 2.3. Let A, $B \in W(X)$. Then A is said to be **more accurate** than B (or B includes A), denoted by $A \subset B$, iff $A(x) \leqslant B(x)$ for each $x \in X$.

The relation $\subset$ induces a partial order on $W(X)$.

Definition 2.4. Let X be an arbitrary set and Y be a metric linear space. F is said to be a **fuzzy mapping** iff F is a mapping from the set X into $W(Y)$, i.e., $F(x) \in W(Y)$ for each $x \in X$.

The following proposition is used in the sequel.

Proposition 2.1. ([4]).*If A, $B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a,b) \leqslant H(A,B)$.*

Following Beg and Ahmed [10], let (X,d) be a metric space. We consider a subcollection of $\Im(X)$ denoted by $W^*(X)$. Each fuzzy set $A \in W^*(x)$, its α-level set is a nonempty compact subset (nonfuzzy) of X for each $\alpha \in [0,1]$. It is obvious that each element $A \in W(X)$ leads to $A \in W^*(X)$ but the converse is not true.

The authors [10] introduced the improvements of the lemmas in Heilpern [3] as follows.

Lemma 2.1. *If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_\alpha(x_0, B) \leqslant D_\alpha(A,B)$ for each $B \in W^*(X)$.*

Lemma 2.2. *$p_\alpha(x,A) \leqslant d(x,y) + p_\alpha(y,A)$ for all x, $y \in X$ and $A \in W^*(X)$.*

Lemma 2.3. *Let $x \in X$, $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$.*

Lemma 2.4. *Let (X,d) be a complete metric space, F: $X \to W^*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Remark 2.1. It is clear that Lemma 2.4 is a generalization of corresponding lemma in Arora and Sharma [1] and Proposition 3.2 in Lee and Cho [7].

Let Ψ be the family of real lower semi-continuous functions F: $[0,\infty)^6 \to R$, $R :=$ the set of all real numbers, satisfying the following conditions:

(ψ_1) F is non-increasing in 3rd, 4th, 5th, 6th coordinate variable,
(ψ_2) there exists $h \in (0,1)$ such that for every u, $v \geqslant 0$ with
(ψ_{21}) $F(u,v,v,u,u+v,0) \leqslant 0$ or (ψ_{22}) $F(u,v,u,v,0,u+v) \leqslant 0$, we have $u \leqslant h\,v$, and
(ψ_3) $F(u,u,0,0,u,u) > 0$ for all $u > 0$.

3. Main results

In 2000, Arora and Sharma [1] proved the following result.

Theorem 3.1. *Let (X,d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W(X)$. If there is a constant q, $0 \leqslant q < 1$, such that, for each x, $y \in X$,*

$$D(T_1(x), T_2(y)) \leqslant q \max\{d(x,y), p(x,T_1(x)), p(y,T_2(y)), p(x,T_2(y)), p(y,T_1(x))\},$$

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Remark 3.1. If there is a constant q, $0 \leqslant q < 1$, such that, for each x, $y \in X$,

$$D(T_1(x), T_2(y)) \leqslant q \max\{d(x,y), p(x,T_1(x)), p(y,T_2(y))\}, \tag{1}$$

then the conclusion of Theorem 3.1 remains valid. This result is considered as a special case of Theorem 3.1.

Beg and Ahmed [10] generalized Theorem 3.1 as follows.

Theorem 3.2. *Let (X,d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W^*(X)$. If there is a $F \in \Psi$ such that, for all x, $y \in X$,*

$$F(D(T_1(x), T_2(y)), d(x,y), p(x,T_1(x)), p(y,T_2(y)), p(x,T_2(y)), p(y,T_1(x))) \leqslant 0, \tag{2}$$

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Widely inspired by a paper of Tas et al. [11], we give another different generalization of Theorem 3.1 with contractive condition (1) as follows.

Theorem 3.3. *Let (X,d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W^*(X)$. Assume that there exist c_1, c_2, $c_3 \in [0,\infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ such that, for all x, $y \in X$,*

$$\begin{aligned} D^2(T_1(x), T_2(y)) \leqslant{}& c_1 \max\{d^2(x,y), p^2(x,T_1(x)), p^2(y,T_2(y))\} \\ &+ c_2 \max\{p(x,T_1(x))p(x,T_2(y)), p(y,T_1(x)) \\ &p(y,T_2(y))\} + c_3 p(x,T_2(y))p(y,T_1(x)). \end{aligned} \tag{3}$$

Then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Proof. Let x_0 be an arbitrary point in X. Then by Lemma 2.4, there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$. For $x_1 \in X$, $(T_2(x_1))_1$ is nonempty compact subset of X. Since $(T_1(x_0))_1$, $(T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0))_1$, then Proposition 2.1 asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1,x_2) \leqslant D_1(T_1(x_0), T_2(x_1))$. So, we obtain from the inequality $D(A,B) \geqslant D_\alpha(A,B)$ $\forall \alpha \in [0,1]$ that

$$\begin{aligned} d^2(x_1,x_2) &\leqslant D_1^2(T_1(x_0),T_2(x_1)) \leqslant D^2(T_1(x_0),T_2(x_1)) \\ &\leqslant c_1 \max\{d^2(x_0,x_1), p^2(x_0,T_1(x_0)), p^2(x_1,T_2(x_1))\} \\ &+c_2 \max\{p(x_0,T_1(x_0))p(x_0,T_2(x_1)), \\ &\quad p(x_1,T_1(x_0))p(x_1,T_2(x_1))\} \\ &+c_3 p(x_0,T_2(x_1))p(x_1,T_1(x_0)) \\ &\leqslant c_1 \max\{d^2(x_0,x_1), d^2(x_1,x_2)\} \\ &+c_2 d(x_0,x_1)[d(x_0,x_1)+d(x_1,x_2)]. \end{aligned}$$

If $d(x_1,x_2) > d(x_0,x_1)$, then we have

$$d^2(x_1,x_2) \leqslant (c_1+2c_2)d^2(x_1,x_2),$$

which is a contradiction. Thus,

$$d(x_1,x_2) \leqslant hd(x_0,x_1),$$

where $h=\sqrt{c_1+2c_2}<1$. Similarly, one can deduce that

$$d(x_2,x_3) \leqslant hd(x_1,x_2).$$

By induction, we have a sequence (x_n) of points in X such that, for all $n \in N \cup \{0\}$,

$$\{x_{2n+1}\} \subset T_1(x_{2n}), \quad \{x_{2n+2}\} \subset T_2(x_{2n+1}).$$

It follows by induction that $d(x_n,x_{n+1}) \leqslant h^n\, d(x_0,x_1)$. Since

$$\begin{aligned} d(x_n,x_m) &\leqslant d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{m-1},x_m) \\ &\leqslant h^n d(x_0,x_1)+h^{n+1}d(x_0,x_1)+\ldots+h^{m-1}d(x_0,x_1) \\ &\leqslant \tfrac{h^n}{1-h}d(x_0,x_1), \end{aligned}$$

then $\lim_{n,\ m\to\infty} d(x_n,x_m)=0$. Therefore, (x_n) is a Cauchy sequence. Since X is complete, then there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset T_i(z)$, $i=1, 2$. Now, we get from Lemmas 2.1 and 2.2 that

$$\begin{aligned} p_\alpha(z,T_2(z)) &\leqslant d(z,x_{2n+1})+p_\alpha(x_{2n+1},T_2(z)) \\ &\leqslant d(z,x_{2n+1})+D_\alpha(T_1(x_{2n}),T_2(z)), \end{aligned}$$

for each $\alpha \in [0,1]$. Taking supremum on α in the last inequality, we obtain that

$$p(z,T_2(z)) \leqslant d(z,x_{2n+1})+D(T_1(x_{2n}),T_2(z)). \tag{4}$$

From the inequality (3), we have that

$$\begin{aligned} D^2(T_1(x_{2n}),T_2(z)) &\leqslant c_1 \max\{d^2(x_{2n},z), p^2(x_{2n},T_1(x_{2n})), \\ &\quad p^2(z,T_2(z))\}+c_2 \max\{p(x_{2n},T_1(x_{2n})) \\ &\quad p(x_{2n},T_2(z)), p(z,T_1(x_{2n}))p(z,T_2(z))\} \\ &\quad + c_3 p(x_{2n},T_2(z))p(z,T_1(x_{2n})) \\ &\leqslant c_1 \max\{d^2(x_{2n},z), d^2(x_{2n},x_{2n+1}), \\ &\quad p^2(z,T_2(z))\}+c_2 \max\{d(x_{2n},x_{2n+1}) \\ &\quad p(x_{2n},T_2(z)), d(z,x_{2n+1})p(z,T_2(z))\} \\ &\quad + c_3 p(x_{2n},T_2(z))d(z,x_{2n+1}). \end{aligned} \tag{5}$$

Letting $n \to \infty$ in the inequalities (4) and (5), it follows that

$$p(z,T_2(z)) \leqslant \sqrt{c_1}\, p(z,T_2(z)).$$

Since $\sqrt{c_1}<1$, we see that $p(z,T_2(z))=0$. So, we get from Lemma 2.3 that $\{z\} \subset T_2(z)$. Similarly, one can be shown that $\{z\} \subset T_1(z)$. □

Remark 3.2.

(I) Condition (3) is not deducible from condition (2) since the function F from $[0,\infty)^6$ into $[0,\infty)$ defined as

$$F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^2-c_1\max\{t_2^2,t_3^2,t_4^2\}-c_2\max\{t_3t_5,t_6t_4\}-c_3t_5t_6,$$

for all $t_1, t_2, t_3, t_4, t_5, t_6 \in [0,\infty)$, where $c_1, c_2, c_3 \in [0,\infty)$ with $c_1+2c_2<1$ and $c_2+c_3<1$, does not generally satisfy condition (ψ_3). Indeed, we have that

$$F(u,u,0,0,u,u)=u^2-c_1u^2-c_3u^2,$$

for all $u>0$ and does not imply that $F(u,u,0,0,u,u)>0$ for all $u>0$.It suffices to consider $c_1=\frac{3}{4}$, $c_2=\frac{1}{9}$, $c_3=\frac{1}{2}$ and then $c_1+2c_2<1$ and $c_2+c_3<1$ but $F(u,u,0,0,u,u)<0$ for all $u>0$. Therefore, Theorems 3.2 and 3.3 are two different generalizations of Theorem 3.1 with contractive condition (1).

(II) If there exist $c_1, c_2, c_3 \in [0,\infty)$ with $c_1+2c_2<1$ and $c_2+c_3<1$ such that, for all $x, y \in X$,

$$\begin{aligned} \delta^2(T_1(x),T_2(y)) &\leqslant c_1\max\{d^2(x,y),p^2(x,T_1(x)),p^2(y,T_2(y))\} \\ &+c_2\max\{p(x,T_1(x))p(x,T_2(y)),p(y,T_1(x)) \\ &p(y,T_2(y))\}+c_3p(x,T_2(y))p(y,T_1(x)), \end{aligned}$$

then the conclusion of Theorem 3.3 remains valid. This result is considered as a special case of Theorem 3.3 because $D(F_1(x), F_2(y)) \leqslant \delta(F_1(x), F_2(y))$ [12, page 414]. Moreover, this result generalizes Theorem 3.3 of Park and Jeong [8].

Example 3.1. Let $X=[0,1]$ endowed with the metric d defined by $d(x,y)=|x-y|$. It is clear that (X,d) is a complete metric space. Let $T_1=T_2=T$. Define a fuzzy mapping T on X such that for all $x \in X$, $T(x)$ is the characteristic function for $\{\frac{3}{4}x\}$. For each $x, y \in X$,

$$\begin{aligned} D^2(T(x),T(y)) &= \frac{9}{16}d^2(x,y) \\ &\leqslant c_1\max\{d^2(x,y),p^2(x,T(x)),p^2(y,T(y))\}+c_2 \\ &\times\max\{p(x,T(x))p(x,T(y)),p(y,T(x))p(y,T(y))\} \\ &+c_3p(x,T(y))p(y,T(x)), \end{aligned}$$

where $c_1=\frac{9}{16}<1$ and $c_2=c_3=0$. The characteristic function for $\{0\}$ is the fixed point of T.

The following theorem generalizes Theorem 3.3 to a sequence of fuzzy contractive mappings.

Theorem 3.4. *Let $(T_n\colon n \in N \cup \{0\})$ be a sequence of fuzzy mappings from a complete metric space (X,d) into $W^*(X)$. Assume that there exist $c_1, c_2, c_3 \in [0,\infty)$ with $c_1+2c_2<1$ and $c_2+c_3<1$ such that, for all $x, y \in X$,*

$$\begin{aligned} D^2(T_0(x),T_n(y)) &\leqslant c_1\max\{d^2(x,y),p^2(x,T_0(x)),p^2(y,T_n(y))\} \\ &+c_2\max\{p(x,T_0(x))p(x,T_n(y)), \\ &p(y,T_0(x))p(y,T_n(y))\} \\ &+c_3p(x,T_n(y))p(y,T_0(x)) \quad \forall n \in N. \end{aligned}$$

Then there exists a common fixed point of the family $(T_n\colon n \in N \cup \{0\})$.

Proof. Putting $T_1=T_0$ and $T_2=T_n$ $\forall n \in N$ in Theorem 3.3. Then, there exists a common fixed point of the family $(T_n\colon n \in N \cup \{0\})$. □

Remark 3.3. If there is a $\phi \in \Phi$ such that, for all $x, y \in X$,

$$D^2(T_0(x), T_n(y)) \leqslant c_1 \max\{d^2(x,y), p^2(x, T_0(x)), p^2(y, T_n(y))\} + c_2 \max\{p(x, T_0(x))p(x, T_n(y)), p(y, T_0(x))p(y, T_n(y))\} + c_3 p(x, T_n(y))p(y, T_0(x)) \quad \forall n \in N.$$

then the conclusion of Theorem 3.4 remains valid. This result is considered as a special case of Theorem 3.4 for the same reason in Remark 3.2(I).

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References

[1] S.C. Arora, V. Sharma, Fixed point theorems for fuzzy mappings, Fuzzy Sets Syst. 110 (2000) 127–130.
[2] L.A. Zadeh, Fuzzy sets, Inform. Contr. 8 (1965) 338–353.
[3] S. Heilpern, Fuzzy mappings and fixed point theorem, J. Math. Anal. Appl. 83 (1981) 566–569.
[4] S.B. Nadler, Multivalued contraction mappings, Pac. J. Math. 30 (1969) 475–488.
[5] I. Beg, A. Azam, Fixed points of asymptotically regular multivalued mappings, J. Austral. Math. Soc. 53 (1992) 313–326.
[6] R.K. Bose, D. Sahani, Fuzzy mappings and fixed point theorems, Fuzzy Sets Syst. 21 (1987) 53–58.
[7] B.S. Lee, S.J. Cho, A fixed point theorems for contractive type fuzzy mappings, Fuzzy Sets Syst. 61 (1994) 309–312.
[8] J.Y. Park, J.U. Jeong, Fixed point theorems for fuzzy mappings, Fuzzy Sets Syst. 87 (1997) 111–116.
[9] V. Popa, Common fixed points for multifunctions satisfying a rational inequality, Kobe J. Math. 2 (1985) 23–28.
[10] I. Beg, M.A. Ahmed, Common fixed point for generalized fuzzy contraction mappings satisfying an implicit relation, Appl. Math. Lett. (2012) (in press).
[11] K. Tas, M. Telci, B. Fisher, Common fixed point theorems for compatible mappings, Int. J. Math. Math. Sci. 19 (1996) 451–456.
[12] T.L. Hicks, Multivalued mappings on probabilistic metric spaces, Math. Japon. 46 (1997) 413–418.