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ORIGINAL ARTICLE

Hermite–Hadamard type integral inequalities for differentiable *m*-preinvex and (α,m)-preinvex functions[☆]

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Abstract In this paper, the notion of m-preinvex and (α, m)-preinvex functions is introduced and then several inequalities of Hermite–Hadamard type for differentiable m-preinvex and (α, m)-preinvex functions are established.

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1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

The following celebrated double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds for convex functions and is well-known in the literature as the Hermite–Hadamard inequality. Both the inequalities in (1.1) hold in reversed direction if f is concave.

The inequality (1.1) has been a subject of extensive research since its discovery and a number of papers have been written providing noteworthy extensions, generalizations and refinements see for example [1–5].

The classical convexity that is stated above was generalized as m-convexity by Toader in [6] as follows:

Definition 1. The function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leqslant tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b^*]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.

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Obviously, for $m = 1$ the Definition 1 recaptures the concept of standard convex functions on $[0, b^*]$.

The notion of m-convexity has been further generalized in [7] as it is stated in the following definition:

Definition 2. The function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1 - t)y) \leqslant t^\alpha f(x) + m(1 - t^\alpha)f(y)$$

for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

It can easily be seen that for $\alpha = 1$, the class of m-convex functions are derived from the above definition and for $\alpha = m = 1$ a class of convex functions are derived

For several results concerning Hermite–Hadamard type inequalities for m-convex and (α, m)-convex functions we refer the interested reader to [9–16].

More recently, a number of mathematicians have attempted to generalize the concept of classical convexity. For example in [17], Hason gave the notion of invexity as significant generalization of classical convexity. Ben-Israel and Mond [18] introduced the concept of preinvex functions, which is a special case of invex functions. Let us first recall the definition of preinvexity and some related results.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot, \cdot)$, if

$$x + t\eta(y, x) \in K, \qquad \forall x, y \in K, \quad t \in [0, 1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called an η-connected set.

Definition 3 [19]. The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v, u)) \leqslant (1 - t)f(u) + tf(v), \qquad \forall u, v \in K, \quad t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [20].

In a recent paper, Noor [21] obtained the following Hermite–Hadamard inequalities for the preinvex functions:

Theorem 1 [21]. *Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leqslant \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \leqslant \frac{f(a) + f(b)}{2}. \tag{1.2}$$

Barani et al. in [23], presented the following estimates of the right-side of a Hermite–Hadamard type inequality in which some preinvex functions are involved.

Theorem 2 [23]. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on K, for every $a, b \in K$ with $\eta(b, a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{|\eta(b, a)|}{8}(|f'(a)| + |f'(b)|). \tag{1.3}$$

Theorem 3 [23]. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{|\eta(b, a)|}{2(1 + p)^{\frac{1}{p}}} \left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.4}$$

For several new results on inequalities for preinvex functions, we refer the interested reader to [22,23] and the references therein.

In the present paper we first give the concept of m-preinvex and (α, m)-preinvex functions, which generalize the concept of preinvex functions, and then we will present new inequalities of Hermite–Hadamard for functions whose derivatives in absolute value are m-preinvex and (α, m)-preinvex. Our results generalize those results presented in very recent paper 23] concerning Hermite–Hadamard type inequalities for preinvex functions. We also present extensions to several variables of some inequalities for m-convex and (α, m)-convex functions which are special cases of our established results.

2. Main results

To establish our main results we first give the following essential definitions and lemmas:

Definition 4. The function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$, is said to be m-preinvex with respect to η if

$$f(u + t\eta(v, u)) \leqslant (1 - t)f(u) + mtf\left(\frac{v}{m}\right)$$

holds for all $u, v \in K$, $t \in [0, 1]$ and $m \in (0, 1]$. The function f is said to be m-preconcave if and only if $-f$ is m-preinvex.

Definition 5. The function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$, is said to be (α, m)-preinvex with respect to η if

$$f(u + t\eta(v, u)) \leqslant (1 - t^\alpha)\, f(u) + mt^\alpha f\left(\frac{v}{m}\right)$$

holds for all $u, v \in K$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$. The function f is said to be (α, m)-preincave if and only if $-f$ is (α, m)-preinvex.

Remark 1. If in Definition 4, $m = 1$, then one obtain the usual definition of preinvexity. If $\alpha = m = 1$, then Definition 5 recaptures the usual definition of the preinvex functions. It is to be noted that every m-preinvex function and (α, m)-preinvex functions are m-convex and (α, m)-convex with respect to $\eta(v, u) = v - u$ respectively.

Lemma 1 [23]. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$, then the following equality holds:*

$$-\frac{f(a)+f(a+\eta(b,a))}{2}+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx = \frac{\eta(b,a)}{2}\int_0^1 (1-2t)f'(a+t\eta(b,a))dt. \tag{2.1}$$

Now we establish results for functions whose derivatives in absolute values raise to some certain power are m-preinvex and (α, m)-preinvex.

Theorem 4. *Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|$ is m-preinvex on K, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{8}\left[|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|\right]. \tag{2.2}$$

Proof. From Lemma 1, we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2}\int_0^1 |1-2t||f'(a+t\eta(b,a))|dt. \tag{2.3}$$

Since $|f'|$ is m-preinvex on K, for every $a, b \in K$ and $t \in [0,1]$, $m \in (0,1]$, we have

$$|f'(a+t\eta(b,a))| \leqslant (1-t)|f'(a)| + mt\left|f'\left(\frac{b}{m}\right)\right|. \tag{2.4}$$

Hence we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2}\left[|f'(a)|\int_0^1 |1-2t|(1-t)dt + m\left|f'\left(\frac{b}{m}\right)\right|\int_0^1 |1-2t|tdt\right]. \tag{2.5}$$

Since

$$\int_0^1 |1-2t|(1-t)dt = \int_0^1 |1-2t|tdt = \int_0^{\frac{1}{2}} (1-2t)(1-t)dt - \int_{\frac{1}{2}}^1 (1-2t)(1-t)dt = \frac{1}{4}.$$

We get the desired inequality from (2.5). This completes the proof of Theorem 4. □

Corollary 1. *If $\eta(b,a) = b - a$ in Theorem 4, then (2.2) reduces to the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{8}\left[|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|\right]. \tag{2.6}$$

Theorem 5. *Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|^q$ is m-preinvex on K for $q > 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Lemma 1 and using the well known Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}. \tag{2.8}$$

Since $|f'|^q$ is m-preinvex on K, for every $a, b \in [a, b]$ with $a < a + \eta(b,a)$ and $m \in (0,1]$, we have

$$|f'(a+t\eta(b,a))|^q \leqslant (1-t)|f'(a)|^q + mt\left|f'\left(\frac{b}{m}\right)\right|^q.$$

Hence

$$\int_0^1 |f'(a+t\eta(b,a))|^q dt \leqslant \int_0^1 \left[(1-t)|f'(a)|^q + mt\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt = \frac{1}{2}|f'(a)|^q + \frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q.$$

Moreover, by using basic calculus we have

$$\int_0^1 |1-2t|^p dt = \int_0^{\frac{1}{2}} (1-2t)^p dt + \int_{\frac{1}{2}}^1 (2t-1)^p dt = \frac{1}{p+1}.$$

A usage of the last two inequalities in (2.8) gives the desired result. This completes the proof of Theorem 5. □

Corollary 2. *If we take $\eta(b,a) = b - a$ in Theorem 5, then (2.7) becomes the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}} \tag{2.9}$$

A similar result may be stated as follows:

Theorem 6. *Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that*

$f' \in L([a, a+\eta(b,a)])$. If $|f'|^q$ is m-preinvex on K for $q \geqslant 1$, then we have the following inequality:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{4}\left[\frac{|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.10}$$

Proof. For $q = 1$, the proof is the same as that of Theorem 4. Suppose now that $q > 1$. Using Lemma 1 and the well-known power-mean integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t||f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}. \tag{2.11}$$

Applying the m-preinvex convexity of $|f'|^q$ on K in the second integral on the right side of (2.11), we have

$$\begin{aligned}\int_0^1 &|1-2t||f'(a+t\eta(b,a))|^q dt \\ &\leqslant \int_0^1 |1-2t|\left[(1-t)|f'(a)|^q + mt\left|f'\left(\frac{b}{m}\right)\right|^q\right]dt \\ &= |f'(a)|^q\int_0^1 |1-2t|(1-t)dt + m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 t|1-2t|dt \\ &= \frac{1}{4}|f'(a)|^q + \frac{m}{4}\left|f'\left(\frac{b}{m}\right)\right|^q. \end{aligned} \tag{2.12}$$

Utilizing inequality (2.12) in (2.11), we get the inequality (2.10). This completes the proof of the theorem. □

Corollary 3. *Suppose $\eta(b,a) = b - a$, then one has the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{4}\left[\frac{|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.13}$$

Remark 2. For $q = 1$, (2.13) reduces to the inequality proved in Theorem 4. If $q = \frac{p}{p-1}$ $(p > 1)$, we have $4^p > p + 1$ for $p > 1$ and accordingly

$$\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}.$$

This reveals that the inequality (2.10) is better than the one given by (2.7) in Theorem 5.

Now we give our results for (α, m)-preinvex functions.

Theorem 7. *Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|$ is (α, m)-preinvex on K, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2}\left[v_2|f'(a)| + mv_1\left|f'\left(\frac{b}{m}\right)\right|\right], \tag{2.14}$$

where $v_1 = \frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)}$ and $v_2 = \frac{1}{2} - v_1$.

Proof. From Lemma 1, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2}\int_0^1 |1-2t||f'(a+t\eta(b,a))|dt. \tag{2.15}$$

Since $|f'|$ is (α, m)-preinvex on K, we have for every $t \in [0,1]$ that

$$\begin{aligned}\int_0^1 &|1-2t||f'(a+t\eta(b,a))|dt \\ &\leqslant |f'(a)|\int_0^1 |1-2t|(1-t^\alpha)dt \\ &\quad + m\left|f'\left(\frac{b}{m}\right)\right|\int_0^1 t^\alpha|1-2t|dt \\ &= \left(\frac{1}{2}-v_1\right)|f'(a)| + mv_1\left|f'\left(\frac{b}{m}\right)\right|, \end{aligned} \tag{2.16}$$

where

$$\int_0^1 |1-2t|t^\alpha dt = \frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)} = v_1$$

and

$$\int_0^1 |1-2t|(1-t^\alpha)dt = \frac{1}{2} - \frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)} = \frac{1}{2} - v_1.$$

Utilizing (2.15) in (2.14), we get the required inequality and hence the proof of the theorem is completed. □

Corollary 4. *If $\eta(b,a) = b - a$ in Theorem 7, the we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{2}\left[v_2|f'(a)| + mv_1\left|f'\left(\frac{b}{m}\right)\right|\right], \tag{2.17}$$

where $v_1 = \frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)}$ and $v_2 = \frac{1}{2} - v_1$.

Theorem 8. *Let $K \subseteq [0, b^*], b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|^q$ is (α, m)-preinvex on $K, q > 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leqslant \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left[\frac{\alpha|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{1+\alpha}\right]^{\frac{1}{q}}, \tag{2.18}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and the Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leqslant \frac{\eta(b,a)}{2}\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}. \quad (2.19)$$

By the (α,m)-preinvexity of $|f'|^q$, we have for every $t\in[0,1]$

$$|f'(a+t\eta(b,a))|^q\leqslant(1-t^\alpha)|f'(a)|^q+mt^\alpha\left|f'\left(\frac{b}{m}\right)\right|^q$$

for $(\alpha,m)\in(0,1]\times(0,1]$. Hence

$$\int_0^1|f'(a+t\eta(b,a))|^q dt\leqslant|f'(a)|^q\int_0^1(1-t^\alpha)dt+m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 t^\alpha dt =\frac{\alpha}{1+\alpha}|f'(a)|^q+\frac{m}{1+\alpha}\left|f'\left(\frac{b}{m}\right)\right|^q.$$

An application of the above inequality in (2.19) and the fact

$$\int_0^1|1-2t|^p dt=\frac{1}{p+1}$$

gives the desired inequality. □

Corollary 5. *If in Theorem* 8, *we take* $\eta(b,a)=b-a$, *we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{\alpha|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{1+\alpha}\right]^{\frac{1}{q}}, \quad (2.20)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 9. *Let* $K\subseteq[0,b^*], b^*>0$ *be an open invex subset with respect to* $\eta:K\times K\to\mathbb{R}$ *and* $a,b\in K$ *with* $a<a+\eta(b,a)$. *Suppose* $f:K\to\mathbb{R}$ *is a differentiable mapping on* K *such that* $f'\in L([a,a+\eta(b,a)])$. *If* $|f'|^q$ *is* (α,m)*-preinvex on* $K, q\geqslant 1$, *then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leqslant\frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}[v_2|f'(a)|^q+mv_1|f'(b)|^q]^{\frac{1}{q}}, \quad (2.21)$$

where $v_2=\frac{1}{2}-v_1$ *and* $v_1=\frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)}$.

Proof. For $q=1$, the proof is similar to that of Theorem 7. Suppose that $q>1$. Using Lemma 1, we have that the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leqslant\frac{\eta(b,a)}{2}\left(\int_0^1|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t||f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}. \quad (2.22)$$

By the (α,m)-preinvexity of $|f'|^q$ on K, for every $t\in[0,1]$ and $(\alpha,m)\in(0,1]\times(0,1]$ we have

$$\int_0^1|1-2t||f'(a+t\eta(b,a))|^q dt\leqslant\int_0^1|1-2t|[(1-t)^\alpha|f'(a)|^q +mt^\alpha|f'(b)|^q]dt=|f'(a)|^q\int_0^1|1-2t|(1-t)^\alpha dt +m|f'(b)|^q\int_0^1|1-2t|t^\alpha dt=v_2|f'(a)|^q+mv_1|f'(b)|^q. \quad (2.23)$$

Using (2.23) in (2.22), we get the required inequality (2.21). This completes the proof of the theorem. □

Corollary 6. *Suppose* $\eta(b,a)=b-a$ *in Theorem* 9, *then one has the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}[v_2|f'(a)|^q+mv_1|f'(b)|^q]^{\frac{1}{q}}, \quad (2.24)$$

where $v_2=\frac{1}{2}-v_1$ *and* $v_1=\frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)}$.

Remark 3. If we take $m=1$ in Theorem 4 an Theorem 5 or if we take $\alpha=m=1$ in Theorems 7 and 8 we get those results proved in Theorems 2 and 3 respectively. This shows that our results are more general than those proved in [23].

Remark 4. If we take $m=1$ in Theorems 4 and 5 or if we take $\alpha=m=1$ in Theorems 7 and 8 with $\eta(b,a)=b-a$, we get those results proved in [1,3].

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