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ORIGINAL ARTICLE

The quasi-uniform character of a topological semigroup

John Mastellos *

Department of Mathematics, University of Patras, 26504 Patras, Greece

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Abstract The topological embedding of a topological semigroup S, commutative with the property of cancelation, into the group $G = S \times S/R$, (R the equivalence $(a,b)R(a',b') \iff ab' = a'b$) to which S is algebraically embedded, was the subject of the search for the mathematicians of a long period. It was based on the fact that S must naturally be a uniform topological space, as every topological group was. The present paper is devoted to the fact that a quasi-uniformity is defined to any topological space, thus to any topological semigroup.

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1. Introduction

1.1. In a series of papers for a long period the mathematicians engaged in *the embedding of a topological commutative semigroup with cancelation to a topological group*. The basic idea was very simple: since a topological group is a *uniform space*, that is a very nice space, it seems a natural demand for a topological semigroup, which embeds to a topological group, to be a uniform space as well. (Cf. the paper of E. Scheiferdecker [12, 1956] and the papers of [11,14,15,4,5,1,2,6] and others). In [3, 2001] the authors refer to a *quasi-uniformity* on a semigroup, that is: a topological semigroup S has a neutral element e and a neighborhood filter $\eta(e)$ of e which gives to S a *quasi-uniform structure*. On the other hand, the operations on the topological semigroups and groups must be continuous.

In the present paper we start with the *quasi-uniformity* which every topological T_0 structure has, hence every topological commutative with cancelation semigroup has. We suppose that the topology of the given *topological semigroup* is *weaker* or *equal* than the one which this structure may has. It is evident that if S is a semigroup and R is an *equivalence relation* on it, the quotient S/R is not a group, not even a semigroup. Meantime, it is defined the *specialization ordering* which has every T_0 but not T_1 topological space. The compatibility of the structures (of topology and of being the space semigroup) and the extension which Szpilran in [13] induces to an ordered space, seem to be obligatory for us.

1.2. In the remaining part of this paragraph we give necessary elements from the relative theory.

* Tel.: +30 2610996751.
E-mail address: kontolat@math.upatras.gr

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A *semigroup* S is called *topological semigroup*, if there is a topology τ such that the function

$$\Phi : S \times S \to S, \Phi(x,y) = x \cdot y \quad \text{(or simply} = xy)$$

is continuous. A *group* G is called *topological group* if the functions Φ and K

$$\Phi : S \times S \to S, \Phi(x,y) = x \cdot y \quad \text{and} \quad K : G \to G, K(g) = g^{-1}$$

are continuous.

A *uniform space* on a set X is a filter $\mathbf{U}$ on $X \times X$ such that: (a) Each member of $\mathbf{U}$ contains the diagonal of $X \times X$. (b) If $U \in \mathbf{U}$, then $V \circ V \subseteq U$ for some $V \in \mathbf{U}$. (c) There is a base of $\mathbf{U}$ from symmetrical elements. The elements of $\mathbf{U}$ are called *entourages*.

If the structure lacks the condition (c), then the space is a *quasi-uniform*. In a semi-group S (resp. a *group* G) by $\tau(\mathbf{U})$ we denote the topology that originated by a *quasi-uniformity* (resp. a *uniformity*) $\mathbf{U}$. Also by $(S, \cdot, \tau(\mathbf{U}))$ we denote the whole structure.

Besides, W.J. Pervin (in [9]) in 1962, firstly published the statement: *"For every topological space there is a quasi-uniformity which induces the given topology"*. Pervin, in the above paper, says that for a topological space (X, τ), the sets

$$U_O = \{(O \times O) \cup [(X \setminus O) \times X] \mid O \in \tau\}$$

define a base for a *quasi-uniformity*, where $O \in \tau$. For every fixed O, the set U_O is an entourage of the quasi-uniformity.

1.3. The *quotient structure* (or *quotient semigroup*) $Q = Q(S, \Sigma)$, (Σ is a commutative sub-semigroup of S), is a set whose elements are of the form $a\alpha^{-1}, a \in S, \alpha \in \Sigma$. So $Q(S, \Sigma) = S \times \Sigma / R$, where R is an equivalence relation defined by: $(a,b)R(c,d) \iff ad = bc$, the operation in $S \times \Sigma$ is component-wise. If the semigroup S is commutative we can write $Q = Q(S,S)$ for the *quotient structure* and the structure $G = S \times S/R$, (R the known relation), is a group to which S is algebraically embedded. This topological embedding of S into the above G is exactly the object of the "embedding" which mathematicians made during the period we have referred to.

1.4. The authors of [3] define a *quasi-uniformity* for a topological commutative semigroup $(S, \cdot, \tau)$. The sets of the form

$$\overline{U} = \{(x,y) \mid y \in xU, U \in \eta(e)\}.$$

are the entourages of the space. The proof of this proposition is based on the fact that for every element U of the $\eta(e)$, there is another element V, such that $V \cdot V \subseteq U(e)$. On the other hand, this construction of a quasi-uniform space is compatible with the one introducing by Pervin.

1.5. In his classical paper [12], Scheiferdecker gave the notion of the *invariance for a uniformity* $\mathbf{U}$. Let $U \in \mathbf{U}$ and $a, b, k \in S$. Then

$$(a,b) \in U \iff (ka, kb) \in U.$$

The main theorem in [12] which we are interesting to, is the following:

1.6. Theorem (Scheiferdecker, [12, p. 375]). *Necessary and sufficient conditions for a topological semigroup $(S, \cdot, \tau)$ (τ the topology of S) to embed into its quotient group $G = S \times S/R$, where R is the known equivalence relation, are the following:*

(a) *The topology τ is the one induced by a uniformity $\mathbf{U}$.*
(b) *The uniform structure may be defined via entourages which fulfill the "invariance" property.* □

Scheiferdecker considered the above G and the structure $(S, \cdot, \tau = \tau(\mathbf{U}))$, where the topology $\tau(\mathbf{U})$ is the one that is induced from the uniformity of $\mathbf{U}$. He proved that the subsets

$$U_1 = \{(A,B) \in Q \times Q | (A = \alpha^{-1}a, B = \beta^{-1}b) \text{ and } (x\alpha = y\beta \in \Sigma \Rightarrow (xa, yb) \in U, U \in \mathbf{U})\}$$

$a, b \in S, \alpha, \beta \in \Sigma$, constitute the entourages of a new *uniformity*, whose the trace on S is the same topology τ. We denote this new uniformity by $\mathbf{U}_1$.

1.7. This paper is divided into 3 paragraphs. More precisely, in 1 the paper's preliminaries are given. In paragraph 2 we present the main part of this research. Especially we examine and investigate many properties of a topological semigroup, without considering the notion of the quasi-uniformity (see for example 2.2, 2.3, 2.5, 2.6, 2.8, etc.). Finally, paragraph 3 refers to the specialization inequality define on a T_0 and not a T_1 space.

2. Quasi-uniform structure in a semigroup

In the sequel, S is always a *commutative semigroup with cancelation*. The condition $aS \cap bS \neq \emptyset, a, b \in S$ ([8]), means that the equivalence relation R such that

$$(a,b)R(c,d) \iff ad = bc, a, b, c, d \in S,$$

is not void. We suppose that this condition is in valid through all the paper. The function

$$\pi : S \times S \to G, \pi((a,b)) = \overline{(a,b)}$$

assigns to each $(a,b) \in S \times S$ the equivalence class in G containing the element (a,b) and which we symbolize by $\overline{(a,b)}$.

2.1. Examples

(1) In the real line we consider additively the set $\mathfrak{R}$, (the set of real numbers), and as topology the one which has as base the intervals $(a, +\infty), a \in \mathfrak{R}$. The set $\mathfrak{R}$ is the set of symbols which finally we construct. We embed this in the set $G = \mathfrak{R} \times \mathfrak{R}/R, R$ the known equivalence relation, which is the natural construction of real numbers with the natural topology. The first topology is weakest of the second.
(2) The same problem in the interval $[0,1]$ with operation the multiplication, the numbers their-selves are the symbols we note and the topology, the one which has as base the set of the form $\{(a,1), a \in [0,1)\}$. It embeds into $G = [0,1) \times [0,1)/R$ of the natural construction of the set of real number and with the natural topology. The former topology is again weaker than the topology of G.
(3) If in 1. we consider as the first and the second topologies the Sorgenfrey topology of $\mathfrak{R}$ (the set of natural numbers) the results are the expected ones. The Sorgenfrey topology of $\mathfrak{R}$ which has as relation the couples: $\{(x,y) \mid x \leqslant y < x + \epsilon\}$.

2.2. Proposition. *If a quasi-uniformity $\mathbf{U}$ is defined on a commutative with cancelation semigroup $(S, \cdot)$ and has the property*

$$(\forall U \in \mathbf{U})(\forall a \in S)[U \subseteq (a,a)U],$$

then S is a topological semigroup.

Proof. It is enough to prove:

$(\forall U)(\exists V)(\forall x \in S)(\forall y \in S)[V(x) \cdot V(y) \subseteq U(xy)]$

or

$(\forall U)(\exists V)(\forall x' \in V(x))(\forall y' \in V(y))[x'y' \in U(xy)]$.

We suppose that U, V belong to the quasi-uniformity $\mathbf{U}$ and fulfill $V \circ V \subseteq U$. Since $(x, x') \in V$ and because of the supposition, it is $(xy, x'y) \in V$. In the same way, since $(y, y') \in V$, it is $(x'y, x'y') \in V$, hence $x'y' \in V \circ V(xy)$ or $x'y' \in U(xy)$. □

2.3. Proposition. *Let $(S, \cdot, \tau(\mathbf{U}))$ be a commutative with cancelation semigroup and $\mathbf{U}$ a quasi-uniformity on S. If the translation $x \mapsto a \cdot x$, a any element of S, is continuous, then S is a topological semigroup.*

Proof. Let $U(x \cdot y), U \in \mathbf{U}$, be a neighborhood of xy. There is a V' such that $V' \circ V' \subseteq U$. Then, there is a V'' such that $(y, y') \in V'' \Rightarrow (ay, ay') \in V'$, for every a. Similarly, there is a V''' such that $(x, x') \in V''' \Rightarrow (xy, x'y) \in V' \cap V''$, where we have put $y = a$ and $x = y$. Put $V = V'' \cap V'''$ and then $(x, x') \in V, (y, y') \in V \Rightarrow (xy, x'y) \in V' \cap V'', (x'y, x'y') \in V' \cap V''$, so $(x, x') \in V$ and $(y, y') \in V$. Therefore $(xy, x'y') \in U$. □

2.4. Remark. We know that *under the conditions of the Proposition 2.3, in the uniform case, if the semigroup S is a group, then it is a topological group.* In fact, we have to prove that the function $x \mapsto x^{-1}$ is continuous. Indeed: $(\forall U)(\exists V)[(x, y) \in V \mapsto (y^{-1}, x^{-1}) \in U]$ and, as the entourages are symmetrical, we conclude a same result.

2.5. Proposition. *Let $(S, \cdot, \tau(\mathbf{U}))$ be a topological commutative with cancelation semigroup, where $\mathbf{U}$ is a quasi-uniformity. The following statements are equivalent:*

(1) *for every a, x in S, for every $U \in \mathbf{U}, aU(x) \subseteq U(ax)$,*
(2) *for every a, x, y in S, for every $U \in \mathbf{U}, (x, y) \in U$ implies $(ax, ay) \in U$.*

Proof. (1) implies (2)
$(x, y) \in U$ implies $y \in U(x)$, that is $ay \in aU(x)$, hence $ay \in U(ax)$, so $(ax, ay) \in U$.
(2) implies (1)

$y \in U(ax)$ implies $y = ak, k \in U(x)$, that is $y = ak, (x, k) \in U$, hence $y = ak, (ax, ak) \in U$ or $(ax, y) \in U$. Thus $y \in U(ax)$. □

2.6. Proposition. *Let $(S, \cdot, \tau(\mathbf{U}))$ be a topological commutative with the property of cancelation semigroup, where $\mathbf{U}$ is a quasi-uniformity. The following statements are equivalent:*

(3) *for every a, x in S, for every $U \in \mathbf{U}, U(ax) \subseteq aU(x)$,*
(4) *for every a, x, y in S, for every $U \in \mathbf{U}, (ax, ay) \in U$ implies $(x, y) \in U$.*

Proof. (3) implies (4)
$(ax, ay) \in U$ implies $ay \in U(ax)$, hence $ay \in aU(x)$ or $ay = a\lambda$ for some $\lambda \in U(x)$. So $y = \lambda$ and $(x, \lambda) \in U$, that is $(x, y) \in U$.

(4) implies (3)
$y \in U(ax)$ implies $(ax, y) \in U$, hence $(ax, aa^{-1}y) \in U$, so $(x, a^{-1}y) \in U$, so $a^{-1}y \in U(x)$, that is $aa^{-1}y \in aU(x)$ and finally $y \in aU(x)$. □

2.7. Remark. The statements (1) and (3) of Propositions 2.5 and 2.6 obtain the existence of the property of the invariance for a semigroup S. Besides, the *invariance property* obtain both the statements (1) and (3), for a semigroup.

2.8. Proposition. *Let $(S, \cdot, \tau(\mathbf{U}))$ be a topological commutative with the property of cancelation semigroup, where $\mathbf{U}$ is a quasi-uniformity. If moreover S has the property:*

for every $a, x, y \in S$, for every $U \in \mathbf{U}, U(ax) \subseteq aU(x)$,

then the translations are continuous and open.

Proof. Since the structure is a topological one, the operation is continuous and the inverse image of an open set A is open. We suppose that $x \in S$ and A be an open subset of S. It is enough to prove that xA is open, that is there is a $U \in \mathbf{U}$ such that $U(ax) \subseteq xA$, for $a \in A$. We have that A is open and $a \in A$, hence there is a $U \in \mathbf{U}$ such that $U(a) \subseteq A$, hence $xU(a) \subseteq xA$. Because of the supposition, there holds the demanded statement. □

2.9. Example. We suppose that in a space: $''y \in U(ax)''$ does not mean that $''y \in aU(x)''$ (in which case we would *invariance*). We consider, as an example, the space of $(0, 1)$ with the natural product and the natural topology. Then, $ax = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$, U being the entourage which corresponds to $d(x, y) < \frac{1}{10}$. Then, $y \in U(ax)$ means that $\frac{3}{10} < y < \frac{5}{10}$, while $''z \in aU(x)''$ gives $\frac{17}{50} < z < \frac{23}{30}$. It means that there are points of $U(a \cdot x)$ which does not belong to $a \cdot U(x)$.

2.10. Let S be a semigroup and $Q = Q(S, \Sigma)$ (Σ is a commutative sub-semigroup of S) the corresponding *quotient structure*. We shall use the following already function $\pi : S \times \Sigma \to Q, \pi(a, b) = \overline{(a, b)}$. If π is continuous, then is also an open function. We also define $P : S \to Q, P(x) = \pi(xb, b)$ and $\rho : S \to S \times \Sigma, \rho(s) = (sb, b), b$ any element of Σ. It is ρ continuous. If π is continuous, then the function $P = \pi \circ \rho$ is continuous too. The image $P(S)$ is exactly the embedding of S to the *quotient structure* $Q = Q(S, \Sigma)$.

We point out that if the semigroup S becomes a *quasi-uniform* space by a topology $\tau(\mathbf{U})$ and has the property of *invariance*, then S is not necessary a topological group. The reason is that the entourages are not obligatory symmetrical.

The following theorem refers to the constructions of the T_0 and not a T_1 *quasi-uniform spaces*. The constructions are the ones of the basic Scheiferdecker's statement cited in the paragraph 1. The Ore conditions are, of course, in valid here.

2.11. Theorem. *Let $(S, \cdot, \tau(\mathbf{U}))$ be a structure, where S is a topological T_0 and not a T_1 commutative with the property of cancelation semigroup, $\mathbf{U}$ be a quasi-uniformity generated by the topology of S and $\mathbf{B}_{\mathrm{U}}$ be a base of $\mathbf{U}$. Let, also, $Q = S \times S/R$, R the known equivalence relation, the quotient structure of S. We also suppose that $\mathbf{U}$ has the invariance property, that is:*

$$(a, b) \in U \Longleftrightarrow (x^*a, x^*b) \in U, x^* \in S, U \in \mathbf{U}. \quad (1)$$

Then, in the quotient structure of a quasi-uniformity $\mathbf{U}_1$ is defined a base $\mathbf{B}_{\mathbf{U}_1}$ which are given by

$$U_1 = \{(A, B) \in Q \times Q | A = a\alpha^{-1}, B = b\beta^{-1}, [x\alpha = y\beta \to (xa, yb) \in U], xa \leqslant yb\}, \quad (2)$$

$a, b, \alpha, \beta \in S$. (The inequality is the specialization relation). Thus, if the quotient structure Q is a T_0 and not a T_1 space,

an inequality is defined in Q which is compatible with the algebraic structure. Moreover, there holds:

(a) $(A,B) \in U_1 \Rightarrow (ZA,ZB) \in U_1, Z,A,B$ in $Q, U_1 \in \mathbf{U}_1$.
(b) $(\forall U \in \mathbf{B}_{\mathrm{U}_1})(\exists U_1 \in \mathbf{B}_{\mathrm{U}_1})[U = U_1 \cap (S \times S)]$.

Proof. The filter $\mathbf{U}_1 = \{U_1 \mid U \in \mathbf{U}\}$ defines a quasi-uniformity. In fact:

(1) *The relation is reflective*: $x\alpha = y\alpha$, hence $x=y$ and $(xa,ya) \in U$, that is $(A,A) \in U_1$.
(2) $(\forall U_1 \in \mathbf{U}_1)(\exists V_1 \in \mathbf{U}_1)[V_1 \circ V_1 \subseteq U_1]$: We suppose A,B are as above and $C = c\gamma^{-1}$.

We consider U and V such that $V \circ V \subseteq U$. If $x\alpha = y\gamma$, then $(xa,yc) \in V$ and $(A,C) \in V_1$. If also $(z\gamma, t\beta) \in V$, we have $(zc,tb) \in V$ and then $(C,B) \in V_1$. Therefore $xz\alpha = zy\gamma$ and $zy\gamma = zt\beta$, so $(xaz,yzc) \in V$ and $(yzc,ztb) \in V$, because of the *invariance property*. Hence $(xza,ztb) \in U$ and finally $(A,C) \in V_1, (C,B) \in V_1$ or $(A,B) \in U_1$.

Since the space is T_0 and not a T_1, every couple belongs to an entourage and gives an inequality.

Proof of (b)

First we prove that $U \subseteq U_1 \cap (S \times S)$

Let $(a,b) \in U$. It is $a = A = (\alpha a)\alpha^{-1}, b = B = (\alpha b)\alpha^{-1}$. Hence $(a,b) = (A,B) \in U_1$. Also, $(a,b) \in S \times S$. Then, $(a,b) \in U_1 \cap (S \times S)$.

Now we show that: $U_1 \cap (S \times S) \subseteq U$

Let $(A,B) \in U_1 \cap (S \times S)$. Then $A = (\alpha a)\alpha^{-1}, B = (\beta b)\beta^{-1}$. Due to (1), for $x\alpha = y\beta$, we have $((x\alpha)a,(y\beta)b) \in U$, thus $(a,b) \in U$.

Proof of (a)

First, we prove the following:

Under the invariance: $(\boldsymbol{\alpha^{-1}a}, \boldsymbol{\alpha^{-1}b}) \in \boldsymbol{U_1} \iff (\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{U}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\alpha} \in \boldsymbol{S}$.

Indeed:

$\Rightarrow$ Let $(A = \alpha^{-1}a, B = \alpha^{-1}b) \in U_1$. If $x\alpha = y\alpha$, then $x = y$ and $(xa,yb) \in U$. Finally $(a,b) \in U$.
$\Leftarrow$ Let $(a,b) \in U$. If $A = \alpha^{-1}a, B = \alpha^{-1}b$, then supposing that $x\alpha = y\alpha$, we have $x = y$ and $(xa,yb) = (xa,xb) = (x,x) \cdot (a,b) \in U_1$.

Now it is enough to prove the following:

(i) $(\gamma^{-1}A, \gamma^{-1}B) \in U_1 \iff (A,B) \in U_1, \gamma \in S$.
(ii) $(\delta A, \delta B) \in U_1 \iff (A,B) \in U_1, \delta \in S$.

(i) Put $A = \alpha^{-1}a, B = \alpha^{-1}b$. Then, $(\gamma^{-1}(\alpha^1 a), \gamma^{-1} \cdot (\alpha^{-1}b)) \in U_1$. By the above lemma we have $(a,b) \in U$.
(ii) The elements A and B are as above. We also suppose that $\delta'\alpha = \alpha'\delta, \alpha, \alpha', \delta, \delta' \in S$. (It is possible, because of (1)). So, $\alpha = \delta'^{-1}\alpha'\delta$ and $\alpha^{-1} = \delta^{-1}\alpha'^{-1} \cdot \delta'$. Hence

$$(\delta A, \delta B) = (\delta\alpha^{-1}\alpha, \delta\alpha^{-1}b) = (\delta\delta^{-1}\alpha'^{-1}\delta'\alpha, \delta\delta^{-1}\alpha'^{-1}\delta' b) = (\alpha'^{-1}\delta'\alpha, \alpha'^{-1}\delta' b).$$

That is: $(\delta A, \delta B) \in U_1 \iff (\alpha'^{-1}\delta'\alpha, \alpha'^{-1}\delta' b) \in U_1$, hence $(\alpha^{-1}a, \alpha^{-1}b) \in U_1$ or $(a,b) \in U_1$. The rest are trivial. □

2.12. The above facts (a) and (b) express the process toward the validity of the *invariance* of the given semi-group to the quotient structure. We have mainly needed, for the sake of brief, the same *denominator*.

All the elements of the quotient space $Q = Q(S,\Sigma)/R$ have the form $a \cdot \alpha^{-1}, a, \alpha \in S$, or the form $\overline{(a,\alpha^{-1})}$. The relations which rule the different operations and equivalences, based on the following, where $b, \beta \in S$:

(1) $a\alpha^{-1} = b\beta^{-1} \iff \alpha\lambda = \beta l, \lambda \in \Sigma, l \in S$.
(2) $a\alpha^{-1}b\beta^{-1} = at(\beta\tau)^{-1}$ with $b\tau = \alpha t, \tau \in \Sigma, t \in S$.

So, in this set Q a commutative semigroup is defined and the cancelation property is an easy consequence.

2.13. Theorem. (Cf. [12, p. 377]). *With the suppositions of the Theorem 2.11 in the semigroup* $(S,\cdot,\tau(\mathbf{U}))$ *the following statements hold:*

(i) $\left(\overline{(a,\alpha)}, \overline{(b,\beta)}\right) \in U_1 \Rightarrow \left(\overline{(ka,\alpha)}, \overline{(kb,\beta)}\right) \in U_1, k \in S$.
(ii) $\left(\overline{(a,\alpha)}, \overline{(b,\beta)}\right) \in U_1 \Rightarrow \left(\overline{(k,k)} \cdot \overline{(a,\alpha)}, \overline{(k,k)} \cdot \overline{(b,\beta)}\right) \in U_1, k \in S$.
(iii) $A = a\alpha^{-1}, B = b\beta^{-1}, (A,B) \in U_1 \Rightarrow (\gamma^{-2}A, \gamma^{-2}B) \in U_1$, *where* $\gamma \in S, \gamma^{-2} = \gamma^{-1} \cdot \gamma^{-1}$.
(iv) *Generally, if there holds the invariance for the* $\mathbf{U}$, *then there holds for the* $\mathbf{U}_1$ *as well.*

Proof. We correspond to every $U \in \mathbf{U}$ the entourage $U_1 \in \mathbf{U}_1$. The demonstrations follow, in a great part, the logic of Scheiferdecker.

(i) If $x_1\alpha = x_2\beta$, then we may prove that $(kx_1a, kx_2b) \in U$, where $k = \overline{(k,1)} \in U_1$. The theorem is true because of the suppositions and the *invariance property*.
(ii) We suppose that $x_1\alpha = x_2\beta$, hence $(x_1a, x_2b) \in U$, so $x_1\alpha = x_2\beta$ and $(x_1ka, x_2kb) \in U_1$ because of the *invariance property* in $\mathbf{U}$.
(iii) It is $\gamma^{-2} = (k, k\gamma^2)$ for any $k \in S$. We have to prove that:

$$\left(\overline{(a,\alpha)}, \overline{(b,\beta)}\right) \in U_1 \Rightarrow \left(\overline{(ka,k\gamma\alpha)}, \overline{(kb,k\gamma\beta)}\right) \in U_1,$$

after the supposition $x_1\alpha = x_2\beta$. From this last relation we have that $x_1k\gamma\alpha = x_2k\gamma\beta$ and from that we conclude to our demand.
(iv) We suppose that $(a,b) \in U$, that is $\left(\overline{(ax,x)}, \overline{(by,y)}\right) \in U_1$. We will prove that $\left(\overline{(kax,x)}, \overline{(kby,y)}\right) \in U_1, k \in S$. It is known that $\left(\overline{(kax,kx)}, \overline{(kby,ky)}\right) \in U_1$. That is, if $\sigma_1kx = \sigma_2ky$, then $(\sigma_1kax, \sigma_2kby) \in U$ or $\sigma_1x = \sigma_2y$ entails $(\sigma_1ax, \sigma_2by) \in U$ or $\left(\overline{(kax,x)}, \overline{(kby,y)}\right) \in U_1$ or $\overline{(ka,kb)} \in U_1$. □

2.14. For every $U \in \mathbf{U}$, the subsets $\pi^{-1}(U)$ consist a base $\mathbf{B}$ for a *quasi-uniformity* structure. In fact:

(a) Let $V \in U, \pi^{-1}(V) = U, V_1 \circ V_1 \subseteq V$. From the latter relation: $\overline{(a,b)} \in V_1, \overline{(b,c)} \in V_1$, that is $\overline{(a,c)} \in V$. So, there holds $\pi^{-1}(V) = V_1$ and if $(x,y) \in \pi^{-1}(V)$, then $\pi(x,y) = \overline{(a,b)}$, besides $\pi(y,z) = \overline{(b,c)}$, that is $V_1 \circ V_1 \subseteq V$.
(b) $\cap\{U \mid U \in \mathbf{B}\} = \cap(\pi^{-1}\{V \mid V \in \mathbf{U}\}) = \pi^{-1}(\cap\{V \mid V \in \mathbf{U}\})$, the latter containing the diagonal of $S \times \Sigma$.

2.15. Proposition. *Let $(S, \cdot, \tau = \tau(\mathbf{U}), e)$ be a topological commutative with the property of cancelation semigroup, e its neutral element and* $\mathbf{U}$ *a quasi-uniformity on S. If $Q = S \times \Sigma / R$ (R the known equivalence relation), is the quotient structure of S and if S has the additional property*:

"if O is open, then $x \cdot O$ is open too",

then S is topologically embedded to Q.

Proof. Let $\pi : S \times \Sigma \to Q, \pi((a,b)) = (a,b)$ and $P : S \to Q$, $P(x) = \pi(xb, b)$. We consider the map:

$\rho : (S, \tau) \to (S \times S, \tau \times \tau), \rho(x) = (x, e)$.

We have:

(i) The map ρ is $1-1$. It is $\rho(x_1) = \rho(x_2)$. Hence, $(x_1, e) = (x_2, e)$ implies $x_1 = x_2$.
(ii) The map ρ is onto $\rho(S)$.
(iii) The set $\rho(S)$ is a sub-semigroup of $S \times S$. It is $(x,e) \cdot (y,e) = (x \cdot y, e) \in \rho(S)$.
(iv) The map $\rho : (S, \tau) \to (S \times S, \tau \times \tau)$, is a homomorphism $(\rho(x,y) = (x \cdot y, e) = (x,e) \cdot (y,e) = \rho(x) \cdot \rho(y))$.
(v) The map: $f_1 : (S^2 \times S^2, \tau^2 \times \tau^2) \to (S \times S, \tau \times \tau), f_1((a,b), (c,d)) = (a \cdot c, b \cdot d)$ is continuous.
Let A be an open sub-set of $S \times S$ such that $(a \cdot c, b \cdot d) \in A$. We have to find a subset B such that $B \subseteq S^2 \times S^2$, $((a,b),(c,d)) \in B, f_1(B) \subseteq A$, where B is open. There are neighborhoods $U_{a \cdot c}, U_{b \cdot d}$ of $a \cdot c, b \cdot d$ respectively, such that $U_{a \cdot c} \times U_{b \cdot d} \subseteq A$. Since the maps $(a,c) \mapsto a \cdot c$ and $(b,d) \mapsto bd$ are continuous, there are neighborhoods V_a, V_b, V_c, V_d of a,b,c,d respectively with $V_a \cdot V_c \subseteq U_{ac}$ and $V_b \cdot V_d \subseteq U_{bd}$. We put: $B = V_a \cdot V_c \times V_b \cdot V_d$.
(vi) The structure $(S \times S, \tau \times \tau)$ is a topological semigroup.
(vii) The structure $(\rho(S), \tau|_{\rho(S)})$ $(\tau|_{\rho(S)})$, the one reduced on $\rho(S)$ is a topological semigroup.
(viii) The map ρ is continuous.Let $A \subseteq S \times S$ be an open set and $(x,e) \in A$. We have to find an open set B such that $B \subseteq S, x \in B, \rho(B) \subseteq A$. For the set A, there are open neighborhoods U_x, U_e of x and e, respectively, such that $U_x \times U_e \subseteq A$. If $B = U_x$, then $\rho(B) = \rho(U_x) = (U_x, e) \subseteq A$.
(ix) The map $\rho : (S, \tau) \to (S \times S, \tau \times \tau|_{\rho(S)})$ is open.
(x) The map $\pi : (S \times S, \tau \times \tau) \to (Q, \tau \times \tau|_Q), \pi(x,y) = \overline{(x,y)}$ is continuous, $(\tau \times \tau|_Q$ is, the natural topology on Q according to its definition).
(xi) The map $P : (S, \tau) \to (Q, \tau \times \tau|_Q), P(x) = \overline{(x,e)}$ is continuous.
We observe that the maps:

$\rho : (S, \tau) \to (S \times S, \tau \times \tau|_{\rho(S)})$

and

$\pi : (S \times S, \tau \times \tau) \to (Q, \tau \times \tau|_Q)$

are continuous, hence the map $\pi|_{\rho(S)}$ is continuous, and therefore the map $P = \rho \circ \pi|_{\rho(S)}$ is continuous.
(xii) The map $P : (S, \tau) \to (Q, \tau \times \tau|_Q)$ is open.

Since the set $S \times S$ is a topological group it is homeomorphic to $S \times S/R$. hence we have an embedding of S into the topological group $S \times S$. □

2.16. Examples

(1) We consider the space $(\Re_+, \tau_+)$ where $\Re_+$ is the set of positive numbers and τ_+ is the usual topology of positive numbers. We consider the space as an additive semigroup and, similarly, we consider the space $(\Re, \tau)$, where $\Re$ is the set of real numbers noted additively and τ the usual topology of real numbers. We note that for an open subset O of the set of positive numbers and for any $x \in \Re$ the set $x \cdot O$ is open. Thus, we have embedding of $\Re_+$ to $\Re$.
(2) Let $J_1 = [0,1]$, under the usual multiplication, $J_2 = [\frac{1}{2}, 1]$ with multiplication defined by $x \circ y = \max(\frac{1}{2}, xy)$ where xy denotes the usual multiplication of real numbers and $J_3 = [0,1]$ with multiplication defined by $x \circ y = \min(x,y)$. J_1 and J_2 have just the two idempotent numbers zero and identity, but in J_3 every element is an idempotent. Every non-idempotent element of J_2 is algebraically nilpotent (see these examples in [12]).Let us consider the topology which has as base the set of the form $\{(a,1] : 0 \leqslant \alpha < 1\}$. We remark that the topology in the three spaces is the topology of a quasi-uniformity. The proposition of 2.15 is in valid for the spaces J_1, J_3 and there does not holds for J_2.
(3) If $(X, \mathbf{U})$ is a quasi-uniform space, $\Delta = \cap_U \mathbf{U}$ and α the canonical mapping from $(X, \mathbf{U})$ onto X/Δ. Given $U \in \mathbf{U}$, select $V \in \mathbf{U}$ such that $V \circ V \circ V \subset U$. Then $(\alpha \times \alpha)(V) : (\alpha \times \alpha)(V) = (\alpha \times \alpha)(V \circ \Delta \circ V) \subset (\alpha \times \alpha)(U)$. Thus $\alpha\mathbf{U}$ is a quasi-uniformity on X/Δ and is $\mathbf{U}$-preserving. (C.f. [7]).

3. The specialization ordering on a semi-group

3.1. A *quasi-uniformity* induces on a space, say X, a relation

$$\cap\{U \in \mathbf{U}\}, \tag{1}$$

which is *reflective* and *transitive*. This relation is a proper inequality for a topological T_0 and not a T_1 space. As we have already said, we denote it by $x \leqslant y$ (or by $x \leqslant^X y$), for $x, y \in X$ and it means that $x \in cl\{y\}$. If in the space exists an operation, symbolized, say, by $\cdot$, then the *compatibility* of relation and operation is given by:

$$x \leqslant y, a \in X \Rightarrow a \cdot x \leqslant a \cdot y. \tag{2}$$

Proposition. *Let $(S, \cdot, \tau = \tau(\mathbf{U}), \leqslant^S)$, ($\leqslant^S$ the specialization order) be a topological commutative with cancelation semigroup, T_0 and not T_1, $\mathbf{U}$ is the quasi-uniformity on S such that $\tau = \tau(\mathbf{U})$. We also assume that:*

$$(\forall a \in S)[U \in \mathbf{U} \Rightarrow (a,a)U \in \mathbf{U}]. \tag{$*$}$$

Then

$$a \leqslant^S b, x \leqslant^S y \Rightarrow x \cdot a \leqslant y \cdot b, a, b, x, y \in S. \tag{$**$}$$

Proof. Let $a \leqslant b$. Then, the element a belongs to every neighborhood of b. Thus, if V, V', U are entourages of $\mathbf{U}$, we have that for every $V, a \in V(b)$, hence $(a,b) \in V$ or -because of the supposition- $(x \cdot a, x \cdot b) \in V$ and thus $x \cdot a \leqslant x \cdot b$. Besides: for every V, V', U with $V \circ V' \subseteq U$ there holds $(a,b) \in V \Rightarrow (a \cdot x, b \cdot x) \in V, x \in V(y) \Rightarrow (x,y) \in V' \Rightarrow (b \cdot x, b \cdot y) \in U$ and thus $(x \cdot a, y \cdot b) \in U$. □

3.2. Remark. The inverse relation of the relation $(*)$ is:

$$(\forall a \in S)[(a,a)U \in \mathbf{U} \Rightarrow U \in \mathbf{U}]. \tag{3}$$

If the specialization order is linear, then the condition $(**)$ implies (3). In fact, if $x \cdot a < x \cdot b$ and $a \geqslant b$, we have $x \cdot a \geqslant x \cdot b$, which is a contradiction.

3.3. During sixties, mathematicians faced the following problem:

"Given a commutative semigroup S with the property of cancelation, ordered by $\leqslant$and given a semigroup Q of quotient structure, under which conditions the order $\leqslant$may be extended to an order $\leqslant^Q$ on Q?"

In fact, the problem was similar to one that the structure Q was a semigroup larger that the given one. Of course, the given inequality is not obligatory, the *specialization*.

The extension of $\leqslant$ on S to an order on Q means that there is another order $\leqslant^Q$ on Q such that

$\leqslant \ \subseteq \leqslant^Q$ and $a \leqslant b \Rightarrow a \leqslant^Q b$.

The first who gave sufficient conditions at the end to extend an order of a semigroup S to its quotient group G, was Puttaswamaiah (cf. [10]). Beyond this, Weinert generalized the study of the order of a semigroup S to the order of any quotient semigroup T. Weinert (in [19]) recapitulated the conditions. The operation in the product $S \times S$ is defined by the relations:

$(a, \alpha) \cdot (b, \beta) = (at, \beta\tau), \tau, t \in S$, with $b\tau = \alpha \cdot t$.

The definition is independent of the choice of the t and τ and the equivalence classes both of them. The proposition below is owed to Puttaswamaiah, it has been generalized by Weinert and the relative idea of a space T_0 but not a T_1 one, is a new one.

3.4. Proposition. *If S is a topological commutative with cancelation semigroup T_0 and not a T_1 ordered by $\leqslant$, ($\leqslant$the specialization order), then this order has an extension $\leqslant^Q$ to every quotient semigroup Q, if there holds:*

$$a\xi \leqslant b\xi, a, b, \xi \in S \Rightarrow a \leqslant b. \qquad \square \tag{4}$$

The extended ordering $\leqslant^Q$ is uniquely determined by the relation (5) below.

The set Q is one of the quotient semigroups and, of course, is probably a group. The new relation $\leqslant^Q$ is defined by the following:

$$a\alpha^{-1} \leqslant^Q b\beta^{-1} \iff \text{thereare } l, \lambda \in S \text{ with } \alpha \cdot \lambda = \beta \cdot l \text{ and } a \cdot \lambda \leqslant b \cdot l \tag{5}$$

So, the Proposition 3.4 may be transformed into the next one.

3.5. Proposition. *Let $(S, \cdot, \tau = \tau(\mathbf{U}))$ be a topological commutative with the property of cancelation T_0 and not T_1 semigroup. We also suppose that S fulfills the statement:*

$$(\forall a \in S)(\forall U \in \mathbf{U})[U \in \mathbf{U} \iff (a, a)U \in \mathbf{U}]. \tag{$***$}$$

Then, the semigroup S is topologically embedded into the quotient structure Q. If moreover the space Q is T_0 and for S there holds the above (3), then the specialization order of S is extended to an inequality of Q.

Proof. If the space S is T_0 and not T_1, then the space has the ordering of *specialization* and the condition $(***)$ assures that the ordering is compatible with the operation in S. For the set Q, it is not secure that it is not T_1.

The condition $a \cdot \xi \leqslant b \cdot \xi, a, b, \xi \in S$ means that for any $U \in \mathbf{U}$, it is $(\xi \cdot a, \xi \cdot b) \in U$, hence $(a, b) \in U$, which, finally, implies that $a \leqslant b$. So, we get the condition which assures the extensibility of $\leqslant$. $\square$

3.6. Remark. The above condition $(***)$ expresses the *invariance* of $\mathbf{U}$ and it is a rather hard condition. We remark that in the basic Proposition 3.1 the above relation $(***)$ is in valid for the structure of the *quotient structure* Q, for the one only direction.

3.7. We refer, now, to the relation $\leqslant$ itself. S.E. Szpilrajn (in [13, 1930]) proved that

"every partially ordered set is extended to a linear ordered one".

For any topological space (X, τ) we consider a *quasi-uniformity* $\mathbf{U}$ such that $\tau = \tau(\mathbf{U})$ and we make use, as usual, of the following symbols: we put $\mathbf{U}^{-1}$ for the class $\{U^{-1} \mid U \in \mathbf{U}\}$ and $\mathbf{U}^* = \mathbf{U} \vee \mathbf{U}^{-1}$, where $\mathbf{U}^*$ is the *supremum* of the *quasi-uniformities* $\mathbf{U}$ and $\mathbf{U}^{-1}$. Let $a, b \in X$. By $a||b$ we denote that the elements a, b are non-comparable to each other.

For any two elements a, b in X and $U \in \mathbf{U}$, we define the subset:

$$\mathfrak{I}^U_{ab} = \{(x, y) \mid (x, y) \in U \text{ or } ((x, a) \in U \text{ and } (b, y) \in U)\}. \tag{6}$$

It is proved that this subset is the entourage of a *quasi-uniformity*. During all the process the relation is the *specialization order*. Of course, the relation must always be compatible and the operation continuous.

We state the following:

3.8. Proposition. *Let be the structure $(S, \cdot, \tau = \tau(\mathbf{U}), \leqslant)$, where S is a topological commutative with the property of cancelation semigroup; moreover $\mathbf{U}$ is a quasi-uniformity with $\tau = \tau(\mathbf{U})$, the space is a T_0 and not a T_1 space and $\leqslant$is the specialization order defined on S. We assume that there holds the above condition $(***)$. For any $U \in \mathbf{U}$ we construct (by the above (6)) the set $\mathfrak{I}^U_{ab}$ and after that the filter*

$$\mathbf{F}_{ab} = \{\mathfrak{I}^U_{ab} \mid U \in \mathbf{U}\}.$$

Then,

(a) *The filter $\mathbf{F}_{ab}$ induces a T_0 quasi-uniformity on S.*
(b) *The point (a, b) belongs to all the elements of $\mathbf{F}_{ab}$, where by $\mathbf{F}_{ab}$ has not the T_1 property; moreover $\cap U \subset \cap \mathbf{F}_{ab}$.*
(c) *$\mathbf{U}^* = \mathbf{F}_{ab} \vee \mathbf{F}^{-1}_{ab}$, (that is $\mathbf{U}^*$ is the supremum of $\mathbf{F}_{ab}, \mathbf{F}^{-1}_{ab}$).*
(d) *If in addition the relation*

$$(\forall k \in S)(\forall U \in \mathbf{U})(\forall (a, b))\left[(k, k)\mathfrak{I}^U_{ab} \in \cap \mathbf{F}_{ab} \to \mathfrak{I}^U_{ab} \in \mathbf{F}_{ab}\right] \tag{7}$$

is fulfilled, then the extension of the relation $\leqslant$is the extension of the $\cap \mathbf{F}_{ab}$.

(e) *In all the levels of the demonstration, the relation $(***)$ holds.*

Proof. (a) *The transitive property*: it comes as follows: Given $U \in \mathbf{U}$, let $V \in \mathbf{U}$ with $V \circ V \circ V \subseteq U$. We also assume that (a, b), as also (b, a), are not contained to U. If $(x, y) \in \mathfrak{I}^V_{ab} \circ \mathfrak{I}^V_{ab}$ there is $z \in S$ such that $(x, z) \in \mathfrak{I}^V_{ab}$ and $(z, y) \in \mathfrak{I}^V_{ab}$. We distinguish the cases: $(x, z) \in V, (z, y) \in V$ or $(x, z) \in V, (z, a) \in V, (b, y) \in V$ *etc*. It is not very difficult for one to prove the result in all the other cases.

The T_0-property is a consequence of the result (c) below, since the T_0-property for U is equivalent to the property of being the structure $\mathbf{U}^*$ a T_1-space.

(b) It is an easy consequence of the definition.

The space is not T_1, since a and b at least belong to all $\mathfrak{I}^U_{ab}, U \in \mathbf{U}$.

(c) We have to prove that for any $\mathfrak{I}^U_{ab}, U \in \mathbf{U}$, there is a V such that

$$\mathfrak{I}^V_{ab} \circ \left(\mathfrak{I}^V_{ab}\right)^{-1} \subseteq U \cap U^{-1}.$$

Let V be such that $V \circ V \circ V \subseteq U$ and (a,b) as well as (b,a), is not contained to U. We distinguish cases, for instance $(x,y) \in V, (x,y) \in V^{-1}$, or $(x,y) \in V, (x,a) \in V$ and $(b,x) \in V$ *etc*. In all the cases the demanded are easily concluded.

(d) Firstly, we denote by the same symbol $\leqslant$ every extension of the given relation. Now, the condition $\xi \cdot a \leqslant \xi \cdot b$ means that for every $U \in \mathbf{U}$ and for every a, b elements of X, there holds $(\xi \cdot a, \xi \cdot b) \in \mathfrak{I}^U_{ab}, \mathfrak{I}^U_{ab} \in \mathbf{F}_{ab}$ which implies-from the relation $(***)$-that there holds $(a,b) \in \mathfrak{I}^U_{xy}$. If the elements are comparable in the starting level, then it is again $a \leqslant b$, if there are non-comparable, they imply again $a \leqslant b$.

(e) Since $\mathfrak{I}^U_{ab} \in \mathbf{U}$, the condition $(***)$ is valid for all the entourages of $\mathbf{F}_{ab}$. $\square$

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