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ORIGINAL ARTICLE

A new relation including ${}_2F_2$ between Laguerre and Hermite matrix polynomials

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Abstract In the present paper, a new relation including hypergeometric matrix function between Laguerre and Hermite matrix polynomials presented in [2,3] is derived.

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1. Introduction

Theory of orthogonal matrix polynomials is a growing field of applied mathematics which owes its development from theoretical and practical examples given below. The property of orthogonality [1–3], Rodrigues formula [1], a second-order Sturm–Liouville differential equation [1], a three-term matrix recurrence formula [4], relation between different orthogonal matrix polynomials [5] and matrix polynomials of several variables [6,18–20] are theoretical examples for orthogonal matrix polynomials. Besides, the practical examples for matrix polynomials can be seen in statistics, group representation theory [7], scattering theory [8], differential equations [3], Fourier series expansions [9], quadrature [10], splines [11] and medical imaging [12].

The aim of this paper is to derive a connection between Laguerre and Hermite matrix polynomials recently presented in [2,3].

Now let us give some known facts and definitions.

If A is a matrix in $\mathbb{C}^{r\times r}$, we denote by $\sigma(A)$ the set of all the eigenvalues of A. If $f(z), g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if $\sigma(A) \subset \mathbb{C}$, we denote by $f(A), g(A)$, respectively, the image by the Riesz–Dunford functional calculus of the functions $f(z), g(z)$, respectively, acting on the matrix A, and

$$f(A)g(A) = g(A)f(A)$$

see [13]. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^N$, $\|y\|_2 = (y^T y)^{1/2}$ is the Euclidean norm of y. For $A, B \in \mathbb{C}^{r\times r}$, this norm also satisfies the following properties

$$\begin{aligned} &\|A+B\| \leqslant \|A\| + \|B\| \\ &\|AB\| \leqslant \|A\|\|B\|. \end{aligned} \tag{1}$$

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Throughout this paper, a matrix polynomial of degree n means an expression of the form

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0,$$

where x is a real variable and A_j $(0 < j < n)$ are $r \times r$ complex matrices. For any matrix A in $\mathbb{C}^{r \times r}$, we denote Pochhammer symbol:

$$(A)_n = A(A+I)\ldots(A+(n-1)I), \quad n \geqslant 1, \ (A)_0 = I. \tag{2}$$

The hypergeometric matrix function $F(A,B;C;z)$ has been given in the form [14]:

$$F(A,B;C;z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} \left[(C)_n\right]^{-1} z^n \tag{3}$$

for matrices A, B and C in $\mathbb{C}^{r\times r}$ such that $C + nI$ is invertible for all integer $n \geqslant 0$ and for $|z| < 1$. In [14], Defez and Jódar show that for matrices $A(k,n)$ and $B(k,n)$ in $\mathbb{C}^{r\times r}$ where $n \geqslant 0, k \geqslant 0$, the following relations are satisfied

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \tag{4}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k) \tag{5}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k). \tag{6}$$

2. Known properties of Laguerre and Hermite matrix polynomials

For the sake of clarity in the presentation, we recall that if A is a matrix in $\mathbb{C}^{r\times r}$ such that

$$-k \text{ is not an eigenvalue of } A \text{ for every integer } k > 0 \tag{7}$$

and λ is a complex number with Re $\lambda > 0$, then the n-th Laguerre matrix polynomial is defined by [3]

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}(A+I)_n (A+I)_k^{-1} x^k; \ n \geqslant 0.$$

Furthermore, the following explicit formula holds:

$$x^n I = \sum_{k=0}^{n} \frac{(-1)^k \lambda^{-n} n! (A+I)_n [(A+I)_k]^{-1}}{(n-k)!} L_k^{(A,\lambda)}(x) \tag{8}$$

see [15]. For the definition of Hermite matrix polynomials, let us suppose that A is a matrix such that

$$\text{Re } z > 0 \text{ for every eigenvalue } z \in \sigma(A) \tag{9}$$

and let us denote $\sqrt{A} = exp((1/2)\ \log A)$ by the image of the function $z^{\frac{1}{2}} = \exp((1/2)\log z)$ by the Riesz–Dunford functional calculus, acting on the matrix A, where $\log z$ denotes the principal branch of the complex logarithm. Then by [2] the n – Hermite matrix polynomial $H_n(x,A)$ is defined by

$$H_n(x,A) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!(n-2k)!} \left(x\sqrt{2A}\right)^{n-2k} \ ; \ n \geqslant 0.$$

Furthermore, the following generating matrix function formula for these matrix polynomials holds:

$$\sum_{n=0}^{\infty} \frac{H_n(x,A)}{n!} t^n = \exp\left(xt\sqrt{2A} - It^2\right). \tag{10}$$

In addition to these facts, the interesting connection between Laguerre and Hermite matrix polynomials is given by [5]

$$\frac{(-1)^n}{\sqrt{\pi}(2n)!}\Gamma(A+(n+1)I)\Gamma^{-1}\left(A+\frac{1}{2}I\right) \times \int_{-1}^{1} (1-t^2)^{A-\frac{1}{2}I} H_{2n}(t\sqrt{x},A)dt = \sum_{k=0}^{n} \frac{x^k}{k!}\left(\lambda I - \frac{1}{2}A\right)^k L_{n-k}^{(A+kI,\lambda)}(x) \ ; \ n \geqslant 0, x > 0.$$

3. A new relation between these matrix polynomials

Let's give a definition of ${}_2F_2$ as generalization of the hypergeometric matrix function.

Definition 1. According to presentation of the hypergeometric function, ${}_2F_2$ hypergeometric matrix function is defined as

$${}_2F_2\left(\begin{matrix} A, & B \\ C, & D \end{matrix} ; x\right) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} (C)_n^{-1} (D)_n^{-1} x^n \tag{11}$$

for matrices A, B, C, D in $\mathbb{C}^{r\times r}$ such that $C + nI$ and $D + nI$ are invertible for all integer $n \geqslant 0$.

Now that find values x for which this series (11) is convergent.

Let's write

$$\left.\begin{matrix} (C+nI)^{-1} = \frac{1}{n}\left(\frac{C}{n}+I\right)^{-1} \\ (D+nI)^{-1} = \frac{1}{n}\left(\frac{D}{n}+I\right)^{-1} \end{matrix}\right\}. \tag{12}$$

If $n > \|C\|$ and $n > \|D\|$, due to perturbation Lemma [13], it follows

$$\left.\begin{matrix} \left\|\left(\frac{C}{n}+I\right)^{-1}\right\| \leqslant \frac{1}{1-\frac{\|C\|}{n}} = \frac{n}{n-\|C\|} \\ \left\|\left(\frac{D}{n}+I\right)^{-1}\right\| \leqslant \frac{1}{1-\frac{\|D\|}{n}} = \frac{n}{n-\|D\|} \end{matrix}\right\}. \tag{13}$$

Let $x \neq 0$ be a complex number and let's consider the expression

$$\frac{\|(A)_{n+1}\|\|(B)_{n+1}\|\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|n!}{\|(A)_n\|\|(B)_n\|\|(C)_n^{-1}\|\|(D)_n^{-1}\|(n+1)!} \frac{|x|^{n+1}}{|x|^n}. \tag{14}$$

Using Pochhammer symbol (2) and (1) in (14), one gets

$$\frac{\|(A)_{n+1}\|\|(B)_{n+1}\|\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|n!}{\|(A)_n\|\|(B)_n\|\|(C)_n^{-1}\|\|(D)_n^{-1}\|(n+1)!} \frac{|x|^{n+1}}{|x|^n} \leqslant \frac{\|A\|_{n+1}\|B\|_{n+1}\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|}{\|A\|_n\|B\|_n\|(C)_n^{-1}\|\|(D)_n^{-1}\|} \frac{|x|}{n+1}. \tag{15}$$

For $n > \|C\|$ and $n > \|D\|$, by (1), (13) and (15) can be written

$$\frac{\|(A)_{n+1}\|\|(B)_{n+1}\|\|(C)_{n+1}^{-1}\|\|(D)_{n+1}^{-1}\|n!}{\|(A)_n\|\|(B)_n\|\|(C)_n^{-1}\|\|(D)_n^{-1}\|(n+1)!} \frac{|x|^{n+1}}{|x|^n} \leqslant (\|A\|+n)(\|B\|+n)\frac{1}{n-\|C\|}\frac{1}{n-\|D\|}\frac{|x|}{n+1} \to 0 \ \text{ as } n \to \infty$$

By ratio test, ${}_2F_2$ hypergeometric matrix series is convergent for any complex number x. Let's start now to show the connection satisfied by Laguerre and Hermite matrix polynomials.

For the principal square root of I(see [17]), by the generating matrix function (10), one can write

$$\sum_{n=0}^{\infty} \frac{H_n\left(x, \frac{I}{2}\right)}{n!} t^n = \exp\left[xtI - t^2 I\right]. \tag{16}$$

Substituting following expression

$$\exp\left[xtI - t^2 I\right] = \exp\left[xtI\right] \exp(-t^2 I)$$

and using Taylor series of functions in the right-hand side of above equation, one gets

$$\exp[xtI - t^2 I] = \sum_{n,s=0}^{\infty} \frac{(-1)^s x^n}{n!\ s!} t^{n+2s} I.$$

From (8), it follows that

$$\exp[xtI - t^2 I] = \sum_{n,s=0}^{\infty} \frac{(-1)^s}{n!\ s!} \times \left\{ \sum_{k=0}^{n} \frac{(-1)^k n! \lambda^{-n} (A+I)_n [(A+I)_k]^{-1}}{(n-k)!} L_k^{(A,\lambda)}(x) \right\} t^{n+2s}. \tag{17}$$

Using (5) and (17) can be written in the form

$$\exp[xtI - t^2 I] = \sum_{n,s,k=0}^{\infty} \frac{(-1)^{s+k}}{n!\ s!} \lambda^{-n-k} (A+I)_{n+k} [(A+I)_k]^{-1} \times L_k^{(A,\lambda)}(x) t^{n+k+2s}.$$

By (6), one gets

$$\exp[xtI - t^2 I] = \sum_{n,k=0}^{\infty} \sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{s+k}}{(n-2s)!\ s!} \lambda^{-n+2s-k} (A+I)_{n-2s+k} \times [(A+I)_k]^{-1} L_k^{(A,\lambda)}(x) t^{n+k}. \tag{18}$$

From Pochhammer symbol in (2), one can write following equations

$$\frac{1}{(n-2s)!} I = \frac{1}{n!} 2^{2s} \left(\frac{-n}{2} I\right)_s \left(\frac{-(n-1)}{2} I\right)_s$$

$$(A+I)_{n-2s+k} = (A+I)_{n+k} \left[\left(\frac{-1}{2}(A+(n+k)I) \right)_s \times \left(\frac{-1}{2}(A+(n+k-1)I) \right)_s 2^{2s} \right]^{-1}.$$

Writing these equations in (18), one have

$$\exp[xtI - t^2 I] = \sum_{n,k=0}^{\infty} \frac{(-1)^k}{n!} \lambda^{-n-k} (A+I)_{n+k} (A+I)_k^{-1} L_k^{(A,\lambda)}(x) \times \sum_{s=0}^{\left[\frac{n}{2}\right]} \left\{ \left(\frac{-n}{2}\right)_s \left(\frac{-(n-1)}{2}\right)_s \times \left[\left(\frac{-1}{2}(A+(n+k)I) \right)_s \times \left(\frac{-1}{2}(A+(n+k-1)I) \right)_s \right]^{-1} \frac{(-1)^s}{s!} \lambda^{2s} \right\} t^{n+k}.$$

Thus, one can write

$$\exp[xtI - t^2 I] = \sum_{n,k=0}^{\infty} \left\{ \frac{(-1)^k}{n!} \lambda^{-n-k} (A+I)_{n+k} (A+I)_k^{-1} L_k^{(A,\lambda)}(x) \times {}_2F_2\left(\begin{matrix} \frac{-n}{2} I, & \frac{-(n-1)}{2} I \\ \frac{-1}{2}(A+(n+k)I), & \frac{-1}{2}(A+(n+k-1)I) \end{matrix} ; -\lambda^2 \right) \right\} t^{n+k}. \tag{19}$$

In (19), by (4), it follows that

$$\exp\left[xtI - t^2 I\right] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\{ \frac{(-1)^k}{(n-k)!} \lambda^{-n} (A+I)_n (A+I)_k^{-1} L_k^{(A,\lambda)}(x) \times {}_2F_2\left(\begin{matrix} \frac{-(n-k)}{2} I, & \frac{-(n-k-1)}{2} I \\ \frac{-1}{2}(A+nI), & \frac{-1}{2}(A+(n-1)I) \end{matrix} ; -\lambda^2 \right) \right\} t^n. \tag{20}$$

Combining (16) and (20) and comparing coefficients of t^n, we have the following desired relation

$$H_n\left(x, \frac{I}{2}\right) = \sum_{k=0}^{n} \lambda^{-n} (-n)_k (A+I)_n\ (A+I)_k^{-1} \times {}_2F_2\left(\begin{matrix} \frac{-(n-k)}{2} I, & \frac{-(n-k-1)}{2} I \\ \frac{-1}{2}(A+nI), & \frac{-1}{2}(A+(n-1)I) \end{matrix} ; -\lambda^2 \right) L_k^{(A,\lambda)}(x)$$

Thus the result has been established:

Theorem 1. *For the principal square root of I, if A is a matrix in $\mathbb{C}^{r\times r}$ satisfying (7) and λ is a complex number Re $(\lambda) > 0$, Laguerre and Hermite matrix polynomials satisfy*

$$H_n\left(x, \frac{I}{2}\right) = \sum_{k=0}^{n} \lambda^{-n} (-n)_k (A+I)_n\ (A+I)_k^{-1} \times {}_2F_2\left(\begin{matrix} \frac{-(n-k)}{2} I, & \frac{-(n-k-1)}{2} I \\ \frac{-1}{2}(A+nI), & \frac{-1}{2}(A+(n-1)I) \end{matrix} ; -\lambda^2 \right) \times L_k^{(A,\lambda)}(x).$$

For the special case of $r = 1$, taking $A = \alpha$, $\lambda = \frac{1}{2}$, $x \to 2x$, the above equation reduces to the known relation between Hermite and Laguerre polynomials (see [16]).

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