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ORIGINAL RESEARCH

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On $\psi_{\mathcal{H}}(.)$-operator in weak structure spaces with hereditary classes

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Abstract

Weak structure space (briefly, *wss*) has master looks, when the whole space is not open, and these classes of subsets are not closed under arbitrary unions and finite intersections, which classify it from typical topology. Our main target of this article is to introduce $\psi_{\mathcal{H}}(.)$-operator in hereditary class weak structure space (briefly, $\mathcal{H}wss$) $(X, w, \mathcal{H})$ and examine a number of its characteristics. Additionally, we clarify some relations that are credible in topological spaces but cannot be realized in generalized ones. As a generalization of w-open sets and w-semiopen sets, certain new kind of sets in a weak structure space via $\psi_{\mathcal{H}}(.)$-operator called $\psi_{\mathcal{H}}$-semiopen sets are introduced. We prove that the family of $\psi_{\mathcal{H}}$-semiopen sets composes a supra-topology on X. In view of hereditary class $\mathcal{H}_0$, wT_1-axiom is formulated and also some of their features are investigated.

Keywords: Weak structures, $\psi_{\mathcal{H}}(.)$-operator, wT_1-spaces, $\psi_{\mathcal{H}}$-semiopen sets

Mathematics Subject Classification: 54A05, 54A20

Introduction

Various topics have been initiated as a result of the interaction between topology and life's problems [1, 2]. The characterizations of the hereditary classes idea $\mathcal{H}$ on a non-empty set X, $S \in \mathcal{H}$ and $\acute{S} \subset S$ implies $\acute{S} \in \mathcal{H}$, which was created by Csaszar in [3], are applicable. In [4], Csaszar inserted the notion of weak structures w on X that is usable in digital topology. However, Modak and Noiri have drawn various relations among different mathematical structures in [5]. In 2012, Zahran et al. [6] utilized the hereditary classes and weak structures on X to extend classical topological concepts and got a new weak structure from old. Furthermore, they investigated some of its properties. Renukadevi and Vimaladevi [7] and Al-Omari et al. [8] extended the study of hereditary classes in the generalized topological spaces. In the current study, we apply the notions of weak structures w and hereditary classes $\mathcal{H}$ on X to acquaint $\psi_{\mathcal{H}}(.)$-operators and discuss some of its characteristic features. With the aid of $\psi_{\mathcal{H}}(.)$, we consider a type of sets in (X, w), which may be referred to as $\psi_{\mathcal{H}}$-semiopen. We show that the family of all $\psi_{\mathcal{H}}$-semiopen sets forms a supra-topology on X [9]. Furthermore, in view of hereditary class $\mathcal{H}_0$, wT_1-axiom is formulated and also some of their features are examined.

Preliminaries

In weak structures, most of the fundamental concepts and facts in ordinary topology are specified analogously. It is predicted that many considerable results in ordinary topology will not be carried over and some of interesting properties will be missing or weakened. Thus, any statement which is true in weak structures is true in ordinary topological spaces. However, in order to attain desirable and interesting inferences, additional terms must be imposed.

If there is no other particularization in the current paper, suppose X is a nonempty set and $P(X)$ be a power set of X. Let $\mathcal{H} \neq \emptyset$ be a hereditary class on a weak structure space (X, w). A set S of X is said to be w-open iff $S \in w$ and w-closed iff the complement of S is w-open in X. For $S \in P(X)$, $i_w(S)$ and $c_w(S)$ denote w-interior, w-closure of a set S, respectively. It is known that $i_w(S)$ is restricting, monotone, and idempotent. The map $c_w(S)$ is enlarging, monotone, and idempotent (see [4]). Note that a subset N of X is w-rare iff $i_w c_w(N) = \emptyset$ [6] and the collection $\mathcal{H}_n = \{N \subseteq X \mid i_w c_w(N) = \emptyset\}$ forms a hereditary class [10]. A weak structure space with a hereditary class $\mathcal{H}$ is called a hereditary class weak structure space (briefly, $\mathcal{H}wss$) and is denoted by $(X, w, \mathcal{H})$.

For a subset S of $(X, w, \mathcal{H})$, define $S^\star_{\mathcal{H}} = \{x \mid x \in N \in w \text{ implies } N \cap S \notin \mathcal{H}\}$ [6]. $S^\star_{\mathcal{H}}$ is a w-closed subset for any $S \in P(X)$ [6]. Note that $S^\star_{\mathcal{H}_n} \neq c_w i_w c_w(S)$ as we see in the next example.

Example 1

Consider $w = \{\emptyset, \{a, b\}, \{a, c\}\}$ is a weak structure on $X = \{a, b, c\}$ with a hereditary class $\mathcal{H}_n = \{\emptyset, \{b\}, \{c\}\}$. If $S = \{b, c\}$, then $S^\star_{\mathcal{H}_n} \neq c_w i_w c_w(S)$.

If $c^\star_{\mathcal{H}}(S) = S \cup S^\star_{\mathcal{H}}$, then there is a weak structure $w^\star_{\mathcal{H}}$ such that $c^\star_{\mathcal{H}}(S)$ is the intersection of all $w^\star_{\mathcal{H}}$-closed supersets of S; $S \in w^\star_{\mathcal{H}}$ iff $c^\star_{\mathcal{H}}(X \backslash S) = (X \backslash S)$. The elements of $w^\star_{\mathcal{H}}$ are called $w^\star_{\mathcal{H}}$-open and their complement are called $w^\star_{\mathcal{H}}$-closed sets. Obviously, a subset S is $w^\star_{\mathcal{H}}$-closed iff $S^\star_{\mathcal{H}} \subset S$. However, Selim et al. in [11] introduced a new local function on any collection of a set X.

Theorem 1 [12] *Let w be a weak structure on X such that w is closed under finite intersection and $S, \acute{S} \in P(X)$. Then, the following hold:*

(1) $c_w(S) \cup c_w(\acute{S}) = c_w(S \cup \acute{S})$.
(2) $i_w(S \cap \acute{S}) = i_w(S) \cap i_w(\acute{S})$.

For getting a deeper insight into further studying some properties of weak structure and generalized topological spaces, see [3, 4, 6, 7, 12–15] for details.

Set operator $\psi_{\mathcal{H}}(.)$

Definition 1 Let (X, w) be a weak structure space. $N \subseteq X$ is said to be w-neighborhood (briefly, $w$nbhood) of a point $x \in X$ if $x \in i_w(N)$. We write $\mathcal{N}_w(x)$ for the collection of all $w$nbhoods of x, i.e., $\mathcal{N}_w(x) = \{N \subset X \mid x \in i_w(N)\}$, $x \in X$. We write $\mathcal{N}^o_w(x)$ for the collection of all w-open nbhoods of x, i.e., $\mathcal{N}^o_w(x) = \{N \in w \mid x \in N\}$.

By using the duality property, one can define an operator $\psi_{\mathcal{H}}(.)$ that is similar to $(.)^{\star}_{\mathcal{H}}$ operator [6].

Definition 2 Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. An operator $\psi_{\mathcal{H}} : P(X) \longrightarrow P(X)$ is defined in such a way:

$$\psi_{\mathcal{H}}(S) = \cup\{N \in w \mid N\backslash S \in \mathcal{H}\}, \quad \text{for every } S \in P(X).$$

Significant findings concerning the behavior of $\psi_{\mathcal{H}}(.)$ are obtained in the coming theorems.

Theorem 2 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. If $S \in P(X)$, then*

(1) $\psi_{\mathcal{H}}(S) = \{x \in X \mid \exists N \in \mathcal{N}^{o}_{w}(x)\ s.t.\ N\backslash S \in \mathcal{H}\}$.
(2) $\cup\{N \in w \mid (N\backslash S) \cup (S\backslash N) \in \mathcal{H}\} \subseteq \psi_{\mathcal{H}}(S)$.
(3) $S^{\star}_{\mathcal{H}} = X\backslash\psi_{\mathcal{H}}(X\backslash S)$.

Proof

Direct to prove. □

Theorem 3 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. If $S \subseteq X$, then the following hold:*

(1) $N \in w, N\backslash S \in \mathcal{H}$ *imply* $N \subset \psi_{\mathcal{H}}(S)$.
(2) *If* $H \in \mathcal{H}$, *then* $\psi_{\mathcal{H}}(X\backslash H) = \mathcal{O}_w$, (*where* $\mathcal{O}_w$ *denotes the union of all w-open sets in* (X, w)),
(3) $\psi_{\mathcal{H}}(X) = \mathcal{O}_w$, *for any* $\mathcal{H}$.

Proof

(1) Let $x \in N$. Since $N \in w$ and $N\backslash S \in \mathcal{H}$, then $x \in \psi_{\mathcal{H}}(S)$.
(2) From $H \in \mathcal{H}$ and Lemma 0.1 of [6], it follows $H^{\star}_{\mathcal{H}} = X\backslash\mathcal{O}_w$. Consequently, $\psi_{\mathcal{H}}(X\backslash H) = \mathcal{O}_w$, i.e., $\mathcal{O}_w = \{x \in X \mid \exists N \in \mathcal{N}^{o}_{w}(x)\text{ s.t. } N \cap H \in \mathcal{H}\}$.
(3) Obvious.

□

Theorem 4 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. For $S \in P(X)$, the following statements hold:*

(1) $i_w(S) \subset \psi_{\mathcal{H}}(S)$,
(2) $\psi_{\mathcal{H}}(S)$ *is a w-open set,*
(3) $\psi_{\mathcal{H}}(S) = \psi_{\mathcal{H}}(S \cap \psi_{\mathcal{H}}(S))$.

Proof

(1) For each $x \in i_w(S)$, there exists $N \in \mathcal{N}_w^o(x)$ s.t. $N \subset S$. This implies $N \backslash S = \emptyset \in \mathcal{H}$. Then, from Theorem 2 (1), $x \in \psi_{\mathcal{H}}(S)$.
(2) Accessible from Proposition 0.3. of [6].
(3) Evidently, $\psi_{\mathcal{H}}(S \cap \psi_{\mathcal{H}}(S)) \subset \psi_{\mathcal{H}}(S)$. Conversely, let $x \in \psi_{\mathcal{H}}(S)$ implies the existence of w-open bhood N of x s.t. $N \backslash S \in \mathcal{H}$. By Theorem 3 (1), $N \subset \psi_{\mathcal{H}}(S)$. So $N \backslash (S \cap \psi_{\mathcal{H}}(S)) = N \backslash S \in \mathcal{H}$. Hence, $x \in \psi_{\mathcal{H}}(S \cap \psi_{\mathcal{H}}(S))$ and so $\psi_{\mathcal{H}}(S) \subset \psi_{\mathcal{H}}(S \cap \psi_{\mathcal{H}}(S))$.

□

Next example shows that the reverse inclusion of Theorem 4 (1) may not achieved:

Example 2

Consider $X = N$; the set of all natural numbers, $w = \{\emptyset, \{1\}, \{2\}, X\}$ and $\mathcal{H} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}\}$. If $S = \{3, 4, \ldots, n, \ldots\}$, then $\psi_{\mathcal{H}}(S) = \{1, 2\}$ and $i_w(S) = \emptyset$. Hence, $\psi_{\mathcal{H}}(S) \neq i_w(S)$.

The proofs of the following lemmas are attainable and then ignored.

Lemma 1 *Let $\mathcal{H}_1, \mathcal{H}_2$ be hereditary classes on (X, w) with $\mathcal{H}_1 \subseteq \mathcal{H}_2$. Then, $\psi_{\mathcal{H}_1}(S) \subseteq \psi_{\mathcal{H}_2}(S)$, for each $S \in P(X)$.*

Lemma 2 *Let w_1, w_2 be two weak structures on X, $\mathcal{H}$ be a hereditary class on X and $S \in P(X)$. If $w_1 \subseteq w_2$, then $\psi_{\mathcal{H}}(w_1)(S) \subseteq \psi_{\mathcal{H}}(w_2)(S)$.*

Theorem 5 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. If $S, \acute{S} \in P(X)$, then the following statements hold:*

(1) *If $S \subset \acute{S}$, then $\psi_{\mathcal{H}}(S) \subset \psi_{\mathcal{H}}(\acute{S})$,*
(2) *$\psi_{\mathcal{H}}(S \cap \acute{S}) \subset \psi_{\mathcal{H}}(S) \cap \psi_{\mathcal{H}}(\acute{S})$ and $\psi_{\mathcal{H}}(S) \cup \psi_{\mathcal{H}}(\acute{S}) \subset \psi_{\mathcal{H}}(S \cup \acute{S})$,*
(3) *If $S \in w$, then $S \subset \psi_{\mathcal{H}}(S)$,*
(4) *$\psi_{\mathcal{H}}(S) \subset \psi_{\mathcal{H}}(\psi_{\mathcal{H}}(S))$,*
(5) *$S \cap \psi_{\mathcal{H}}(S) = i^{\star}_{\mathcal{H}}(S)$ (where $i^{\star}_{\mathcal{H}}(S)$ is interior of S with respect to $w^{\star}_{\mathcal{H}}$).*

Proof
(1) and (2) are obvious.

3 Let $S \in w$, then by using Theorem 4 (1), $S = i_w(S) \subset \psi_{\mathcal{H}}(S)$ and so $S \subset \psi_{\mathcal{H}}(S)$.
4 Let $x \in \psi_{\mathcal{H}}(S)$, then there exists $N \in \mathcal{N}_w^o(x)$ s.t. $N \backslash S \in \mathcal{H}$. From Theorem 3, $N \subset \psi_{\mathcal{H}}(S)$ and so $N \backslash \psi_{\mathcal{H}}(S) = \emptyset \in \mathcal{H}$. Hence, $x \in \psi_{\mathcal{H}}(\psi_{\mathcal{H}}(S))$.
5 Intelligible.

□

Remark 1

Let w be closed under finite intersection and $\mathcal{H}$ be an ideal on (X, w). Then, $\psi_{\mathcal{H}}(S \cap \acute{S}) = \psi_{\mathcal{H}}(S) \cap \psi_{\mathcal{H}}(\acute{S})$, for any sets $S, \acute{S}$ of X.

Now we give an example to show that the reverse inclusion of Theorem 5 (2) fails to hold in general:

Example 3

Consider $w = \{\emptyset, \{d\}, \{a,b\}, \{b,c\}\}$ is a weak structure on $X = \{a,b,c,d\}$ with a hereditary class $\mathcal{H} = \{\emptyset, \{c\}, \{d\}\}$.

(1) If $S = \{a,b\}$ and $\acute{S} = \{b,c\}$, then $\psi_{\mathcal{H}}(S) = \{a,b,d\}$, $\psi_{\mathcal{H}}(\acute{S}) = \{b,c,d\}$ and $\psi_{\mathcal{H}}(S \cap \acute{S}) = \{d\}$. Hence, $\psi_{\mathcal{H}}(S \cap \acute{S}) \neq \psi_{\mathcal{H}}(S) \cap \psi_{\mathcal{H}}(\acute{S})$.
(2) If $S = \{a\}$ and $\acute{S} = \{b,c,d\}$, then $\psi_{\mathcal{H}}(S) = \{d\}$, $\psi_{\mathcal{H}}(\acute{S}) = \{b,c,d\}$ and $\psi_{\mathcal{H}}(S \cup \acute{S}) = X$. Hence,
$\psi_{\mathcal{H}}(S \cup \acute{S}) \neq \psi_{\mathcal{H}}(S) \cup \psi_{\mathcal{H}}(\acute{S})$.

In Theorem 5 (4), the reverse inclusion fails to hold in general as we can see by the next example.

Example 4

Consider $w = \{\emptyset, \{d\}, \{a,b\}, \{b,c\}\}$ is a weak structure on $X = \{a,b,c,d\}$ with a hereditary class $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. If $S = \{c,d\}$, then $\psi_{\mathcal{H}}(S) = \{b,c,d\}$ and $\psi_{\mathcal{H}}\psi_{\mathcal{H}}(S) = X$. Hence, $\psi_{\mathcal{H}}\psi_{\mathcal{H}}(S) \neq \psi_{\mathcal{H}}(S)$.

Also, Example 4 shows that Theorem 5 (3) fails to hold. If $S = \{c,d\}$, then $\psi_{\mathcal{H}}(S) = \{b,c,d\}$. It is clear that S is not a w-open set but it satisfies $S \subset \psi_{\mathcal{H}}(S)$.

Theorem 6 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. If $\Theta = \{S \subset X \mid S \subset \psi_{\mathcal{H}}(S)\}$, then Θ is a generalized topology on X and $\Theta = w^{\star}_{\mathcal{H}}$.*

Proof

Let $\Theta = \{S \subset X \mid S \subset \psi_{\mathcal{H}}(S)\}$. It is clear that $\emptyset \in \Theta$. Now if $S_r \in \Theta$ for each $r \in \Upsilon$, so $S_r \subset \psi_{\mathcal{H}}(S_r) \subset$

$\psi_{\mathcal{H}}(\cup_r S_r)$ for each $r \in \Upsilon$. This implies that $\cup_r S_r \subset \psi_{\mathcal{H}}(\cup_r S_r)$. Hence, $\cup_r S_r \in \Theta$. This shows that Θ is a generalized topology on X. Let $S \in \Theta$, then $S \subset \psi_{\mathcal{H}}(S) = X \backslash (X \backslash S)^{\star}_{\mathcal{H}}$ which implies that $(X \backslash S)^{\star}_{\mathcal{H}} \subset (X \backslash S)$. Therefore, $(X \backslash S)$ is $w^{\star}_{\mathcal{H}}$-closed and so S is $w^{\star}_{\mathcal{H}}$-open. Therefore, $\Theta \subset w^{\star}_{\mathcal{H}}$. Conversely, $S \in w^{\star}_{\mathcal{H}}$ and $x \in S$. Then, there exists $N \in w$ and $H \in \mathcal{H}$ s.t. $x \in (N \backslash H) \subset S$. Now $(N \backslash H) \subset S$ implies that $(N \backslash S) \subset H$ which in turn implies that $(N \backslash S) \in \mathcal{H}$ and so $x \in \psi_{\mathcal{H}}(S)$. Therefore, $w^{\star}_{\mathcal{H}} \subset \Theta$. Hence, $\Theta = w^{\star}_{\mathcal{H}}$. □

Proposition 1 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$ and $S \in P(X)$. If N is a nonempty w-open subset of $\psi_{\mathcal{H}}(S) \backslash \psi_{\mathcal{H}}(X \backslash S)$ with $\psi_{\mathcal{H}}(S) \backslash S \in \mathcal{H}$, then $N \backslash S \in \mathcal{H}$ and $N \cap S \notin \mathcal{H}$.*

Proof

Suppose $N \subset \psi_{\mathcal{H}}(S) \backslash \psi_{\mathcal{H}}(X \backslash S)$. If $N \subset \psi_{\mathcal{H}}(S)$ and $\psi_{\mathcal{H}}(S) \backslash S \in \mathcal{H}$, then $N \backslash S \subset \psi_{\mathcal{H}}(S) \backslash S$ and so $N \backslash S \in \mathcal{H}$. If $N \subset X \backslash \psi_{\mathcal{H}}(X \backslash S)$, hence $x \notin \psi_{\mathcal{H}}(X \backslash S)$ for every $x \in N$. Since N is a w-open set containing x, then $N \backslash (X \backslash S) \notin \mathcal{H}$ and so $N \cap S \notin \mathcal{H}$. □

Next segment is interested to introduce and examine the concepts w-codense (resp. strongly w-codense, $*$-strongly w-codense) hereditary class in $(X, w, \mathcal{H})$ with some of their effects.

Definition 3 [6] Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. $\mathcal{H}$ is w-codense iff $w \cap \mathcal{H} = \{\emptyset\}$ iff $X = X^{\star}_{\mathcal{H}}$.

Proposition 2 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. $\mathcal{H}$ is w-codense iff $\psi_{\mathcal{H}}(\emptyset) = \emptyset$.*

Proof

Obvious, from Proposition 0.5 of [6] and Theorem 2. □

The proof of the following lemmas follows directly from Proposition 2.

Lemma 3 *Let $\mathcal{H}_1, \mathcal{H}_2$ be hereditary classes on (X, w) with $\mathcal{H}_1 \subseteq \mathcal{H}_2$. Then, $\mathcal{H}_1$ is w-codense, if $\mathcal{H}_2$ is w-codense.*

Lemma 4 *Let $\mathcal{H}$ be w-codense and $A \subseteq X$. Then, $A \notin \mathcal{H}$, if $x \in i_w(A)$.*

Theorem 7 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. If $\psi_{\mathcal{H}}(S) \subset S^{\star}_{\mathcal{H}}$ for every $S \subset X$, then $\mathcal{H}$ is w-codense.*

Proof

We shall prove $X \subset X^{\star}_{\mathcal{H}}$. Let $x \notin X^{\star}_{\mathcal{H}}$, then $x \notin \psi_{\mathcal{H}}(X)$. Hence, for every w-open nbd. N of x, $N \backslash X \notin \mathcal{H}$. So $\emptyset \notin \mathcal{H}$, which is a contradiction. Consequently, $X = X^{\star}_{\mathcal{H}}$, i.e., $\mathcal{H}$ is w-codense. □

Remark 2

For any $x \in X$, $\{A \subset X \mid x \notin A\}$ is a hereditary class denoted by $\mathcal{H}_0$.

In view of hereditary class $\mathcal{H}_0$, wT_1-axiom is formulated and also some of their features are investigated.

Theorem 8 *In a $\mathcal{H}wss$ $(X, w, \mathcal{H})$, the following statements are equivalent:*

(1) $\cap_{x \in X} \mathcal{N}^o_w(x) = \{x\}$,
(2) *$(X, w, \mathcal{H})$ is a wT_1-space* [16],
(3) *If $\mathcal{H}_0 \cap \mathcal{N}^o_w(y) = \{\emptyset\}$, then $x = y$.*

Proof

(1) $\Longrightarrow$ (2) Let $x, y \in X$, and $x \neq y$. From (1), there exist w-open sets M and N s.t. $x \in M$ while $y \notin M$ and $y \in N$ while $x \notin N$.

(2) $\Longrightarrow$ (3) Let $x \neq y$. In view of (2), there exist w-open sets M and N s.t. $x \in M$ while $y \notin M$ and $y \in N$ while $x \notin N$. Hence, there exists $N \in \mathcal{N}^o_w(y)$ and $N \in \mathcal{H}_0$ and so $\mathcal{H}_0 \cap \mathcal{N}^o_w(y) \neq \{\emptyset\}$.

(3) $\Longrightarrow$ (1) Let $y \in \cap_{x\in X}\mathcal{N}^o_w(x)$ and $y \neq x$. Then, for every $N \in \mathcal{N}^o_w(x)$; $y \in N$, i.e., $N \in \mathcal{N}^o_w(y)$. Also, from (3), $\mathcal{H}_0 \cap \mathcal{N}^o_w(y) \neq \{\emptyset\}$; there exists M s.t. $M \in \mathcal{N}^o_w(y)$ and $M \in \mathcal{H}_0$. Hence, $x \notin M$, i.e., $M \notin \mathcal{N}^o_w(x)$, which is a contradiction, so $x = y$.

□

Definition 4 If N, S are w-open, w-closed sets, respectively, and $N \setminus S \in \mathcal{H}$ implies $N \subset S$, then a hereditary class $\mathcal{H}$ is said to be strongly w-codense on (X, w).

Proposition 3 *$\mathcal{H}$ is strongly w-codense on (X, w) iff $\psi_{\mathcal{H}}(S) \subset i_w(S)$, for every w-closed set S.*

Proof

($\Longrightarrow$) Let S be a w-closed set. Suppose $x \in \psi_{\mathcal{H}}(S)$, then there exists $N \in w$ s.t. $x \in N$ and $N\setminus S \in \mathcal{H}$. Since $\mathcal{H}$ is strongly w-codense, thus $N \subset S$ and so $x \in i_w(S)$. Hence, $\psi_{\mathcal{H}}(S) \subset i_w(S)$.

($\Longleftarrow$) Let N, S be w-open, w-closed sets, respectively, and $N\setminus S \in \mathcal{H}$. From Theorem 3, $N \subset \psi_{\mathcal{H}}(S)$. Since $\psi_{\mathcal{H}}(S) \subset i_w(S)$, then $N \subset S$. □

As a straightforward consequence to Theorem 4 and Proposition 3, the next corollaries are fulfilled.

Corollary 1 *If $\mathcal{H}$ is strongly w-codense on (X, w), then $\psi_{\mathcal{H}}(S)\setminus S = \emptyset$, for each w-closed set S of X.*

Corollary 2 *If S is a w-closed subset of X, then $\mathcal{H}$ is strongly w-codense on (X, w) iff $\psi_{\mathcal{H}}(S) = i_w(S)$. Equivalently, if S is a w-open subset of X, then $\mathcal{H}$ is strongly w-codense on (X, w) iff $S^\star_{\mathcal{H}} = c_w(S)$.*

Definition 5 The boundary $\partial_w(S)$ of a subset S of (X, w) is defined as $\partial_w(S) = c_w(S)\setminus i_w(S)$.

The next result is a direct consequence of Definition 5 and Theorem 4 (1), so its proof is disregarded.

Lemma 5 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}$wss. For any subset S of X, the following statements hold:*

(1) $S\backslash\partial_w(S) \subset \psi_{\mathcal{H}}(S)$,
(2) $X = \psi_{\mathcal{H}}(S) \cup \psi_{\mathcal{H}}(X\backslash S) \cup \partial_w(S)$.

In view of Theorem 1 and Corollary 2, the following theorem holds.

Theorem 9 *Let w be closed under finite intersection and a hereditary class $\mathcal{H}$ be strongly w-codense on X. If $\psi_{\mathcal{H}}(S) \cup \psi_{\mathcal{H}}(\acute{S}) = \psi_{\mathcal{H}}(S \cup \acute{S})$, for any w-closed sets $S, \acute{S}$ of (X, w), then* $\partial_w(S \cup \acute{S}) = (\partial_w(S)\backslash\psi_{\mathcal{H}}(\acute{S})) \cup (\partial_w(\acute{S})\backslash\psi_{\mathcal{H}}(S))$

Proof

$$
\begin{aligned}
&(\partial_w(S)\backslash\psi_{\mathcal{H}}(\acute{S})) \cup (\partial_w(\acute{S})\backslash\psi_{\mathcal{H}}(S))\\
&\quad= [\partial_w(S) \cup \partial_w(\acute{S})] \cap [\partial_w(S) \cup X\backslash\psi_{\mathcal{H}}(S)]\\
&\quad\cap [\partial_w(\acute{S}) \cup X\backslash\psi_{\mathcal{H}}(\acute{S})] \cap [X\backslash(\psi_{\mathcal{H}}(S) \cap \psi_{\mathcal{H}}(\acute{S}))]\\
&\quad= [\partial_w(S) \cup \partial_w(\acute{S})]\\
&\quad\cap [X\backslash\psi_{\mathcal{H}}(S)] \cap [X\backslash\psi_{\mathcal{H}}(\acute{S})] \cap [X\backslash(\psi_{\mathcal{H}}(S) \cap \psi_{\mathcal{H}}(\acute{S}))]\\
&\quad= [\partial_w(S) \cup \partial_w(\acute{S})] \cap [X\backslash(\psi_{\mathcal{H}}(S) \cup \psi_{\mathcal{H}}(\acute{S}))] \cap [X\backslash(\psi_{\mathcal{H}}(S) \cap \psi_{\mathcal{H}}(\acute{S}))]\\
&\quad= [\partial_w(S) \cup \partial_w(\acute{S})] \cap [X\backslash(\psi_{\mathcal{H}}(S) \cup \psi_{\mathcal{H}}(\acute{S}))]\\
&\quad= [\partial_w(S) \cup \partial_w(\acute{S})] \cap [X\backslash(\psi_{\mathcal{H}}(S \cup \acute{S}))]\\
&\quad\subseteq [c_w(S) \cup c_w(\acute{S})] \cap [X\backslash(\psi_{\mathcal{H}}(S \cup \acute{S}))]\\
&\quad\subseteq [c_w(S \cup \acute{S})] \cap [X\backslash(\psi_{\mathcal{H}}(S \cup \acute{S}))]\\
&\quad= c_w(S \cup \acute{S})\backslash\psi_{\mathcal{H}}(S \cup \acute{S}) = \partial_w(S \cup \acute{S})\\
&\partial_w(S \cup \acute{S}) = c_w(S \cup \acute{S})\backslash\psi_{\mathcal{H}}(S \cup \acute{S})\\
&\quad\subseteq [c_w(S) \cup c_w(\acute{S})]\backslash[\psi_{\mathcal{H}}(S) \cup \psi_{\mathcal{H}}(\acute{S})]\\
&\quad= [c_w(S) \cup c_w(\acute{S})] \cap [X\backslash\psi_{\mathcal{H}}(S)] \cap [X\backslash\psi_{\mathcal{H}}(\acute{S})]\\
&\quad= [c_w(S) \cap X\backslash\psi_{\mathcal{H}}(S) \cap X\backslash\psi_{\mathcal{H}}(\acute{S})] \cup [c_w(\acute{S}) \cap X\backslash\psi_{\mathcal{H}}(S) \cap X\backslash\psi_{\mathcal{H}}(\acute{S})]\\
&\quad= (\partial_w(S)\backslash\psi_{\mathcal{H}}(\acute{S})) \cup (\partial_w(\acute{S})\backslash\psi_{\mathcal{H}}(S))
\end{aligned}
$$

Hence, $\partial_w(S \cup \acute{S}) = (\partial_w(S)\backslash\psi_{\mathcal{H}}(\acute{S})) \cup (\partial_w(\acute{S})\backslash\psi_{\mathcal{H}}(S))$.

□

Definition 6 For N, $\acute{N} \in w$ and for all $S \subseteq X$, $(N \cap \acute{N})\backslash S \in \mathcal{H}$ and $N \cap \acute{N} \cap S \in \mathcal{H}$ implies $N \cap \acute{N} = \emptyset$, then a hereditary class $\mathcal{H}$ is said to be $*$-strongly w-codense on (X, w).

Lemma 6 *If a hereditary class $\mathcal{H}$ is $*$-strongly w-codense on (X, w), then the following statements hold:*

(1) *$\mathcal{H}$ is w-codense,*
(2) *$\mathcal{H}$ is strongly w-codense.*

Proof

(1) We shall prove only $X \subseteq X^{*}$. Let $x \notin X^{*}$, then there exists $N \in \mathcal{N}_{w}^{o}(x)$ s.t. $N \cap X \in \mathcal{H}$. Since $N \backslash X = \emptyset \in \mathcal{H}$ and $\mathcal{H}$ is $*$-strongly w-codense, then $N = \emptyset$. It is a contradiction, and so $X = X^{*}$.

(2) Let N, S be w-open, w-closed sets, respectively, and $N \backslash S \in \mathcal{H}$. Suppose $\acute{N} = X \backslash S$, then $(N \cap \acute{N}) \backslash S \in \mathcal{H}$ and $N \cap \acute{N} \cap S = \emptyset \in \mathcal{H}$. Since $\mathcal{H}$ is $*$-strongly w-codense, hence $N \cap \acute{N} = \emptyset$. So $N \subset S$.

□

In Lemma 6, the reverse implications fail to hold in general as we can see by the next examples.

Example 5

(1) Consider $w = \{\emptyset, \{a, b\}, \{a, c\}\}$ is a weak structure on $X = \{a, b, c, d\}$ with $\mathcal{H} = \{\emptyset, \{a\}, \{d\}\}$. Clearly, $\psi_{\mathcal{H}}(\emptyset) = \emptyset$. If $N = \{a, b\}$, $\acute{N} = \{a, c\}$ are w-open sets, then $(N \cap \acute{N}) \backslash S \in \mathcal{H}$ and $N \cap \acute{N} \cap S \in \mathcal{H}$ for every $S \subseteq X$ but $N \cap \acute{N} \neq \emptyset$. Then, $\mathcal{H}$ is w-codense but it is not $*$-strongly w-codense on (X, w).

(2) Consider $w = \{\emptyset, \{a, b, c\}, \{a, b, d\}\}$ is a weak structure on $X = \{a, b, c, d\}$ with $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. consider $N = \{a, b, c\}$, $\acute{N} = \{a, b, d\}$ are w-open sets. Obviously, $\mathcal{H}$ is a strongly w-codense on (X, w). For $S = \{a, c\}$, $(N \cap \acute{N}) \backslash S \in \mathcal{H}$ and $N \cap \acute{N} \cap S \in \mathcal{H}$ but $N \cap \acute{N} \neq \emptyset$. Then, $\mathcal{H}$ is strongly w-codense but it is not $*$-strongly w-codense on (X, w).

Theorem 10 *Let (X, w) be a weak structure space, and let a hereditary class $\mathcal{H}$ be $*$-strongly w-codense on X. Then, $\psi_{\mathcal{H}}(S) \subset S^{\star}_{\mathcal{H}}$, for every $S \in P(X)$.*

Proof

Suppose there exists an element $x \in \psi_{\mathcal{H}}(S)$ and $x \notin S^{\star}_{\mathcal{H}}$. Then, there exist w-open sets N, $\acute{N}$ s.t. $x \in N \cap \acute{N}$, $N \backslash S \in \mathcal{H}$ and $\acute{N} \cap S \in \mathcal{H}$. Hence, $(N \cap \acute{N}) \backslash S \in \mathcal{H}$ and $N \cap \acute{N} \cap S \in \mathcal{H}$. Since $\mathcal{H}$ is $*$-strongly w-codense, then $N \cap \acute{N} = \emptyset$. But this contradicts the fact that $x \in N \cap \acute{N}$. Consequently, $\psi_{\mathcal{H}}(S) \subset S^{\star}_{\mathcal{H}}$. □

The following example shows that the reverse inclusions of Theorem 10 fail to hold.

Example 6

Consider $w = \{\emptyset, \{a, c\}\}$ is a weak structure on $X = \{a, b, c\}$. If $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$ is a $*$-strongly w-codense on (X, w). For $S = \{b, c\}$, $\psi_{\mathcal{H}}(S) = \{a, c\}$, and $S^{\star}_{\mathcal{H}} = X$, i.e., $S^{\star}_{\mathcal{H}} \nsubseteq \psi_{\mathcal{H}}(S)$.

Remark 3

The reverse direction of Theorem 10 is true if we assume the following condition: The intersection of any two w-open sets is w-open.

Theorem 11 *Let w be a weak structure on X, $\mathcal{H}$ be a $*$-strongly w-codense hereditary class and $S \subset X$. Then, $\psi_{\mathcal{H}}(S) = \emptyset$, if $S \in \mathcal{H}$.*

Proof

Let $S \in \mathcal{H}$ and $\mathcal{H}$ be a $*$-strongly w-codense hereditary class, then in view of Lemma 0.1 of [6] and Theorem 10 $\psi_{\mathcal{H}}(S) \subset S^{\star}_{\mathcal{H}} = X \backslash \mathcal{O}_w$. Since $\psi_{\mathcal{H}}(S)$ is a w-open set, then from definition of $\mathcal{O}_w$, $\psi_{\mathcal{H}}(S) = \psi_{\mathcal{H}}(S) \cap \mathcal{O}_w = \emptyset$. □

Corollary 3 *If $\mathcal{H}$ is a $*$-strongly w-codense hereditary class on (X, w) and $X \backslash S \in \mathcal{H}$, then $S^{\star}_{\mathcal{H}} = X$.*

Example 7

Let X be any nonempty set endowed with the weak structure $w = \{\emptyset\}$ and any hereditary class $\mathcal{H}$. Then, the following hold:

(1) $\psi_{\mathcal{H}}(S) = \emptyset$, for every $S \subseteq X$.
(2) $\mathcal{H}$ is a strongly w-codense hereditary class.
(3) $\mathcal{H}$ is a $*$-strongly w-codense hereditary class.

Presently, we introduce and study the concepts $\psi_{\mathcal{H}}$-open, $\psi_{\mathcal{H}}$-semiopen and $\psi^*_{\mathcal{H}}$ -semiopen sets with some of their properties.

Definition 7 Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. A subset S of X is said to be

(1) $\psi_{\mathcal{H}}$-open (or $w^{\star}_{\mathcal{H}}$-open), if $S \subset \psi_{\mathcal{H}}(S)$.
(2) $\psi_{\mathcal{H}}$-semiopen if $S \subset c_w \psi_{\mathcal{H}}(S)$.
(3) $\psi^*_{\mathcal{H}}$-semiopen if $S \subset (\psi_{\mathcal{H}}(S))^*$.

The class of all $\psi_{\mathcal{H}}$-open (resp. $\psi_{\mathcal{H}}$-semiopen, $\psi^*_{\mathcal{H}}$-semiopen) sets in $(X, w, \mathcal{H})$ is denoted by $\psi_{\mathcal{H}}O(X, w)$ (resp. $\psi_{\mathcal{H}}SO(X, w)$, $\psi^*_{\mathcal{H}}SO(X, w)$).

We can say that $\psi^* : (X, w, \mathcal{H}) \longrightarrow C(X, w)$, (where $C(X, w)$ is a class of all w-closed sets in (X, w)), is a set operator and it is defined as $\psi^*(S) = (\psi(S))^*$ for $A \subset X$. Although, Modak and Bandyopadhyay in [17], Modak in [18] and Modak and Islam in [13] have introduced similar types of set $\psi_{\mathcal{H}}$-semiopen, $\psi^*_{\mathcal{H}}$-semiopen and the similar type of operator ψ^* in the ideal topological space.

Lemma 7 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$ and $S \subset X$. Then,*

(1) $c_w\psi_{\mathcal{H}}(S)$ *is* $\psi_{\mathcal{H}}$*-semiopen.*
(2) $\psi_{\mathcal{H}}c_w(S)$ *is* $\psi_{\mathcal{H}}$*-semiopen.*
(3) *Every w-regular closed set is* $\psi_{\mathcal{H}}$*-semiopen.*
(4) X *is* $\psi_{\mathcal{H}}$*-semiopen.*

Proof

(1) Since $\psi_{\mathcal{H}}(S) \subset c_w\psi_{\mathcal{H}}(S)$, then $\psi_{\mathcal{H}}(S) \subset \psi_{\mathcal{H}}\psi_{\mathcal{H}}(S) \subset \psi_{\mathcal{H}}c_w\psi_{\mathcal{H}}(S)$. Hence, $c_w\psi_{\mathcal{H}}(S) \subset c_w\psi_{\mathcal{H}}c_w\psi_{\mathcal{H}}(S)$ and so $c_w\psi_{\mathcal{H}}(S)$ is $\psi_{\mathcal{H}}$-semiopen.
(2) From (4) of Theorem 5, $\psi_{\mathcal{H}}c_w(S) \subset \psi_{\mathcal{H}}\psi_{\mathcal{H}}c_w(S)$. Then, $\psi_{\mathcal{H}}c_w(S) \subset c_w\psi_{\mathcal{H}}\psi_{\mathcal{H}}c_w(S)$. So $\psi_{\mathcal{H}}c_w(S)$ is $\psi_{\mathcal{H}}$-semiopen.
(3) Let S be a w-regular closed set, then $S = c_w i_w(S)$. Then, by using Theorem 4 (1), S is $\psi_{\mathcal{H}}$-semiopen.
(4) Obvious.

□

Theorem 12 *Let* $(X, w, \mathcal{H})$ *be a* $\mathcal{H}wss$*. If* $S \subset G \subset \psi_{\mathcal{H}}(S)$*, then* G *is* $\psi_{\mathcal{H}}$*-semiopen.*

Proof

Let $S \subset G \subset \psi_{\mathcal{H}}(S)$*. Since* $S \subset G$*, then* $c_w\psi_{\mathcal{H}}(S) \subset c_w\psi_{\mathcal{H}}(G)$*. Hence,* $G \subset \psi_{\mathcal{H}}(S) \subset c_w\psi_{\mathcal{H}}(S) \subset c_w\psi_{\mathcal{H}}(G)$*. Therefore,* G *is* $\psi_{\mathcal{H}}$*-semiopen set.* □

The proof of the following theorem is obvious and then omitted.

Theorem 13 *Let* $(X, w, \mathcal{H})$ *be a* $\mathcal{H}wss$*. Then,*

$$\begin{array}{ccccc} \psi^*_{\mathcal{H}}\text{-semiopen} & \Rightarrow & \psi_{\mathcal{H}}\text{-semiopen} & \Leftarrow & w\text{-semiopen.} \\ & & \Uparrow & & \Uparrow \\ & & \psi_{\mathcal{H}}\text{-open} & \Longleftarrow & w\text{-open} \end{array}$$

In the next, we give examples which show that the reverse implications are not true.

Example 8

Let $X = \{a, b, c\}$*,* $w = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}\}$ *and* $\mathcal{H} = \{\emptyset, \{b\}\}$*. Then,*

(1) A set $S = \{c\}$ is $\psi_{\mathcal{H}}$-semiopen, but it is not w-semiopen.
(2) A set $S = \{b\}$ is $\psi_{\mathcal{H}}$-semiopen, but it is not $\psi^*_{\mathcal{H}}$-semiopen.

Example 9

Let $X = \{a, b, c, d\}$, a weak structure $w = \{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}\}$ and $\mathcal{H} = \{\emptyset, \{b\}, \{d\}\}$. If $S = \{a, c, d\}$, then $\psi_{\mathcal{H}}(S) = \{a, b, c\}$ and $c_w\psi_{\mathcal{H}}(S) = X$. Hence, $\psi_{\mathcal{H}}$-semiopen set may not be $\psi_{\mathcal{H}}$-open.

The proof of the coming theorems are evident and then omitted.

Theorem 14 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}$wss. If $\mathcal{H}$ is the class of all w-rare sets, then every α-w-open set is $\psi_{\mathcal{H}}$-open.*

Theorem 15 *Let $\mathcal{H}$ be a strongly w-codense hereditary class on (X, w) and S be a w-closed set. If S is $\psi_{\mathcal{H}}$-open, then $S = i_w(S)$ and so it is w-semiopen.*

Theorem 16 *Let $\mathcal{H}$ be a strongly w-codense hereditary class on (X, w) and S be a w-open set. Then, every $\psi_{\mathcal{H}}$-semiopen set is $\psi^*_{\mathcal{H}}$-semiopen.*

Proof

Let S be a $\psi_{\mathcal{H}}$-semiopen set, then $S \subset c_w\psi_{\mathcal{H}}(S)$. Since $\psi_{\mathcal{H}}(S)$ is a w-open set and $\mathcal{H}$ is strongly w-codense, hence by Corollary 2, $c_w\psi_{\mathcal{H}}(S) = (\psi_{\mathcal{H}}(S))^\star$. So $S \subset (\psi_{\mathcal{H}}(S))^*$. □

Theorem 17 *Let $\mathcal{H}$ be a strongly w-codense hereditary class on (X, w) and S be a w-closed set. If S is $\psi_{\mathcal{H}}$-semiopen, then it is w-βopen.*

Proof

Let S be a $\psi_{\mathcal{H}}$-semiopen set, then $S \subset c_w\psi_{\mathcal{H}}(S)$ and so $S \subset c_w\psi_{\mathcal{H}}c_w(S)$. Since S is a w-closed set and $\mathcal{H}$ is strongly w-codense, hence $\psi_{\mathcal{H}}(S) = i_w(S)$. Therefore $S \subset c_wi_wc_w(S)$. □

Example 10

In Example 9, $\{b\}$ is a w-βopen set but it is not $\psi_{\mathcal{H}}$-semiopen.

Theorem 18 *Let $\mathcal{H}$ be a $*$-strongly w-codense hereditary class on (X, w). Then, the following statements hold:*

(1) *$\psi_{\mathcal{H}}(S) \subset i_wc_w(S)$, for any set S of X.*
(2) *Every $\psi_{\mathcal{H}}$-open set is w-βopen.*
(3) *$\psi_{\mathcal{H}}(S) \subset S$, for every w-closed set S.*

Proof

Clear. □

Remark 4

Let $\mathcal{H}$ be a $*$-strongly w-codense hereditary class on (X, w). Then, for any set S of X, $i_w(S) \subset \psi_{\mathcal{H}}(S) \subset S^{\star}_{\mathcal{H}} \subset c_w(S)$.

Theorem 19 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. Then, a subset S of X is $\psi_{\mathcal{H}}$-semiopen if there exists w-open set G s.t. $G \subset S \subset c_w(G)$.*

Proof

Suppose there exists w-open set G s.t. $G \subset S \subset c_w(G)$. Then, $G \subset S$ implies that $c_w\psi_{\mathcal{H}}(G) \subset c_w\psi_{\mathcal{H}}(S)$. Since G is w-open, then $S \subset c_w(G) \subset c_w\psi_{\mathcal{H}}(G) \subset c_w\psi_{\mathcal{H}}(S)$. Therefore, S is $\psi_{\mathcal{H}}$-semiopen. □

Theorem 20 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. Then, the union of $\psi_{\mathcal{H}}$-semiopen sets is also $\psi_{\mathcal{H}}$-semiopen.*

Proof

Let S_j be $\psi_{\mathcal{H}}$-semiopen set for $j \in J$. For each $j \in J$, $S_j \subset c_w\psi_{\mathcal{H}}(S_j) \subset c_w\psi_{\mathcal{H}}(\cup_j S_j)$. Hence, $\cup_j S_j$ is $\psi_{\mathcal{H}}$-semiopen. □

Next example shows that the intersection of two $\psi_{\mathcal{H}}$-semiopen sets in $(X, w, \mathcal{H})$ may not be $\psi_{\mathcal{H}}$-semiopen.

Example 11

In Example 9, let $S = \{a, b\}$, $\acute{S} = \{b, c\}$ be two $\psi_{\mathcal{H}}$-semiopen sets. Then, $S \cap \acute{S} = \{b\}$ and $c_w\psi_{\mathcal{H}}(S \cap \acute{S}) = \{d\}$. So the intersection of two $\psi_{\mathcal{H}}$-semiopen sets is not $\psi_{\mathcal{H}}$-semiopen.

Corollary 4 *Let $(X, w, \mathcal{H})$ be a $\mathcal{H}wss$. Then, the family of all $\psi_{\mathcal{H}}$-semiopen sets forms a supra-topology on X.*

Conclusion

Diverse topics had appeared as the results of the interaction between topology and life's problems. One of this topics, studies $\psi_{\mathcal{H}}(.)$-operator relating the concepts weak structures with hereditary classes, which is useful in generalizing the most basic properties in general topology. As a generalization of w-open sets and w-semiopen sets, certain new kind of sets in a weak structure space via $\psi_{\mathcal{H}}(.)$-operator called $\psi_{\mathcal{H}}$-semiopen sets were introduced. Also, via hereditary class $\mathcal{H}_0$, wT_1-axiom was formulated and also some of their features were examined.

Abbreviations

wss: Weak structure space; $\mathcal{H}wss$: Hereditary class weak structure space; *w*nbhood: *w*-neighborhood.

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Authors' contributions
HMA-D investigated a certain new kind of sets via operators and contributed to the interpretation of the results. RAH conceived of the presented idea, developed operators, verified the theories and corollaries, and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

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