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SHORT COMMUNICATION

On *b*-chromatic number of sun let graph and wheel graph families

J. Vernold Vivin [a,*], M. Vekatachalam [b]

[a] *Department of Mathematics, University College of Engineering Nagercoil, Anna University, Tirunelveli Region, Nagercoil 629 004, Tamil Nadu, India*
[b] *Department of Mathematics, RVS Educational Trust's Group of Institutions, RVS Faculty of Engineering, Coimbatore 641 402, Tamil Nadu, India*

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Abstract A proper coloring of the graph assigns colors to the vertices, edges, or both so that proximal elements are assigned distinct colors. Concepts and questions of graph coloring arise naturally from practical problems and have found applications in many areas, including Information Theory and most notably Theoretical Computer Science. A *b*-coloring of a graph G is a proper coloring of the vertices of G such that there exists a vertex in each color class joined to at least one vertex in each other color class. The *b*-chromatic number of a graph G, denoted by $\varphi(G)$, is the maximal integer k such that G may have a *b*-coloring with k colors. In this paper, we obtain the *b*-chromatic number for the sun let graph S_n, line graph of sun let graph $L(S_n)$, middle graph of sun let graph $M(S_n)$, total graph of sun let graph $T(S_n)$, middle graph of wheel graph $M(W_n)$ and the total graph of wheel graph $T(W_n)$.

1991 MATHEMATICS SUBJECT CLASSIFICATION: 05C15; 05C75; 05C76

1. Introduction

All graphs considered in this paper are nontrivial, simple and undirected. Let G be a graph with vertex set V and edge set E. A k-coloring of a graph G is a partition $P = \{V_1, V_2, \ldots, V_k\}$ of V into independent sets of G. The minimum cardinality k for which G has a k-coloring is the chromatic number $\chi(G)$ of G. The b-chromatic number $\varphi(G)$ [1–4] of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring. The b-chromatic number was introduced by Irving and Manlove [1] by considering proper colorings that are minimal with respect to a partial order defined on the set of all partitions of $V(G)$. They have shown that determination of $\varphi(G)$ is NP-hard for general graphs, but polynomial for trees. There has been an increasing interest in the study of b-coloring since the publication of [1].

* Corresponding author. Tel.: +91 9566326299.
E-mail addresses: vernoldvivin@yahoo.in, vernold_vivin@yahoo.com (J. Vernold Vivin), venkatmaths@gmail.com (M. Vekatachalam).

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Irving and Manlove [1] have also proved the following upper bound of $\varphi(G)$

$$\varphi(G) \leqslant \Delta(G) + 1 \quad (1)$$

Kouider and Mahéo [5] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. Kratochvíl et al. [6] characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper b-coloring even for connected bipartite graphs with $k = \Delta(G) + 1$.

Effantin and Kheddouci studied [7–9] the b-chromatic number for the complete caterpillars, the powers of paths, cycles, and complete k-ary trees.

Faik [10] was interested in the continuity of the b-coloring and proved that chordal graphs are b-continuous.

Corteel et al. [11] proved that the b-chromatic number problem is not approximable within $120/133 - \epsilon$ for any $\epsilon > 0$, unless $P = NP$.

Hoáng and Kouider characterized in [12], the bipartite graphs and the P_4-sparse graphs for which each induced subgraph H of G has $\varphi(H) = \chi(H)$.

Kouider and Zaker [13] proposed some upper bounds for the b-chromatic number of several classes of graphs in function of other graph parameters (clique number, chromatic number, biclique number).

Kouider and El Sahili proved in [14] by showing that if G is a d-regular graph with girth 5 and without cycles of length 6, then $\varphi(G) = d + 1$.

Jakovac and Klavžar [2], proved that the b-chromatic number of cubic graphs is four with the exception of Petersen graph, $K_{3,3}$, prism over K_3, and sporadic with 10 vertices.

Effantin and Kheddouci [15] proposed a discussion on relationships between this parameter and two other coloring parameters (the Grundy and the partial Grundy numbers). The property of the dominating nodes in a b-coloring is very interesting since they can communicate directly with each partition of the graph.

Recently, Vernold Vivin and Venkatachalam [4] have proved the b-chromatic number for corona of any two graphs. The authors also investigated the b-chromatic number of star graph families [3].

2. Preliminaries

The $n-$sun let graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.

For any integer $n \geqslant 4$, the wheel graph W_n is the n-vertex graph obtained by joining a vertex v_1 to each of the $n-1$ vertices $\{w_1, w_2, \ldots, w_{n-1}\}$ of the cycle graph C_{n-1}.

The line graph [16] of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [16] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

3. *b*-coloring on Sunlet Graph and its line, middle and total graphs

Theorem 1. *Let $n \geqslant 6$. Then, the b-chromatic number of the sun let graph is $\varphi(S_n) = 4$.*

Proof. Let S_n be the sun let graph on $2n$ vertices. Let $V(S_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ where v_i's are the vertices of cycles taken in cyclic order and u_i's are pendant vertices such that each v_iu_i is a pendant edge. Consider the following 4-coloring (c_1, c_2, c_3, c_4) of S_n, assign the color c_1 to v_n, c_2 to v_1, c_3 to v_2, c_4 to v_3, c_2 to v_4, c_4 to v_{n-1}, c_3 to u_n, c_4 to u_1 and for $2 \leqslant i \leqslant n-1$, assign the color c_1 to u_i. For $5 \leqslant i \leqslant n-2$, if any, assign to vertex v_i one of the allowed colors - such color exists, because $deg(v_i) = 3$. An easy check shows that this coloring is a b - coloring. Therefore, $\varphi(S_n) \geqslant 4$ (see Fig. 1).

Since $\Delta(S_n) = 3$, using (1.1), we get that $\varphi(S_n) \leqslant 4$. Hence, $\varphi(S_n) = 4$, for all $n \geqslant 6$. $\square$

Theorem 2. *Let $n \geqslant 6$. Then, the b-chromatic number of the line graph of sun let graph is $\varphi(L(S_n)) = 5$.*

Proof. Let $V(S_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(S_n) = \{e'_i : 1 \leqslant i \leqslant n\} \cup \{e_i : 1 \leqslant i \leqslant n-1\} \cup \{e_n\}$ where e_i is the edge v_iv_{i+1} $(1 \leqslant i \leqslant n-1), e_n$ is the edge v_nv_1 and e'_i is the edge v_iu_i $(1 \leqslant i \leqslant n)$. By the definition of line graph $V(L(S_n)) = E(S_n) = \{u'_i : 1 \leqslant i \leqslant n\} \cup \{v'_i : 1 \leqslant i \leqslant n-1\} \cup \{v'_n\}$ where v'_i and u'_i represents the edge e_i and e'_i $(1 \leqslant i \leqslant n)$ respectively. Consider the following 5-coloring $(c_1, c_2, c_3, c_4, c_5)$ of $L(S_n)$, assign the color c_5 to u'_1, c_4 to u'_2, c_5 to u'_3, c_1 to u'_4, c_2 to u'_5, c_1 to u'_6, c_3 to v'_n, c_3 to v'_6 and for $1 \leqslant i \leqslant 5$, assign the color c_i to v'_i. For $7 \leqslant i \leqslant n$, assign the color c_2 to u'_i. For

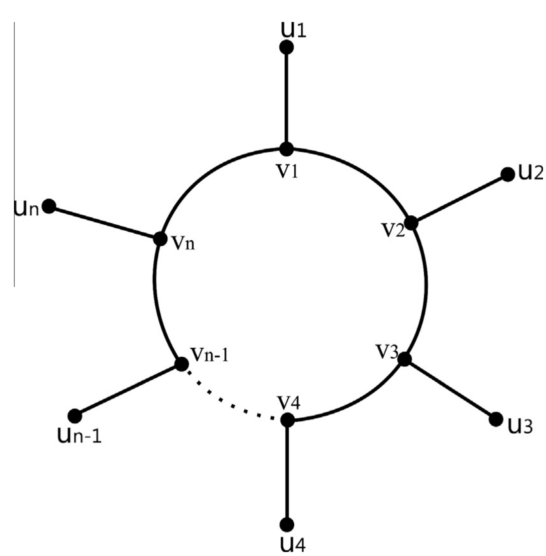

Figure 1 Sunlet Graph S_n.

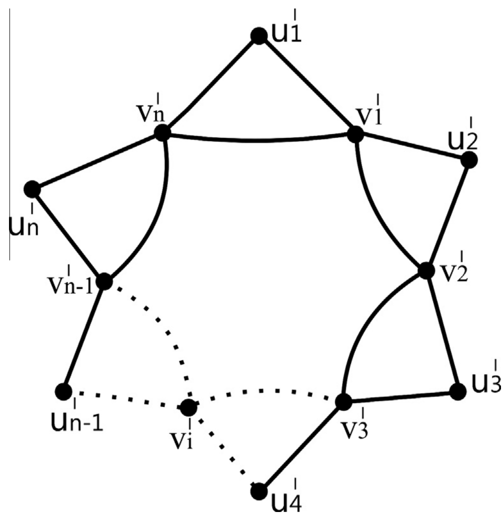

Figure 2 Line graph of Sunlet Graph $L(S_n)$.

$7 \leqslant i \leqslant n-1$, if any, assign to vertex v'_i one of the allowed colors - such color exists, because $deg(v'_i) = 4$. An easy check shows that this coloring is a b - coloring. Therefore, $\varphi(L(S_n)) \geqslant 5$.

Since $\Delta(L(S_n)) = 4$, using (1.1), we get that $\varphi(L(S_n) \leqslant 5$. Hence, $\varphi(L(S_n)) = 5$, for all $n \geqslant 6$ (see Fig. 2). □

Theorem 3. *Let $n \geqslant 9$. Then, the b-chromatic number of the middle graph of sun let graph is $\varphi(M(S_n)) = 7$.*

Proof. Let $V(S_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(S_n) = \{e'_i : 1 \leqslant i \leqslant n\} \cup \{e_i : 1 \leqslant i \leqslant n-1\} \cup \{e_n\}$ where e_i is the edge $v_i v_{i+1} (1 \leqslant i \leqslant n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leqslant i \leqslant n)$. By the definition of middle graph $V(M(S_n)) = V(S_n) \cup E(S_n) = \{v_i : 1 \leqslant i \leqslant n\} \cup \{u_i : 1 \leqslant i \leqslant n\} \cup \{v'_i : 1 \leqslant i \leqslant n\} \cup \{u'_i : 1 \leqslant i \leqslant n\}$, where v'_i and u'_i represents the edge e_i and e'_i $(1 \leqslant i \leqslant n)$ respectively. Consider the following 7-coloring $(c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ of $M(S_n)$, assign the color c_7 to v'_n, c_6 to u'_1, c_1 to v'_1, c_3 to v_1, c_5 to u'_2, c_2 to v'_2, c_4 to v_2, c_3 to v'_3, c_7 to u'_3, c_6 to v_3, c_1 to v_4, c_5 to u'_4, c_4 to v'_4, c_2 to v_5, c_7 to u'_5, c_6 to v'_5, c_3 to u'_6, c_1 to v_6, c_5 to v'_6, $c_4$to u'_7, c_2 to v_7, c_7 to v'_7, c_3 to u'_8, c_1 to v_8, c_6 to v'_8 and for $1 \leqslant i \leqslant n$, assign to vertex u_i one of the allowed colors - such color exists, because $deg(u_i) = 1$. For $9 \leqslant i \leqslant n$, if any, assign to vertex v_i and to vertex u_i one of the allowed colors - such color exists, because $deg(v_i) = 3$. For $9 \leqslant i \leqslant n-1$, assign to vertex v'_i one of allowed colors-such color exists, because $deg(u_i) = 6$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(S_n)) \geqslant 7$.

Since $\Delta(M(S_n)) = 6$, using (1.1), we get that $\varphi(M(S_n)) \leqslant 7$. Hence, $\varphi(M(S_n)) = 7$, for all $n \geqslant 9$ (see Fig. 3). □

Theorem 4. *Let $n \geqslant 9$. Then, the b-chromatic number of the total graph of sun let graph is $\varphi(T(S_n)) = 7$.*

Proof. Consider the coloring of $M(S_n)$ introduced on the proof of Theorem 3. An easy check shows that this coloring is a b-coloring of $T(S_n)$. Hence, $\varphi(T(S_n)) \geqslant 7$, for all $n \geqslant 9$.

Since $\Delta(T(S_n)) = 6$, using (1.1), we get that $\varphi(T(S_n)) \leqslant 7$. Hence, $\varphi(T(S_n)) = 7$, for all $n \geqslant 9$ (see Fig. 4). □

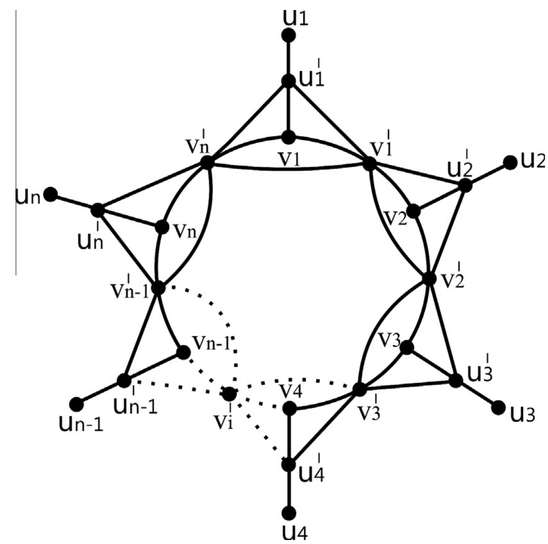

Figure 3 Middle graph of Sunlet Graph $M(S_n)$.

4. *b*-coloring of the middle and total graph of a wheel graph

Theorem 5. *Let $n \geqslant 7$. Then, the b-chromatic number of the middle graph of wheel graph is $\varphi(M(W_n)) = n$, n is number of vertices in W_n.*

Proof. Let $V(W_n) = \{v, v_1, v_2, \ldots v_{n-1}\}$ and let $V(M(W_n)) = \{v, v_1, v_2, \ldots v_{n-1}\} \cup \{e_1, e_2, \ldots e_{n-1}\} \cup \{u_1, u_2, \ldots u_{n-1}\}$. Where u_i is the vertex of $M(W_n)$ corresponding to the edge $v_i v_{i+1}$ of $W_n (1 \leqslant i \leqslant n-1)$ and e_i is the vertex of $M(W_n)$ corresponding to the edge vv_i of $W_n (1 \leqslant i \leqslant n-1)$. By the definition of middle graph, the vertices v and $\{e_i : (1 \leqslant i \leqslant n-1)\}$ induces a clique of order n in $M(W_n)$. Therefore, $\varphi(M(W_n)) \geqslant n$. Consider the following n-coloring of $M(W_n)$. For $1 \leqslant i \leqslant n-1$, assign the color c_i to e_i and assign the color c_n to v. For $1 \leqslant i \leqslant n-1$, assign to vertex u_i one of the allowed colors - such color exists, because $deg(u_i) = 6$. For $1 \leqslant i \leqslant n-1$, if

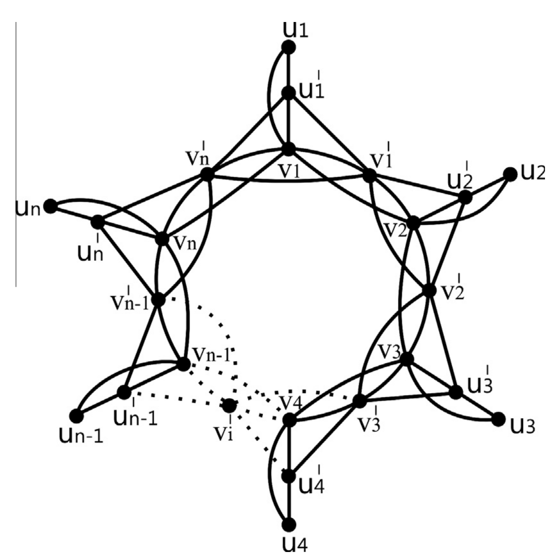

Figure 4 Total graph of Sunlet Graph $T(S_n)$.

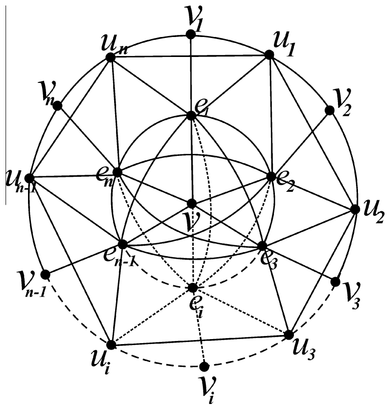

Figure 5 Middle graph of a wheel $M(W_{n+1})$.

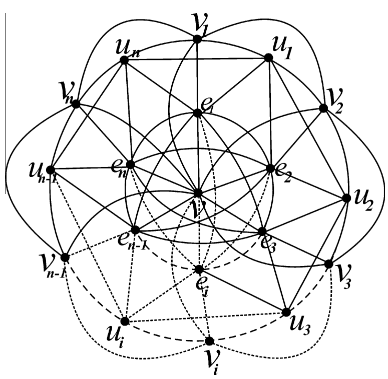

Figure 6 Total graph of a wheel $T(W_{n+1})$.

any, assign to vertex v_i one of the allowed colors - such color exists, because $deg(v_i) = 3$. An easy check shows that this coloring is a b-coloring. Therefore, $\varphi(M(W_n)) \geqslant n$.

Let us assume that $\varphi(M(W_n))$ is greater than n, there must be at least $n+1$ vertices of degree n in $M(W_n)$, all with distinct colors, and each adjacent to vertices of all of the other colors. For $n \geqslant 6$, this is a contradiction. Thus, we have $\varphi(M(W_n)) \leqslant n$. Hence, $\varphi(M(W_n)) = n, \forall\ n \geqslant 7$ (see Fig. 5). □

Theorem 6. *Let $n \geqslant 7$. Then, the b-chromatic number of the total graph of wheel graph is $\varphi(T(W_n)) = n$, n is number of vertices in W_n.*

Proof. Consider the coloring of $M(W_n)$ introduced on the proof of Theorem 5. An easy check shows that this coloring is a b-coloring of $T(W_n)$. Hence, $\varphi(T(W_n)) = n, \forall\ n \geqslant 6$ (see Fig. 6). □

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References

[1] R.W. Irving, D.F. Manlove, The b-chromatic number of a graph, Discrete Appl. Math. 91 (1999) 127–141.
[2] M. Jakovac, S. Klavžar, The b-chromatic number of cubic graphs, Graphs Comb. 26 (1) (2010) 107–118.
[3] M. Venkatachalam, J. Vernold Vivin, The b-chromatic number of star graph families, Le Matematiche 65 (1) (2010) 119–125.
[4] J. Vernold Vivin, M. Venkatachalam, The b-chromatic number of corona graphs, Utilitas Math. 88 (2012) 299–307.
[5] M. Kouider, M. Mahéo, Some bounds for the b-chromatic number of a graph, Discrete Math. 256 (1-2) (2002) 267–277.
[6] J. Kratochvíl, Z. Tuza, M. Voigt, On the b-chromatic number of graphs, in: 28th International Workshop on Graph-Theoretic Concepts in Computer Science, vol. 2573, Cesky Krumlov, Czech Republic, LNCS, 2002, pp. 310–320.
[7] B. Effantin, The b-chromatic number of power graphs of complete caterpillars, J. Discrete Math. Sci. Cryptogr. 8 (3) (2005) 483–502.
[8] B. Effantin, H. Kheddouci, The b-chromatic number of some power graphs, Discrete Math. Theor. Comput. Sci. 6 (2003) 45–54.
[9] B. Effantin, H. Kheddouci, Exact values for the b-chromatic number of a power complete k-ary tree, J. Discrete Math. Sci. Cryptogr. 8 (1) (2005) 117–129.
[10] T. Faik, About the b-continuity of graphs, Electron. Notes Discrete Math. 17 (2004) 151–156.
[11] S. Corteel, M. Valencia-Pabon, J.C. Vera, On approximating the b-chromatic number, Discrete Appl. Math. 146 (2005) 106–110.
[12] C. Hoáng, M. Kouider, On the b-dominating coloring of graphs, Discrete Appl. Math. 152 (2005) 176–186.
[13] M. Kouider, M. Zaker, Bounds for the b-chromatic number of some families of graphs, Discrete Math. 306 (2006) 617–623.
[14] M. Kouider, A. El Sahili, About b-colouring of regular graphs, Rapport de Recherche No 1432, CNRS-Université Paris Sud-LRI, 2006, 02/2006.
[15] B. Effantin, H. Kheddouci, Discussion on the (partial) Grundy and b-chromatic numbers of graphs, Utilitas Math. 80 (2009) 65–89.
[16] F. Harary, Graph Theory, Narosa Publishing Home, New Delhi, 1969.