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Original Article

Symmetries of $f-$associated standard static spacetimes and applications

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ABSTRACT

The purpose of this note is to study and explore some collineation vector fields on standard static spacetimes $I_f \times M$(also called $f-$ associated SSST). Conformal vector fields, Ricci and matter collineations are studied. Many implications for the existence of these collineations on $f-$associated SSSTs are obtained. Moreover, Ricci soliton structures on $f-$ associated SSSTs admitting a potential conformal vector field are considered.

1. An introduction

Warped product manifolds are extensively studied as a method to generate general exact solutions to Einstein's field equation [1–3]. A standard static spacetime (also called $f-$associated SSST) is often pictured in the form of a Lorentzian warped product manifold $I_f \times M$ [4,5]. An $f-$ associated standard static spacetime is, to some extent, a generalization of some well-known classical spacetimes such as the Einstein static universe and Minkowski spacetime [1,6].

The study of spacetime symmetries is essential for solving Einstein field equation and for providing further insight into conservative laws of dynamical systems (see [7] an important reference for symmetries of classical spacetimes). Collineation vector fields, in general, enable physicists to portray the geometry of a spacetime [8–10]. The presence of a non-trivial collineation vector field on a spacetime is sufficient to guarantee some kind of symmetry. Vector fields which preserve a certain feature or quantity of a spacetime along their local flow lines are called collineations. The Lie derivative of aforesaid feature or quantity vanishes in direction of a collineation vector field. The most important collineations are those preserving metric, curvature and Ricci curvature by virtue of their essential role in general relativity. An extensive work has been done in the last two decades studying collineations and their generalizations on classical 4−dimensional spacetimes.

The main purpose of this work is to study and explore some collineation vector fields on $f-$associated standard static spacetimes. Many answers are given to the following questions: Under what condition(s) is a vector field on an $f-$associated standard static spacetime a certain collineation or a conformal vector field? What does the base factor submanifold M inherit from an $f-$associated standard static spacetime $I_f \times M$ admitting a collineation or a conformal vector field? Ricci soliton structures on $f-$ associated standard static spacetimes admitting a potential conformal vector field are considered.

The distribution of this article is as follows. In Section 2, the basic definitions and related formulas of both $f-$associated standard static spacetimes and collineation vector fields are considered. Section 3 carries a study of collineation vector fields on $f-$associated standard static spacetimes. Finally, we study Ricci soliton structures on $f-$associated standard static spacetimes admitting conformal vector fields in Section 4.

2. Preliminaries

A standard static spacetime (also called $f-$associated SSST) is a Lorentzian warped product manifold $\bar{M} = I_f \times M$ furnished with

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the metric $\bar{g} = -f^2 dt^2 \oplus g$, where (M, g) is a Riemannian manifold, $f: M \to (0, \infty)$ is smooth and $I = (t_1, t_2)$ with $-\infty \le t_1 < t_2 \le \infty$. We use the same notation for a vector field $X \in \mathfrak{X}(M)$ and for its lift to the $f-$associated SSST $\bar{M}$. Likewise, a function ω on M will be identified with $\omega \circ \pi$ on $\bar{M}$, where $\pi: I_f \times M \to M$ is the natural projection map of $I \times M$ onto M. Note that we use $\mathrm{grad}\omega \in \mathfrak{X}(M)$ for the gradient vector field of ω on M and for its lift to $\mathfrak{X}(\bar{M})$ [2,3]. The submanifold $\{t\} \times M$ and M are isomorphic for every $t \in M$. Accordingly, we refer to this factor submanifold as M.

Let $\bar{M} = I_f \times M$ be a standard static spacetime equipped with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. Then the Levi–Civita connection $\bar{D}$ on $\bar{M}$ is given by

$$\bar{D}_{\partial_t}\partial_t = f\,\mathrm{grad}f \qquad \begin{array}{l}\bar{D}_{\partial_t}X = \bar{D}_X\partial_t = X(\ln f)\partial_t \\ \bar{D}_X Y = D_X Y\end{array} \tag{2.1}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$, where D is the Levi–Civita connection on M. The Riemannian curvature tensor $\bar{\mathrm{R}}$ is given by

$$\bar{\mathrm{R}}(X, \partial_t)\partial_t = -fD_X\mathrm{grad}f \qquad \begin{array}{l}\bar{\mathrm{R}}(\partial_t, \partial_t)\partial_t = \bar{\mathrm{R}}(\partial_t, \partial_t)X = \bar{\mathrm{R}}(X, Y)\partial_t = 0 \\ \bar{\mathrm{R}}(\partial_t, X)Y = \frac{1}{f}H^f(X, Y)\partial_t \\ \bar{\mathrm{R}}(X, Y)Z = \mathrm{R}(X, Y)Z,\end{array} \tag{2.2}$$

where R is the curvature tensor of M and $H^f(X, Y) = g(D_X\mathrm{grad}f, Y)$ is the Hessian of f. Finally, the Ricci curvature tensor, $\bar{\mathrm{R}}\mathrm{ic}$, of the $f-$ associated SSST $\bar{M}$ is as follows

$$\bar{\mathrm{R}}\mathrm{ic}(\partial_t, \partial_t) = f\triangle f \qquad \begin{array}{l}\bar{\mathrm{R}}\mathrm{ic}(X, \partial_t) = 0 \\ \bar{\mathrm{R}}\mathrm{ic}(X, Y) = \mathrm{Ric}(X, Y) - \frac{1}{f}H^f(X, Y),\end{array} \tag{2.3}$$

where $\triangle f$ denotes the Laplacian of f on M.

The Lie derivative $\mathcal{L}_\zeta$ in direction of ζ is given by

$$\left(\mathcal{L}_\zeta g\right)(X, Y) = g(D_X\zeta, Y) + g(X, D_Y\zeta) \tag{2.4}$$

for any $X, Y \in \mathfrak{X}(M)$.

Now, we will recall the definitions of conformal vector fields and some collineations on an arbitrary pseudo-Riemannian manifold (M, g, D) with metric g and the Levi-Civita connection D on M. A vector field $\zeta \in \mathfrak{X}(M)$ is called a conformal vector field if $\mathcal{L}_\zeta g = \rho g$ for some smooth function $\rho: M \to \mathbb{R}$, where $\mathcal{L}_\zeta$ is the Lie derivative in direction of ζ. In particular, $\zeta \in \mathfrak{X}(M)$ is called homothetic if ρ is constant and Killing if $\rho = 0$. The symmetry of Eq. (2.4) implies that ζ is a Killing vector field if and only if $g(D_X\zeta, X) = 0$ for any vector field $X \in \mathfrak{X}(M)$. A pseudo-Riemannian $n-$ dimensional manifold has at most $n(n+1)/2$ independent Killing vector fields and at most $(n+1)(n+2)/2$ independent conformal vector fields. The symmetry generated by a Killing vector field ζ on M is called isometry. A pseudo-Riemannian manifold which permits a maximum aforementioned symmetry has a constant curvature. Also, ζ is called a concircular vector field if $D_X\zeta = \rho X$ for any $X \in \mathfrak{X}(M)$[11]. A concircular vector field $\zeta \in \mathfrak{X}(M)$ on M is a conformal vector field with conformal factor 2ρ . A concircular vector field is also a parallel vector field if $\rho = 0$. Moreover, for a constant factor ρ, we have $R(X, Y)\zeta = 0$. A vector field ζ on a pseudo-Riemannian manifold (M, g) is called a curvature collineation if the Lie derivative of the curvature tensor R vanishes in the direction of $\zeta \in \mathfrak{X}(M)$, that is, $\mathcal{L}_\zeta \mathrm{R} = 0$. Similarly, M is said to admit a Ricci curvature collineation if there is a vector field $\zeta \in \mathfrak{X}(M)$ such that $\mathcal{L}_\zeta \mathrm{Ric} = 0$, where Ric is the Ricci curvature tensor. One may notice that every Killing field is a curvature collineation and every curvature collineation is a Ricci curvature collineation. The converse is not generally true. A vector field $\zeta \in \mathfrak{X}(M)$ is called a conformal Ricci collineation if

$$\left(\mathcal{L}_\zeta \mathrm{Ric}\right)(X, Y) = \rho g(X, Y)$$

for some smooth function ρ on M. Finally, a spacetime M is said to admit a matter collineation if there is a vector field $\zeta \in \mathfrak{X}(M)$ such that $\mathcal{L}_\zeta \mathrm{T} = 0$, where T is the energy-momentum tensor. For $(n+1)-$ dimensional spacetime, the Einstein field equation is given by

$$\mathrm{Ric} - \frac{r}{2}g = k_n T,$$

where k_n is called the multidimensional gravitational constant, r is the scalar curvature and λ is the cosmological constant [12]. Suppose that ζ is a Killing vector field, then $\mathcal{L}_\zeta T = 0$, i.e., ζ is a matter collineation field. Note that a matter collineation is not necessarily Killing.

3. Symmetries of a standard static spacetime

In this section, we explore several types of collineations on an $f-$ associated SSST $\bar{M} = I_f \times M$ equipped with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. Necessary and sufficient conditions are derived for a standard static spacetime to admit a conformal vector field or a collineation.

3.1. Conformal vector fields

Assume that $h\partial_t, x\partial_t, y\partial_t \in \mathfrak{X}(I)$ and $\zeta, X, Y \in \mathfrak{X}(M)$, then

$$\left(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g}\right)\left(\bar{X}, \bar{Y}\right) = \left(\mathcal{L}_\zeta g\right)(X, Y) - 2xyf^2\left(\dot{h} + \zeta(\ln f)\right), \tag{3.1}$$

where $\bar{\zeta} = h\partial_t + \zeta$, $\bar{X} = x\partial_t + X$ and $\bar{Y} = y\partial_t + Y$. This formula (3.1) is a particular case of a notable one on warped product manifolds. The following result yields immediately from Eq. (3.1).

Theorem 1. *Let $\bar{M} = I_f \times M$ be a standard static spacetime equipped with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. Then a time-like vector field $\bar{\zeta} = h\partial_t \in \mathfrak{X}(\bar{M})$ is a Killing vector field on $\bar{M}$ if and only if $\dot{h} = 0$. Moreover, assume that $\zeta(f) = 0$. Then a space-like vector field $\bar{\zeta} = \zeta$ on $\bar{M}$ is Killing if and only if $\zeta \in \mathfrak{X}(M)$ is a Killing vector field on M.*

Corollary 1. *Let $\bar{M} = I_f \times M$ be a standard static spacetime equipped with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. Then a time-like vector field $\bar{\zeta} = h\partial_t \in \mathfrak{X}(\bar{M})$ is a matter collineation on $\bar{M}$ if $\dot{h} = 0$. Also, a space-like vector field $\bar{\zeta} = \zeta \in \mathfrak{X}(\bar{M})$ is a matter collineation on $\bar{M}$ if $\zeta \in \mathfrak{X}(M)$ is a Killing vector field on M and $\zeta(f) = 0$.*

Theorem 2. *A vector field $\bar{\zeta} = h\partial_t + \zeta$ on a standard static spacetime $\bar{M} = I_f \times M$ is a conformal vector field if and only if ζ is a conformal vector field on M with conformal factor $\bar{\rho} = 2\left(\dot{h} + \zeta(\ln f)\right)$.*

Proof. Let $\bar{\zeta} = h\partial_t + \zeta$ be a conformal vector field on $\bar{M} = I_f \times M$ with factor $\bar{\rho}$, then Eq. (3.1) implies that

$$-\bar{\rho}f^2xy + \bar{\rho}g(X, Y) = \left(\mathcal{L}_\zeta g\right)(X, Y) - 2xyf^2\left(\dot{h} + \zeta(\ln f)\right).$$

Thus

$$\begin{aligned}\left(\mathcal{L}_\zeta g\right)(X, Y) &= \bar{\rho}g(X, Y) \\ -\bar{\rho}f^2xy &= -2xyf^2\left(\dot{h} + \zeta(\ln f)\right)\end{aligned}$$

and consequently ζ is a conformal vector field on M with conformal factor $\bar{\rho} = 2\left(\dot{h} + \zeta(\ln f)\right)$. The converse is direct. □

It is noted that the metric $\bar{g} = -f^2 dt^2 \oplus g$ on $\bar{M}$ can be expressed as a conformal metric to a product one on $I \times M$. The metric $\bar{g}$ may be rewritten as follows

$$\bar{g} = f^2\left(-dt^2 + \frac{1}{f^2}g\right) = f^2\tilde{g},$$

where $\tilde{g} = -dt^2 + \hat{g}$ and $\hat{g} = \frac{1}{f^2}g$. Now, we examine the effects of replacing $\bar{g}$ on M by $\tilde{g} = -dt^2 + \hat{g}$. Similar discussions on

4−dimensional spacetimes and on some warped spacetimes are provided in [7, Chapter 11] and [13,14] respectively. Suppose that $\bar{\zeta} = h\partial_t + \zeta$ is a conformal vector field on $(\bar{M}, \tilde{g})$ with conformal factor $\tilde{\rho}$, then

$$\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g} = [2\zeta(\ln f) + \tilde{\rho}]\bar{g}.$$

Thus, $\bar{\zeta}$ is a conformal vector field on $(\bar{M}, \bar{g})$ with factor $\bar{\rho} = \tilde{\rho}+2\zeta(\ln f)$. Likewise, a conformal vector field ζ on (M, g) with conformal factor ρ is a conformal vector field on $(M, \bar{g})$, where $\rho = \hat{\rho} + 2\zeta(\ln f)$. The above discussions and results in [15, Theorem 1] imply the following.

Theorem 3. *Let $\bar{M} = I_f \times M$ be a standard static spacetime equipped with the metric tensor $\bar{g} = -f^2 \mathrm{d}t^2 \oplus g$ and let $\tilde{g} = -\mathrm{d}t^2 + \hat{g}$ and $\hat{g} = \frac{1}{f^2} g$. Then,*

1. *a Killing vector field ζ on $(M, \hat{g})$ is a Killing vector field on $(\bar{M}, \tilde{g})$,*
2. *$(\bar{M}, \tilde{g})$ admits a homothetic vector field if and only if $(M, \hat{g})$ admits a homothetic vector field,*
3. *each conformal vector field on $(\bar{M}, \tilde{g})$ is a conformal vector field on $(\bar{M}, \bar{g})$.*

The following result is a direct consequence of the above result.

Theorem 4. *Let $\zeta \in \mathfrak{X}(M)$ be a homothetic vector field on (M, g) with factor c and let $\zeta(f) = 0$. Then $\bar{\zeta} = c(at + b)\partial_t + 2a\zeta$ is a homothetic vector field on $(\bar{M}, \bar{g})$ with factor $2ac$. Moreover, $\bar{\zeta}$ is a Killing vector field on $(\bar{M}, \bar{g})$ if $a = 0$ or $c = 0$.*

The study of Killing vector fields of constant length is remarkable in that they correspond to isometries of constant displacement. Consequently, these vector fields are in relation with Clifford-Wolf translation in Riemannian manifolds [16]. In the following, Killing vector fields of constant length on a standard static spacetime $\bar{M} = I_f \times M$ are considered.

Theorem 5. *A Killing vector field $\bar{\zeta} = h\partial_t + \zeta$ on a standard static spacetime $\bar{M} = I_f \times M$ has a constant length if and only if ζ satisfies*

$$D_\zeta \zeta + h^2 f \,\mathrm{grad} f = 0 \text{ and } h\dot{h} + 2h\zeta(\ln f) = 0. \tag{3.2}$$

Corollary 2. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a Killing vector field of constant length on a standard static spacetime $\bar{M} = I_f \times M$. Then the flow lines of ζ are geodesics on M if and only if f is constant or $h = 0$.*

Theorem 6. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a Killing vector field on a standard static spacetime $\bar{M} = I_f \times M$, where f is constant, and let $\alpha(s)$, $s \in \mathbb{R}$, be a geodesic on $(\bar{M}, \bar{g})$ with tangent vector field $\bar{X} = x\partial_t + X$. Then h is constant and $\zeta \in \mathfrak{X}(M)$ is a Jacobi vector field along the integral curves of X.*

Theorem 7. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a conformal vector field along a curve $\alpha(s)$ with unit tangent vector $\bar{U} = u\partial_t + U$ on a standard static spacetime $\bar{M} = I_f \times M$. Then the conformal factor $\bar{\rho}$ of $\bar{\zeta}$ is given by*

$$\bar{\rho} = 2\left[-u^2\left(\dot{h}f^2 + f\zeta(f)\right) + g(D_U\zeta, U)\right].$$

Now, the structure of concircular vector fields on a standard static spacetimes is considered.

Theorem 8. *A vector field $\bar{\zeta} \in \mathfrak{X}(\bar{M})$ on a standard static spacetime $\bar{M} = I_f \times M$ is a concircular vector field if and only if ζ is a concircular vector field on M with factor $\rho = \dot{h}$ and f is constant.*

Proof. It is clear that

$$\bar{D}_{\bar{X}}\bar{\zeta} = \left(x\dot{h} + hX(\ln f) + x\zeta(\ln f)\right)\partial_t + xhf\,\mathrm{grad} f + D_X\zeta$$

for any vector field $\bar{X} = x\partial_t + X \in \mathfrak{X}(\bar{M})$. Suppose that f is constant and ζ is a concircular vector field on M with factor $\rho = \dot{h}$, then

$$\bar{D}_{\bar{X}}\bar{\zeta} = \rho\bar{X},$$

i.e., $\bar{\zeta} = h\partial_t + \zeta$ is a concircular vector field on a standard static spacetime $\bar{M} = I_f \times M$.

Conversely, we assume that ρ is a scalar function. Then

$$\bar{D}_{\bar{X}}\bar{\zeta} - \rho\bar{X} = \left(x\dot{h} + hX(\ln f) + x\zeta(\ln f) - x\rho\right)\partial_t + xhf\,\mathrm{grad} f + D_X\zeta - \rho X$$

for any vector field $\bar{X} = x\partial_t + X \in \mathfrak{X}(\bar{M})$. Suppose that $\bar{\zeta}$ is concircular on $\bar{M}$, then

$$x\dot{h} + hX(\ln f) + x\zeta(\ln f) - x\rho = 0,$$
$$xhf\,\mathrm{grad} f + D_X\zeta - \rho X = 0.$$

If f is constant, we get that

$$x\left(\dot{h} - \rho\right) = 0,$$
$$D_X\zeta - \rho X = 0,$$

i.e., ζ is a concircular vector field on M with factor $\rho = \dot{h}$. □

3.2. Ricci collineations

Let us consider Ricci collineations on a standard static spacetime.

Proposition 1. *Let $\bar{\zeta} = h\partial_t + \zeta$ be a vector field on a standard static spacetime $\bar{M} = I_f \times M$, then*

$$\left(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathrm{Ric}}\right)\left(\bar{X}, \bar{Y}\right) = \left(\mathcal{L}_\zeta \mathrm{Ric}\right)(X, Y) + \left(2xy\dot{h}\right)f\triangle f + xy\zeta(f\triangle f) - \zeta\left(\frac{1}{f}H^f(X, Y)\right) + \frac{1}{f}H^f([\zeta, X], Y) + \frac{1}{f}H^f(X, [\zeta, Y])$$

for any $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$.

Proof. Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$, then

$$\begin{aligned}\left(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{\mathrm{Ric}}\right)\left(\bar{X}, \bar{Y}\right) &= \bar{\zeta}\left(\bar{\mathrm{Ric}}\left(\bar{X}, \bar{Y}\right)\right) - \bar{\mathrm{Ric}}\left(\left[\bar{\zeta}, \bar{X}\right], \bar{Y}\right) - \bar{\mathrm{Ric}}\left(\bar{X}, \left[\bar{\zeta}, \bar{Y}\right]\right) \\ &= \bar{\zeta}\left(\mathrm{Ric}(X, Y) - \frac{1}{f}H^f(X, Y) + xyf\triangle f\right) + 2\left(xy\dot{h}\right)f\triangle f \\ &\quad - \mathrm{Ric}(X, [\zeta, Y]) + \frac{1}{f}H^f(X, [\zeta, Y]) \\ &\quad - \mathrm{Ric}([\zeta, X], Y) + \frac{1}{f}H^f([\zeta, X], Y) \\ &= \left(\mathcal{L}_\zeta \mathrm{Ric}\right)(X, Y) + \left(2xy\dot{h}\right)f\triangle f + xy\zeta(f\triangle f) \\ &\quad - \zeta\left(\frac{1}{f}H^f(X, Y)\right) + \frac{1}{f}H^f([\zeta, X], Y) + \frac{1}{f}H^f(X, [\zeta, Y])\end{aligned}$$

for any vector fields $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$. □

The above proposition leads directly to the following results.

Theorem 9. *Let $\bar{\zeta} = h\partial_t \in \mathfrak{X}(\bar{M})$ be a vector field on a standard static spacetime $\bar{M} = I_f \times M$. Then, $\bar{\zeta}$ is a Ricci collineation on $\bar{M}$ if and only if $\dot{h} = 0$ or $\triangle f = 0$.*

Theorem 10. *Let $\bar{\zeta} = \zeta \in \mathfrak{X}(\bar{M})$ be a vector field on a standard static spacetime $\bar{M} = I_f \times M$ and assume that $H^f = 0$. Then, $\bar{\zeta}$ is a Ricci collineation on $\bar{M}$ if and only if ζ is a Ricci collineation on M.*

4. Ricci soliton on standard static spacetimes

A smooth vector field ζ on a pseudo-Riemannian manifold (M, g) is said to define a Ricci soliton if

$$\frac{1}{2}\left(\mathcal{L}_\zeta g\right)(X, Y) + \mathrm{Ric}(X, Y) = \lambda g(X, Y),$$

where Ric is the Ricci curvature, $\mathcal{L}_\zeta$ denotes the Lie derivative of the metric tensor g and λ is a constant [17,18]. In this section, we consider Ricci solitons on standard static spacetimes. Let

$(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, where $\bar{M} = I_f \times M$ is a standard static spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$. It is clear that a potential field $\bar{\zeta}$ is conformal on $(\bar{M}, \bar{g})$ if and only if $(\bar{M}, \bar{g})$ is an Einstein manifold.

Theorem 11. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I_f \times M$ is a standard static spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ and assume that $H^f = 0$. Then (M, g, ζ, λ) is a Ricci soliton.*

Proof. Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, then

$$\frac{1}{2}(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) + \bar{\mathrm{Ric}}(\bar{X}, \bar{Y}) = \lambda\bar{g}(\bar{X}, \bar{Y}),$$

where $\bar{X} = x\partial_t + X$ and $\bar{Y} = y\partial_t + Y$ are vector fields on $\bar{M}$. By using Eqs. (3.1) and (2.3), we get

$$\frac{1}{2}(\mathcal{L}_{\zeta}g)(X, Y) + \mathrm{Ric}(X, Y) - \frac{1}{f}H^f(X, Y) = \lambda g(X, Y).$$

Suppose now that $H^f = 0$, then

$$\frac{1}{2}(\mathcal{L}_{\zeta}g)(X, Y) + \mathrm{Ric}(X, Y) = \lambda g(X, Y).$$

Thus (M, g, ζ, λ) is a Ricci soliton. □

Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, where $\bar{M} = I_f \times M$ is a standard static spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$. Then

$$\frac{1}{2}(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) + \bar{\mathrm{Ric}}(\bar{X}, \bar{Y}) = \lambda\bar{g}(\bar{X}, \bar{Y}).$$

Thus

$$\frac{1}{2}(\mathcal{L}_{\zeta}g)(X, Y) + \mathrm{Ric}(X, Y) - \frac{1}{f}H^f(X, Y) = \lambda g(X, Y), \tag{4.1}$$

$$\dot{h}f + \zeta(f) - \triangle f = \lambda f. \tag{4.2}$$

Suppose that grad$f$ is a concircular vector field with factor ρ, then

$$\frac{1}{2}(\mathcal{L}_{\zeta}g)(X, Y) + \mathrm{Ric}(X, Y) = \left(\lambda + \frac{\rho}{f}\right)g(X, Y).$$

Theorem 12. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, where $\bar{M} = I_f \times M$ is a standard static spacetime, and assume that* grad$f$ *is a concircular vector field with factor ρ. Then $\left(M, g, \zeta, \lambda + \frac{\rho}{f}\right)$ is a Ricci soliton whenever $\frac{\rho}{f}$ is constant.*

Theorem 13. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, where $\bar{M} = I_f \times M$ is a standard static spacetime, $H^f = 0$ and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ is a conformal vector field on $\bar{M}$ with factor 2ρ. Then (M, g) is Ricci flat and $\lambda = \rho$.*

Proof. Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I_f \times M$ is a standard static spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a conformal vector field on $\bar{M}$. Then

$$\bar{\mathrm{Ric}}(\bar{X}, \bar{Y}) = (\lambda - \rho)\bar{g}(\bar{X}, \bar{Y})$$

and hence $\triangle f = -(\lambda - \rho)f$ and

$$\mathrm{Ric}(X, Y) - \frac{1}{f}H^f(X, Y) = (\lambda - \rho)g(X, Y).$$

Assuming that $H^f = 0$, we have $\mathrm{Ric}(X, Y) = 0$. □

Corollary 3. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, where $\bar{M} = I_f \times M$ is a standard static spacetime, $H^f = 0$ and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ is a Killing vector field on $\bar{M}$. Then $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ is a steady Ricci soliton, i.e., $\lambda = 0$.*

Corollary 4. *Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton, where $\bar{M} = I_f \times M$ is a standard static spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$, and assume that $H^f = 0$ and (M, g) is Ricci flat. Then $\bar{\zeta}$ is conformal with factor 2λ.*

Proof. Let $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ be a Ricci soliton where $\bar{M} = I_f \times M$ is a standard static spacetime and $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$. Then

$$\frac{1}{2}(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) + \bar{\mathrm{Ric}}(\bar{X}, \bar{Y}) = \lambda\bar{g}(\bar{X}, \bar{Y})$$

for any vector fields $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$. Eqs. (2.3) imply that

$$(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) = 2\lambda\bar{g}(\bar{X}, \bar{Y}),$$

i.e., $\bar{\zeta}$ is a conformal vector field with factor 2λ. □

Theorem 14. *Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a vector field on a standard static spacetime $\bar{M} = I_f \times M$. Then $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ is a Ricci soliton if*

1. ζ is a conformal vector field on M with conformal factor 2ρ,
2. (M, g) is Einstein manifold with factor μ,
3. $H^f = 0$,
4. $\dot{h} + \zeta(\ln f) = \rho + \mu = \lambda$.

Proof. Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$, then Eqs. (3.1) and (2.3) imply that

$$\begin{aligned}\frac{1}{2}(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) + \bar{\mathrm{Ric}}(\bar{X}, \bar{Y}) = {} & \frac{1}{2}(\mathcal{L}_{\zeta}g)(X, Y) + \mathrm{Ric}(X, Y) \\ & - xyf^2(\dot{h} + \zeta(\ln f)) \\ & + xyf\triangle f - \frac{1}{f}H^f(X, Y).\end{aligned}$$

Since ζ is a conformal vector field with conformal factor 2ρ, $H^f = 0$ and (M, g) is Einstein manifold with factor μ,

$$\frac{1}{2}(\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) + \bar{\mathrm{Ric}}(\bar{X}, \bar{Y}) = (\rho + \mu)g(X, Y) - xyf^2(\dot{h} + \zeta(\ln f)).$$

The last condition implies that

$$\frac{1}{2}(\mathcal{L}_{\bar{\zeta}}\bar{g})(\bar{X}, \bar{Y}) + \bar{\mathrm{Ric}}(\bar{X}, \bar{Y}) = \lambda\bar{g}(\bar{X}, \bar{Y})$$

and the proof is complete. □

A similar discussion leads to the following result.

Theorem 15. *Let $\bar{\zeta} = h\partial_t + \zeta \in \mathfrak{X}(\bar{M})$ be a vector field on a standard static spacetime $\bar{M} = I_f \times M$. Then $(\bar{M}, \bar{g}, \bar{\zeta}, \lambda)$ is a Ricci soliton if*

1. (M, g, ζ, λ) is a Ricci soliton,
2. $H^f = 0$,
3. $\dot{h} + \zeta(\ln f) = \lambda$.

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