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ORIGINAL ARTICLE

L-fuzzy *G*-subalgebras of *G*-algebras

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Abstract In this paper, the *L*-fuzzification of *G*-subalgebras are considered and some related properties are investigated. A characterization of *L*-fuzzy *G*-algebras are given. We classified the *G*-subalgebras by their family of level subalgebras of *G*-algebras.

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1. Introduction

The study of *BCK*/*BCI*-algebras [1,2] was initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Hu and Li [3] introduced a wide class of abstract algebras: *BCH*-algebras. They have shown that the class of *BCI*-algebras is a proper subclass of the class of *BCH*-algebras. Neggers et al. [4] introduced *Q*-algebras and generalized some theorems discussed in *BCK*/*BCI*-algebras. Ahn et al. [5] introduced a new notion, called QS-algebras and discussed some properties of the *G*-part of *QS*-algebras. Neggers and Kim [6] introduced a new notion, called *B*-algebras which is related to several classes of algebras of interest such as *BCK*/*BCI*/*BCH*-algebras. Kim and Kim [7] introduced the notion of *BG*-algebras, which is a generalization of *B*-algebras. Senapati et al. [8–15] done lot of works on *B*-algebras and *BG*-algebras. Walendziak [16] introduced a new notion, called a *BF*-algebra which is a generalization of *B*-algebra and obtained several results.

Bandru and Rafi [17] introduced a new notion, called *G*-algebras, which is a generalization of *QS*-algebras and discussed relationship between these algebras with other related algebras such as *Q*-algebras, *BCI*-algebras, *BCH*-algebras, *BF*-algebras and *B*-algebras. They introduced the concept of 0-commutative, *G*-part and medial of *G*-algebras and studied their related properties.

The objective of this paper is to introduce the concept of *L*-fuzzy set [18] to *G*-subalgebras of *G*-algebras. The notion of *L*-fuzzy *G*-subalgebras of *G*-algebras is defined and lot of properties are investigated. We classified the *G*-subalgebras by their family of level subalgebras of *G*-algebras. We prove that if every *L*-fuzzy *G*-subalgebras has the finite image, then

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every descending chain of G-subalgebras terminates at finite step. In addition to it we observe that every ascending chain of G-subalgebras terminates at finite step if the set of values of any L-fuzzy G-subalgebras is a well ordered subset of L.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included. Throughout this paper $(L, \leqslant, \vee, \wedge)$ denotes a complete distributive lattice with maximal element 1 and minimal element 0 respectively.

Definition 2.1 ([17] G-algebra). A non-empty set X with a constant 0 and a binary operation $*$ is said to be G-algebra if it satisfies the following axioms

$G1.\ x * x = 0$

$G2.\ x * (x * y) = y, \quad \text{for all } x, y \in X.$

A G-algebra is denoted by $(X, *, 0)$.

Now, we introduce the concept of G-subalgebra over a crisp set X and the binary operation $*$ in the following. The definition of G-subalgebra is given below.

Definition 2.2 ([17] G-subalgebra). A non-empty subset S of a G-algebra X is called a G-subalgebra of X if $x * y \in S$, for all $x, y \in S$.

From this definition it is observed that, if a subset S of a G-algebra and it is closed, then S becomes a G-subalgebra.

Our main objective is to investigate the idea of G-subalgebras on L-fuzzy set. In L-fuzzy set, the membership values of the elements are written together along with the elements. The definition of this set is given below.

Definition 2.3 ([18]). Let X be a non-empty set. A L-fuzzy set $A = \{\langle x, \alpha_A(x)\rangle : x \in X\}$ of X is a function $\alpha_A : X \to L$.

The intersection of two L-fuzzy sets $A = \{< x, \alpha_A(x) >: x \in X\}$ and $B = \{< x, \alpha_B(x) >: x \in X\}$ in X is defined as $A \cap B = \alpha_A(x) \wedge \alpha_A(y)$ for all $x \in X$.

3. Main results

In what follows, let X denote a G-algebra unless otherwise specified. Combined the definitions of G-subalgebra over crisp set and the idea of L-fuzzy set we define L-fuzzy G-subalgebra, which is defined below.

Definition 3.1. Let $A = \{\langle x, \alpha_A(x)\rangle : x \in X\}$ be a L-fuzzy set in X, where X is a G-subalgebra, then the set A is L-fuzzy G-subalgebra over the binary operator $*$ if it satisfies the condition $\alpha_A(x * y) \geqslant \alpha_A(x) \wedge \alpha_A(y)$ for all $x, y \in X$.

If A is a L-fuzzy G-subalgebra in X, then $\alpha_A(0)$ is the upper bound of $\alpha_A(x)$, for all $x \in X$, i.e. $\alpha_A(0) \geqslant \alpha_A(x)$. Also, it is easily proved that $\alpha_A(0 * x) \geqslant \alpha_A(x)$ for all $x \in X$. Let $\{x_n\}$ be a sequence of X. Then $\alpha_A(0) \geqslant \alpha_A(x_n)$ or $1 \geqslant \alpha_A(0) \geqslant \alpha_A(x_n)$. If $\lim_{n\to\infty} \alpha_A(x_n) = 1$, then $\alpha_A(0) = 1$.

Like other subalgebras, the intersection of two L-fuzzy G-subalgebras of X is also a L-fuzzy G-subalgebra of X. More generally, intersection of infinite number of L-fuzzy G-subalgebras of X is also a L-fuzzy G-subalgebra of X.

If A is a L-fuzzy G-subalgebra of X, then it is easy to verify that the set $I_{\alpha_A} = \{x \in X : \alpha_A(x) = \alpha_A(0)\}$ is a G-subalgebra of X.

Theorem 3.2. *Let B be a non-empty subset of X and A be a L-fuzzy set in X defined by*

$$\alpha_A(x) = \begin{cases} \lambda, & \text{if } x \in B \\ \tau, & \text{otherwise} \end{cases}$$

for all $\lambda, \tau \in L$ with $\lambda \geqslant \tau$. Then A is a L-fuzzy G-subalgebra of X if and only if B is a G-subalgebra of X. Moreover, $I_{\alpha_A} = B$.

Proof. Let A be a L-fuzzy G-subalgebra of X. Let $x, y \in X$ be such that $x, y \in B$. Then $\alpha_A(x * y) \geqslant \alpha_A(x) \wedge \alpha_A(y) = \lambda \wedge \lambda = \lambda$. So $x * y \in B$. Hence, B is a G-subalgebra of X.

Conversely, suppose that B is a G-subalgebra of X. Let $x, y \in X$. Consider two cases:

Case (i) If $x, y \in B$ then $x * y \in B$, thus $\alpha_A(x * y) = \lambda = \alpha_A(x) \wedge \alpha_A(y)$.

Case (ii) If $x \notin B$ or, $y \notin B$, then $\alpha_A(x * y) \geqslant \tau = \alpha_A(x) \wedge \alpha_A(y)$.

Hence, A is a L-fuzzy G-subalgebra of X.

Also, $I_{\alpha_A} = \{x \in X, \alpha_A(x) = \alpha_A(0)\} = \{x \in X, \alpha_A(x) = \lambda\} = B$. □

Definition 3.3. Let A is a L-fuzzy G-subalgebra of X. For $s \in L$, the set $U(\alpha_A : s) = \{x \in X : \alpha_A(x) \geqslant s\}$ is called a level subset of A.

Obviously, this level subset $U(\alpha_A : s)$ is a G-subalgebra of X.

Theorem 3.4. *Let A be a L-fuzzy set in X, such that the set $U(\alpha_A : s)$ is G-subalgebra of X for every $s \in L$. Then A is a L-fuzzy G-subalgebra of X.*

Proof. Let for every $s \in L$, $U(\alpha_A : s)$ is subalgebra of X. In contrary, let $x_0, y_0 \in X$ be such that $\alpha_A(x_0 * y_0) < \alpha_A(x_0) \wedge \alpha_A(y_0)$. Let $\alpha_A(x_0) = \theta_1, \alpha_A(y_0) = \theta_2$ and $\alpha_A(x_0 * y_0) = s$. Then $s < \theta_1 \wedge \theta_2$. Let us consider, $s_1 = \frac{1}{2}[\alpha_A(x_0 * y_0) + \alpha_A(x_0) \wedge \alpha_A(y_0)]$. We get that $s_1 = \frac{1}{2}(s + \theta_1 \wedge \theta_2)$. Therefore, $\theta_1 > s_1 = \frac{1}{2}(s + \theta_1 \wedge \theta_2) > s$ and $\theta_2 > s_1 = \frac{1}{2}(s + \theta_1 \wedge \theta_2) > s$. Hence, $\theta_1 \wedge \theta_2 > s_1 > s = \alpha_A(x_0 * y_0)$, so that $x_0 * y_0 \notin U(\alpha_A : s)$ which is a contradiction, since $\alpha_A(x_0) = \theta_1 \geqslant \theta_1 \wedge \theta_2 > s_1$ and $\alpha_A(y_0) = \theta_2 \geqslant \theta_1 \wedge \theta_2 > s_1$. This implies $x_0, y_0 \in U(\alpha_A : s)$. Thus $\alpha_A(x * y) \geqslant \alpha_A(x) \wedge \alpha_A(y)$ for all $x, y \in X$. Hence, A is a L-fuzzy G-subalgebra of X. □

Theorem 3.5. *Any subalgebra of X can be realized as a level subalgebra of some L-fuzzy G-subalgebra of X.*

Proof. Let P be a L-fuzzy G-subalgebra of X, and A be a L-fuzzy set on X defined by

$$\alpha_A(x) = \begin{cases} \lambda, & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases}$$

for all $\lambda \in L$. We consider the following cases:

Case (i): If $x, y \in P$, then $\alpha_A(x) = \lambda, \beta_A(x) = \tau$. Thus, $\alpha_A(x * y) = \lambda = \lambda \wedge \lambda = \alpha_A(x) \wedge \alpha_A(y)$.

Case (ii): If $x \in P$ and $y \notin P$ then $\alpha_A(x) = \lambda$ and $\alpha_A(y) = 0$. Thus, $\alpha_A(x * y) \geqslant 0 = \lambda \wedge 0 = \alpha_A(x) \wedge \alpha_A(y)$.

Case (iii): If $x \notin P$ and $y \in P$ then $\alpha_A(x) = 0$ and $\alpha_A(y) = \lambda$. Thus, $\alpha_A(x * y) \geqslant 0 = 0 \wedge \lambda = \alpha_A(x) \wedge \alpha_A(y)$.

Case (iv): If $x \notin P$ and $y \notin P$ then $\alpha_A(x) = 0$ and $\alpha_A(y) = 0$. Now $\alpha_A(x * y) \geqslant 0 = 0 \wedge 0 = \alpha_A(x) \wedge \alpha_A(y)$.

Therefore, A is a L-fuzzy G-subalgebra of X. □

As a generalization of Theorem 3.5, we prove the following theorem:

Theorem 3.6. *Let X be a G-algebra. Then given any chain of G-subalgebras $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_r = X$, there exists a L-fuzzy G-subalgebra A of X whose level subalgebras are exactly the G-subalgebras of this chain.*

Proof. Consider a set of numbers $s_0 > s_1 > \cdots > s_r$, where each $s_i \in [0,1]$. Let $A = \{< x, \alpha_A(x) >: x \in X\}$ be a L-fuzzy set defined by

$$\alpha_A(x) = \begin{cases} s_0, & \text{if } x \in P_0 \\ s_i, & \text{if } x \in P_i - P_{i-1},\ 0 < i \leqslant r. \end{cases} \tag{1}$$

We consider the following two cases:

Case (i): Let $x, y \in P_i - P_{i-1}$. Therefore, by (1), $\alpha_A(x) = \alpha_A(y) = s_i$. Since P_i is a G-subalgebra, we have $x * y \in P_i$, and so either $x * y \in P_i - P_{i-1}$ or $x * y \in P_{i-1}$. In any case we conclude that $\alpha_A(x * y) \geqslant s_i = \alpha_A(x) \wedge \alpha_A(y)$.

Case (ii): Let $x \in P_i - P_{i-1}$ and $y \in P_j - P_{j-1}$ for $i > j$. Therefore, by (1), $\alpha_A(x) = s_i$ and $\alpha_A(y) = s_j$. Then $x * y \in P_i$ since P_i is a G-subalgebra of X and $P_j \subset P_i$. Hence $\alpha_A(x * y) \geqslant s_j = \alpha_A(x) \wedge \alpha_A(y)$.

Thus A is a L-fuzzy G-subalgebra of X. From (1), it follows that $Im(\alpha_A) = \{s_0, s_1, \ldots, s_r\}$. Hence, the level subalgebras of A are given by the chain of G-subalgebras

$$U(\alpha_A : s_0) \subset U(\alpha_A : s_1) \subset \cdots \subset U(\alpha_A : s_r) = X.$$

Now $U(\alpha_A : s_0) = \{x \in X : \alpha_A(x) \geqslant s_0\} = P_0$. Finally we prove that $U(\alpha_A : s_i) = P_i$ for $0 < i \leqslant r$. Clearly $P_i \subseteq U(\alpha_A : s_i)$. If $x \in U(\alpha_A : s_i)$, then $\alpha_A(x) \geqslant s_i$ which implies that $x \notin P_j$ for $j > i$. Hence $\alpha_A(x) \in \{s_0, s_1, \ldots, s_i\}$, and so $x \in P_k$ for some $k \leqslant i$. As $P_k \subseteq P_i$, it follows that $x \in P_i$. Therefore, $U(\alpha_A : s_i) = P_i$ for $0 \leqslant i \leqslant r$. This completes the proof. □

Note that if X is a finite G-algebra, then the number of G-subalgebras of X is finite where as the number of level subalgebras of a L-fuzzy G-subalgebra A appears to be infinite. But since every level subalgebra is indeed a G-subalgebra of X, not all these G-subalgebras are distinct. The next theorem characterizes this aspect.

Theorem 3.7. *Let A be a L-fuzzy G-subalgebra of a G-algebra X. Two level subalgebras $U(\alpha_A : s_1), U(\alpha_A : s_2)$ (with $s_1 < s_2$) of A are equal if and only if there is no $x \in X$ such that $s_1 \leqslant \alpha_A(x) < s_2$.*

Proof. Assume that $U(\alpha_A : s_1) = U(\alpha_A : s_2)$ for $s_1 < s_2$ and assume that there exists $x \in X$ such that $s_1 \leqslant \alpha_A(x) < s_2$. Then $U(\alpha_A : s_2)$ is a proper subset of $U(\alpha_A : s_1)$, which is a contradiction.

Conversely, suppose that there is no $x \in X$ such that $s_1 \leqslant \alpha_A(x) < s_2$. Since $s_1 < s_2$, we have $U(\alpha_A : s_2) \subseteq U(\alpha_A : s_1)$. If $x \in U(\alpha_A : s_1)$, then $\alpha_A(x) \geqslant s_1$ and so $\alpha_A(x) \geqslant s_2$, because $\alpha_A(x)$ does not lie between s_1 and s_2. Hence $x \in U(\alpha_A : s_2)$, which implies that $U(\alpha_A : s_1) \subseteq U(\alpha_A : s_2)$. This completes the proof. □

Remark 3.8. As a consequence of Theorem 3.7, the level subalgebras of a L-fuzzy G-subalgebra A of a finite G-algebra X form a chain. But $\alpha_A(x) \leqslant \alpha_A(0)$ for all $x \in X$. Therefore $U(\alpha_A : s_0)$, where $s_0 = \alpha_A(0)$, is the smallest level subalgebra but not always $U(\alpha_A : s_0)$ as shown in the following example, and so we have the chain $U(\alpha_A : s_0) \subset U(\alpha_A : s_1) \subset \cdots \subset U(\alpha_A : s_r) = X$, where $s_0 > s_1 > \cdots > s_r$.

Theorem 3.9. *Let X be a finite G-algebra and A be a L-fuzzy G-subalgebra of X. If $Im(\alpha_A) = \{s_1, \ldots, s_n\}$, then the family of G-subalgebras $U(\alpha_A : s_i), 1 \leqslant i \leqslant n$ constitutes all the level subalgebras of A.*

Proof. Let $s \in [0,1]$ and $s \notin Im(\alpha_A)$. Suppose $s_1 < s_2 < \cdots < s_n$ without loss of generality. If $s \leqslant s_1$, then $U(\alpha_A : s_1) = X = U(\alpha_A : s)$. If $s > s_n$, then obviously $U(\alpha_A : s) = \phi$. If $s_{i-1} < s < s_i$, then by Theorem 3.7, we get $U(\alpha_A : s) = X = U(\alpha_A : s_i)$. Thus for any $s \in L$, the level subalgebra is one of $\{U(\alpha_A : s_i) | i = 1, 2, \ldots, n\}$. □

It is easy to very that two L-fuzzy G-subalgebras of a G-algebra may have an identical family of level subalgebras but the L-fuzzy G-subalgebras may not be equal.

Theorem 3.10. *Let X be a G-algebra and A be a L-fuzzy G-subalgebra of X. If $Im(\alpha_A)$ is finite, say $\{s_1, s_2, \ldots, s_n\}$, then for any $s_i, s_j \in Im(\alpha_A), U(\alpha_A : s_i) = U(\alpha_A : s_j)$ implies $s_i = s_j$.*

Theorem 3.11. *Let $A = \{\langle x, \alpha_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \alpha_B(x)\rangle : x \in X\}$ be two L-fuzzy G-subalgebras of a finite G-algebra X with identical family of level subalgebras. If $Im(\alpha_A) = \{t_0, t_1, \ldots, t_r\}$ and $Im(\alpha_B) = \{s_0, s_1, \ldots, s_k\}$ where $t_0 > t_1 > \cdots > t_r$ and $s_0 > s_1 > \ldots > s_k$ then we have*

(i) $r = k$

(ii) $U(\alpha_A : t_i) = U(\alpha_B : s_i), 0 \leqslant i \leqslant k$

(iii) if $x \in X$ *such that* $\alpha_A(x) = t_i$, *then* $\alpha_B(x) = s_i, 0 \leqslant i \leqslant k$.

Proof

(i) By Theorem 3.9, the only subalgebras of A and B are the two families $U(\alpha_A : t_i)$ and $U(\alpha_B : s_i)$. Since A and B have the same family of level subalgebras, it follows that $r = k$.

(ii) Using Remark 3.8 and (i), we have chains of level subalgebras $U(\alpha_A : t_0) \subset U(\alpha_A : t_1) \subset \cdots \subset U(\alpha_A : t_k) = X$ and $U(\alpha_B : t_0) \subset U(\alpha_B : t_1) \subset \cdots \subset U(\alpha_B : t_k) = X$. It follows clearly that if $t_i, t_j \in Im(\alpha_A)$ such that $t_i > t_j$ and $s_i, s_j \in Im(\alpha_B)$ such that $s_i > s_j$ then

$$U(\alpha_A : t_i) \subset U(\alpha_A : t_j) \quad \text{and} \quad U(\alpha_B : s_i) \subset U(\alpha_B : s_j). \tag{2}$$

Since the two families of level subalgebras are identical, it is clear that $U(\alpha_A : t_0) = U(\alpha_B : s_0)$. By hypothesis $U(\alpha_A : t_1) = U(\alpha_B : s_j)$ for some $j > 0$. Assume that $U(\alpha_A : t_1) \neq U(\alpha_B : s_1)$. Then $U(\alpha_A : t_1) = U(\alpha_B : s_j)$ for some $j > 1$, and $U(\alpha_B : s_1) = U(\alpha_A : t_i)$ for some $t_i < t_1$. Thus by (2) we obtain that $U(\alpha_B : s_j) = U(\alpha_A : t_1) \subset U(\alpha_A : t_i)$ and $U(\alpha_A : t_i) = U(\alpha_B : s_1) \subset U(\alpha_B : s_j)$. This is a contradiction. Hence $U(\alpha_A : t_1) = U(\alpha_B : s_1)$. By induction on $i, 0 \leqslant i \leqslant k$, we finally obtain that $U(\alpha_A : t_i) = U(\alpha_B : s_i), 0 \leqslant i \leqslant k$.

(iii) Let $x \in X$ be such that $\alpha_A(x) = t_i$ and let $\alpha_B(x) = s_j$, where $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant k$. It is sufficient to show that $s_j = s_i$. Now $x \in U(\alpha_A : t_i) = U(\alpha_B : s_i)$ implies that $\alpha_B(x) = s_j \geqslant s_i$. This gives from (2) that $U(\alpha_B : s_j) \subseteq U(\alpha_B : s_i)$. Since $x \in U(\alpha_B : s_j)$, it follows from (ii) that $x \in U(\alpha_A : t_j)$ and so $\alpha_A(x) = t_i \geqslant t_j$, which implies that $U(\alpha_A : t_i) \subseteq U(\alpha_A : t_j)$ by (2). Using (ii), we have $U(\alpha_B : s_i) = U(\alpha_A : t_i) \subseteq U(\alpha_A : t_j) = U(\alpha_B : s_j)$. Thus $U(\alpha_B : s_i) = U(\alpha_B : s_j)$, and by Theorem 3.10, $s_j = s_i$. This completes the proof. □

Theorem 3.12. *Let A and B be two L-fuzzy G-subalgebras of a finite G-algebra X such that the families of level subalgebras of A and B are identical. Then A = B if and only if* $Im(\alpha_A) = Im(\alpha_B)$.

Proof. If $A = B$, then clearly $Im(\alpha_A) = Im(\alpha_B)$. Conversely, assume that $Im(\alpha_A) = Im(\alpha_B)$. For convenience, let us denote $Im(\alpha_A) = \{t_0, t_1, \ldots, t_r\}$ and $Im(\alpha_B) = \{s_0, s_1, \ldots, s_k\}$, where $t_0 > t_1 > \cdots > t_r$ and $s_0 > s_1 > \cdots > s_r$. Then $s_0 \in Im(\alpha_B) = Im(\alpha_A)$. Thus $s_0 = t_{n_0}$ for some n_0. Assume that $t_{n_0} \neq t_0$. So $t_{n_0} < t_0$. Now $s_1 \in Im(\alpha_A)$, and hence $s_1 = t_{n_1}$ for some n_1. Since $s_0 > s_1$, we have $t_{n_0} > t_{n_1}$. Continuing in this way, we have $t_{n_0} > t_{n_1} > \cdots > t_{n_r}$. Since $s_0 = t_{n_0} < t_0$, this contradicts to the fact that $Im(\alpha_A) = Im(\alpha_B)$. Hence we must have $s_0 = t_0$. Proceeding this manner, we get that $s_i = t_i, 0 \leqslant i \leqslant r$. Now let $x_0, x_1, \ldots, x_r$ be distinct elements of X such that $\alpha_A(x_i) = t_i, 0 \leqslant i \leqslant r$. By Theorem 3.11, $\alpha_B(x_i) = s_i, 0 \leqslant i \leqslant r$. Since $s_i = t_i$, it follows that $\alpha_A(x) = \alpha_B(x)$ for each $x \in X$. Therefore $A = B$. □

Theorem 3.13. *Let* $T_1 \supseteq T_2 \supseteq T_3 \cdots$ *be a descending chain of G-subalgebras of X which terminates at finite step. For a L-fuzzy G-subalgebra A of X, if a sequence of elements of* $Im(\alpha_A)$ *is strictly increasing, then A is finite valued.*

Proof. Assume that A is infinite valued. Let $\{\phi_n\}$ be a strictly increasing sequence of elements of $Im(\alpha_A)$. Then $0 \leqslant \phi_1 < \phi_2 < \cdots < 1$. Note that $U(\alpha_A : \phi_t)$ is G-subalgebras of X for $t = 1, 2, 3, \ldots$. Let $x \in U(\alpha_A : \phi_t)$ for $t = 2, 3, \ldots$. Then $\alpha_A(x) \geqslant \phi_t > \phi_{t-1}$, which implies that $x \in U(\alpha_A : \phi_{t-1})$. Hence $U(\alpha_A : \phi_t) \subseteq U(\alpha_A : \phi_{t-1})$ for $t = 2, 3, \ldots$. Since $\phi_{t-1} \in Im(\alpha_A)$ there exists x_{t-1} such that $\alpha_A(x_{t-1}) = \phi_{t-1}$. It follows that $x_{t-1} \in U(\alpha_A : \phi_{t-1})$, but $x_{t-1} \notin U(\alpha_A : \phi_t)$. Thus $U(\alpha_A : \phi_t) \subsetneq U(\alpha_A : \phi_{t-1})$, and so we obtain a strictly descending chain $U(\alpha_A : \phi_1) \supsetneq U(\alpha_A : \phi_2) \supsetneq \cdots$ of G-subalgebras of X which is not terminating. This is impossible. Therefore, A is finite valued. □

Now we consider the converse of Theorem 3.13.

Theorem 3.14. *If every L-fuzzy G-subalgebra A of X has the finite image, then every descending chain of G-subalgebras of X terminates at finite step.*

Proof. Suppose there exists a strictly descending chain $T_0 \supsetneq T_1 \supsetneq T_2 \cdots$ of G-subalgebras of X which does not terminate at finite step. Define a L-fuzzy set A in X by

$$\alpha_A(x) = \begin{cases} \frac{n}{n+1} & \text{if} \quad x \in T_n \setminus T_{n+1}, \\ 1 & \text{if} \quad x \in \bigcap_{n=0}^{\infty} T_n, \end{cases}$$

where $n = 0, 1, 2, \ldots$ and T_0 stands for X. Let $x, y \in X$. Now, we consider the following cases:

If x and $y \in T_n$, then $x * y \in T_n$ because T_n is a G-subalgebra of X. Hence,

$$\alpha_A(x * y) \geqslant \frac{n}{n+1} = \alpha_A(x) \wedge \alpha_A(y).$$

If $x \in T_n \setminus T_{n+1}$ and $x \in T_m \setminus T_{m+1}$, where $n > m$, then $x * y \in T_m$. Hence,

$$\alpha_A(x * y) \geqslant \frac{m}{m+1} = \alpha_A(x) \wedge \alpha_A(y).$$

If $x \in T_n \setminus T_{n+1}$ and $x \in T_m \setminus T_{m+1}$, where $n < m$, then $x * y \in T_n$. Hence,

$$\alpha_A(x * y) \geqslant \frac{n}{n+1} = \alpha_A(x) \wedge \alpha_A(y).$$

This proves that A is a L-fuzzy G-subalgebra with an infinite number of different values, which is a contradiction. This completes the proof. □

Theorem 3.15. *Every ascending chain of G-subalgebras of X terminates at finite step if and only if the set of values of any L-fuzzy G-subalgebra is a well ordered subset of L.*

Proof. Let A be a L-fuzzy G-subalgebra of X. Suppose that the set of values of A is not a well-ordered subset of L. Then there exists a strictly decreasing sequence $\{\gamma_n\}$ such that $\alpha_A(x_n) = \gamma_n$. It follows that $U(\alpha_A : \gamma_1) \subsetneq U(\alpha_A : \gamma_2) \subsetneq U(\alpha_A : \gamma_3) \subsetneq \cdots$ is a strictly ascending chain of G-subalgebras of X which is not terminating. This is impossible.

To prove the converse suppose that there exists a strictly ascending chain

$$T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq \cdots \tag{3}$$

of closed ideals of X which does not terminate at finite step. Note that $T = \cup_{n \in \mathbb{N}} T_n$ is a closed ideal of X. Define an ILFS $A = (\alpha_A, \beta_A)$ in X by

$$\alpha_A(x) = \begin{cases} \frac{1}{k} & \text{where} \quad k = \min\{n \in \mathbb{N} | x \in T_n\} \\ 0 & \text{if} \quad x \notin T_n \end{cases}$$

By using similar method as Theorem 3.14, we can prove that A is a L-fuzzy G-subalgebra of X. Since the chain (3) is not terminating, A has a strictly descending sequence of values. This contradicts that the value set of any L-fuzzy G-subalgebra is well-ordered. This completes the proof. □

4. Conclusions

To investigate the structure of an algebraic system, it is clear that subalgebras with special properties play an important role. In the present paper, we considered the notions of L-fuzzy G-subalgebras of G-algebras and investigated some of their useful properties. It is our hope that this work would other foundations for further study of the theory of G-algebras.

In our future study of L-fuzzy structure of G-algebra, may be the following topics should be considered: (i) to find interval-valued L-fuzzy G-subalgebras, (ii) to find intuitionistic L-fuzzy and interval-valued intuitionistic L-fuzzy G-subalgebras, (iii) to find intuitionistic (T, S)-fuzzy G-subalgebras, where S and T are given imaginable triangular norms, (iv) to find $(\epsilon, \epsilon \vee q)$-fuzzy and intuitionistic L-fuzzy G-subalgebras of G-algebras.

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