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Original Article

New exact solutions of coupled generalized regularized long wave equations

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ABSTRACT

In the present paper, we established a traveling wave solution by using sine-cosine functions algorithm and the Kudryashov Method for nonlinear partial differential equations. Different methods are used to obtain the exact solutions for different types of nonlinear partial differential equations such as, coupled generalized regularized long wave equation (CGRLW). We plot the exact solutions for these equations at different time levels.

1. Introduction

Exact soliton solutions for nonlinear partial differential equations (NPDEs) play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, optics, condensed matter physics, plasma physics, and so on. The achievements and the research direction in the research area are the construction of the exact soliton solutions of these nonlinear PDEs using many mathematical methods. Now, with the help of computerized symbolic computations and with progress also in the science of numerical analysis many researchers found many methods to solve nonlinear partial differential equations, which were difficult to resolve in the past, such as finite difference method [1,2]. Also, there exist some mathematical models that describe some of the complex phenomena in physics, chemistry and other various scientific fields had to be and there are many methods to solve these models and get the exact results of these models non-linear waves' equations, which plays an important role in the study of many phenomena physical. There are many numerical methods that were used in solving such phenomena such as finite element methods [3–6], Hirota's method [7], extended tanh-function method [8], sine-cosine method [9], Variational iterative method [10] and so on. The coupled equal width wave equation given by Ali et al. [11] solved by EL-Sayed et al. using classical method [12]. In this paper, we will apply the sine-cosine function algorithms and the Kudryashov method to find the new exact solutions of some important nonlinear partial differential equations. To illuminate the utility of these methods, we will apply them to the coupled generalized regularized long wave (CGRLW) equations.

The rest of this paper is organized as follows: In Section 2, we describe the sine-cosine function algorithms. In Section 3, we apply the proposed sine-cosine function algorithms to establish the exact solutions for the coupled generalized regularized long wave equations. In Section 4, we apply the proposed Kudryashov method to establish the exact solutions for CGRLW equations. Conclusions are presented in Section 5.

2. The sine-cosine functions method

Consider the system of nonlinear partial differential equations in the form

$$H(w, w_t, w_x, w_{xx}, w_{xxt}, g, g_t, g_x, g_{xx}, g_{xxt}, \dots) = 0, \tag{1}$$

where $w(x, t)$ and $g(x, t)$ are solutions for the system of nonlinear partial differential equations (1). We use the transformations,

$$\begin{aligned} w(x,t) &= h(\zeta), \\ g(x,t) &= l(\zeta), \end{aligned} \tag{2}$$

where $\zeta = x - st$ and s represent the constant velocity of a wave traveling in the positive direction of x-axis. Then, from Eqs. (2), we get

$$\begin{aligned} w_t &= -sh'(\zeta), \quad w_x = h'(\zeta), \quad w_{xx} = h''(\zeta), \quad w_{xxt} = -sh'''(\zeta), \ \dots \\ g_t &= -sl'(\zeta), \quad g_x = l'(\zeta), \quad g_{xx} = l''(\zeta), \quad g_{xxt} = -sl'''(\zeta), \ \dots . \end{aligned} \tag{3}$$

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Use Eqs. (3) to transfer the system of nonlinear partial differential equations (1) to nonlinear ordinary differential equations

$$G(h', h'', h''', l', l'', l''', \dots) = 0, \tag{4}$$

In the Sine-Cosine technique, the solutions of (4) may be expressed in the form

$$\begin{aligned} h(\zeta) &= a\sin^m(\eta_1\zeta),\\ l(\zeta) &= b\sin^n(\eta_2\zeta), \end{aligned} \tag{5}$$

$$\begin{aligned} h(\zeta) &= a\cos^m(\eta_1\zeta),\\ l(\zeta) &= b\cos^n(\eta_2\zeta), \end{aligned} \tag{6}$$

where a, η_1, η_2, b, n and m are parameters to be determined. From Eqs. (5), we get the following derivatives with respect to ζ:

$$\begin{aligned} h(\xi) &= a\sin^m(\eta_1\zeta),\\ h'(\xi) &= ma\eta_1\sin^{m-1}(\eta_1\zeta)\cos(\eta_1\zeta),\\ h''(\xi) &= -m^2a\eta_1^2\sin^m(\eta_1\zeta) + m(m-1)a\eta_1^2\sin^{m-2}(\eta_1\zeta),\\ l(\xi) &= b\sin^n(\eta_2\zeta),\\ l'(\xi) &= nb\eta_2\sin^{n-1}(\eta_2\zeta)\cos(\eta_2\zeta),\\ l''(\xi) &= -n^2b\eta_2^2\sin^n(\eta_2\zeta) + n(n-1)b\eta_2^2\sin^{n-2}(\eta_2\zeta). \end{aligned} \tag{7}$$

Similarly, from Eqs. (6), we get

$$\begin{aligned} h(\xi) &= a\cos^m(\eta_1\zeta),\\ h'(\xi) &= -ma\eta_1\cos^{m-1}(\eta_1\zeta)\sin(\eta_1\zeta),\\ h''(\xi) &= -m^2a\eta_1^2\cos^m(\eta_1\zeta) + m(m-1)a\eta_1^2\cos^{m-2}(\eta_1\zeta),\\ l(\xi) &= b\cos^n(\eta_2\zeta),\\ l'(\xi) &= -nb\eta_2\cos^{n-1}(\eta_2\zeta)\sin(\eta_2\zeta),\\ l''(\xi) &= -n^2b\eta_2^2\cos^n(\eta_2\zeta) + n(n-1)b\eta_2^2\cos^{n-2}(\eta_2\zeta). \end{aligned} \tag{8}$$

Substituting (8) into the system of nonlinear ordinary differential equations (4) gives a trigonometric equation of $\cos^m(\eta_1\zeta)$ and $\cos^n(\eta_2\zeta)$ terms. To determine the parameters we need make balancing between m and n powers in (8). Then we collect all terms with the same power in $\cos^m(\eta_1\zeta)$, $\cos^n(\eta_2\zeta)$ and put to zero their coefficients to get a system of algebraic equations in the unknowns a, η_1, η_2, b, n and m. Similarly, substituting (7) into the system of nonlinear ordinary differential equation (4) gives a trigonometric equation of $\sin^m(\eta_1\zeta)$ and $\sin^n(\eta_2\zeta)$ terms. We can obtain the unknown parameters a, η_1, η_2, b, n and m by the same way. Hence, the solutions considered in (5) and (6) are obtained.

3. Applications

3.1. Test problem

The coupled generalized regularized long wave equations (CGRLWE) consider the following problem: Find functions $w(x, t)$ and $g(x, t)$ satisfying CGRLWE in the form [12],

$$\begin{aligned} &w_t + w_x + \varepsilon w^p w_x - \mu w_{xxt} + \varepsilon g^p g_x = 0,\\ &g_t + g_x + \varepsilon g^p g_x - \mu g_{xxt} = 0, \end{aligned} \tag{9}$$

where p is a positive integer.

Using (2) and (3) into (9) gives the following system of ordinary differential equations

$$\begin{aligned} &(1-s)h' + \varepsilon h^p h' + \mu s h''' + \varepsilon l^p l' = 0,\\ &(1-s)l' + \varepsilon l^p l' + \mu s l''' = 0. \end{aligned} \tag{10}$$

Integrating (10) once and taking the constant of integration to be zero, we find

$$\begin{aligned} &(1-s)h + \varepsilon\frac{1}{p+1}h^{p+1} + \mu s h'' + \varepsilon\frac{1}{p+1}l^{p+1} = 0,\\ &(1-s)l + \varepsilon\frac{1}{p+1}l^{p+1} + \mu s l'' = 0. \end{aligned} \tag{11}$$

Substituting Eq. (8) into (10) gives:

$$\begin{aligned} &(1-s)a\cos^m(\eta_1\zeta) + \frac{\varepsilon}{(p+1)}a^{(p+1)}\cos^{(p+1)m}(\eta_1\zeta)\\ &\quad - s\mu m^2 a\eta_1^2\cos^m(\eta_1\zeta) + s\mu m(m-1)a\eta_1^2\cos^{m-2}(\eta_1\zeta)\\ &\quad + \frac{\varepsilon}{(p+1)}b^{(p+1)}\cos^{(p+1)n}(\eta_2\zeta) = 0,\\ &(1-s)b\cos^n(\eta_2\zeta) + \frac{\varepsilon}{(p+1)}b^{(p+1)}\cos^{(p+1)n}(\eta_2\zeta)\\ &\quad + s\mu n^2 b\eta_2^2\cos^n(\eta_2\zeta) + s\mu n(n-1)b\eta_2^2\cos^{n-2}(\eta_2\zeta) = 0, \end{aligned} \tag{12}$$

The last system is satisfied only if the following system of algebraic equations holds:

$$\begin{aligned} &m-1 \neq 0,\\ &m-2 = (p+1)m = (p+1)n,\\ &\frac{\varepsilon}{(p+1)}a^{p+1} + \frac{\varepsilon}{(p+1)}b^{p+1} = -s\mu m(m-1)a\eta_1^2,\\ &(1-s)a = s\mu m^2 a\eta_1^2,\\ &n-1 \neq 0,\\ &n-2 = (p+1)n,\\ &\frac{\varepsilon}{(p+1)}b^{p+1} = -s\mu n(n-1)b\eta_2^2,\\ &(1-s)b = s\mu n^2 b\eta_2^2. \end{aligned} \tag{13}$$

We can get the same system (12) by substituting Eq. (7) into (10). To solve this system, we must take concrete values for p. In what follows we take $p = 1, 2, 3$.

Now we study these cases case by case.

3.1.1. Case one (the exact solution of CRLWE at $p = 1$)

Using MATHEMATICA package software for solving the system (12) at $p = 1$ we obtain:

$$\begin{aligned} &m = n = -2, \quad \eta_1 = \eta_2 = \pm\frac{1}{2}\sqrt{\frac{1-s}{\mu s}},\\ &b = \frac{3(s-1)}{\varepsilon}, \quad a_1 = \frac{3(s-1)}{2\varepsilon}\left(1 - i\sqrt{3}\right),\\ &a_2 = \frac{3(s-1)}{2\varepsilon}\left(1 + i\sqrt{3}\right). \end{aligned} \tag{14}$$

By back substitution (13) into (5) and (6) with (2) and setting $s = s + 1, s > 0$, we obtain the exact solution of the CRLWE in the form

$$\begin{aligned} g(x,t) &= \frac{3s}{\varepsilon}\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),\\ g(x,t) &= -\frac{3s}{\varepsilon}\operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),\\ w_1(x,t) &= \frac{3s}{2\varepsilon}(1-i\sqrt{3})\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),\\ w_1(x,t) &= -\frac{3s}{2\varepsilon}(1-i\sqrt{3})\operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),\\ w_2(x,t) &= \frac{3s}{2\varepsilon}(1+i\sqrt{3})\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),\\ w_2(x,t) &= -\frac{3s}{2\varepsilon}(1+i\sqrt{3})\operatorname{csc}h^2\left(\frac{1}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right). \end{aligned}$$

Now we can plot these solutions at different time levels and we can show the motion of solitary waves on Fig. 1.

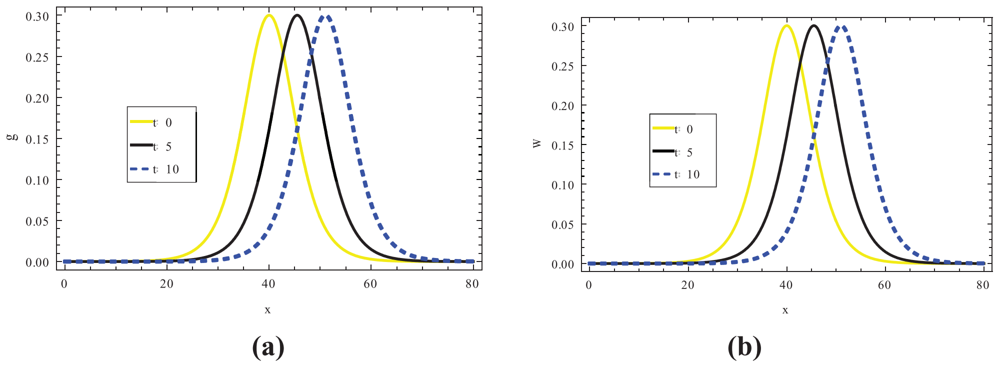


Fig. 1. Absolute values of the exact solutions for the CRLWE at different time levels.

3.1.2. Case two (the exact solution of CMRLWE at $p = 2$)

Using MATHEMATICA package software for solving the system (12) at $p = 2$ we obtain:

$$m = n = -1,$$

$$\eta_1 = \eta_2 = \pm\sqrt{\frac{1-s}{\mu s}},\ b = \sqrt{\frac{6(s-1)}{\varepsilon}},\ a_1 = \frac{2s\varepsilon + \sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}},$$

$$a_2 = \frac{-2(1+i\sqrt{3})s\varepsilon-(1-i\sqrt{3})\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{2\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}},$$

$$a_3 = \frac{-2(1-i\sqrt{3})s\varepsilon-(1+i\sqrt{3})\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{2\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}}. \tag{15}$$

By back substitution (14) into (5) and (6) with (2) and setting $s = s + 1, s > 0$, we obtain the exact solution of the CMRLWE in the form

$$g(x,t) = \sqrt{\frac{6s}{\varepsilon}}\,\mathrm{sec}\,h\left(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),$$

$$g(x,t) = (-i)\sqrt{\frac{6s}{\varepsilon}}\,\mathrm{csc}\,h\left(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),$$

$$w_1(x,t) = \frac{2s\varepsilon + \sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}}\,\mathrm{sec}\,h(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)),$$

$$w_1(x,t) = (-i)\frac{2s\varepsilon + \sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}}\,\mathrm{csc}\,h(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)),$$

$$w_2(x,t) = \frac{-2(1+i\sqrt{3})s\varepsilon-(1-i\sqrt{3})\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{2\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}}$$
$$\times\,\mathrm{sec}\,h\left(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),$$

$$w_2(x,t) = (-i)\frac{-2(1+i\sqrt{3})s\varepsilon-(1-i\sqrt{3})\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{2\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}}$$
$$\times\,\mathrm{csc}\,h\left(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),$$

$$w_3(x,t) = \frac{-2(1-i\sqrt{3})s\varepsilon-(1+i\sqrt{3})\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{2\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}}$$
$$\times\,\mathrm{sec}\,h\left(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right),$$

$$w_3(x,t) = (-i)\frac{-2(1-i\sqrt{3})s\varepsilon-(1+i\sqrt{3})\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)^2}}{2\varepsilon\sqrt[3]{\left(\sqrt{46s^3\varepsilon^3}-3\sqrt{6\left(\frac{s}{\varepsilon}\right)^3\varepsilon^3}\right)}}$$
$$\times\,\mathrm{csc}\,h\left(\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right).$$

Now we can plot these solutions at different time levels and we can show the motion of solitary waves on Fig. 2.

3.1.3. Case three (the exact solution of CGRLWE) at $p = 3$

Using MATHEMATICA package software for solving the system (12) at $p = 3$ we obtain:

$$m = n = -\frac{2}{3},$$

$$\eta_1 = \eta_2 = \pm\frac{3}{2}\sqrt{\frac{1-s}{\mu}},\qquad b = \sqrt[3]{\frac{10(s-1)}{\varepsilon}},$$

$$a_1 = -\frac{\sqrt{\dfrac{8\sqrt[6]{2}\,\sqrt[3]{3^2}\,\sqrt[6]{5^5}\sqrt{\left(\frac{s}{\varepsilon}\right)^3}\varepsilon^4+\sqrt[3]{6\left(45s^2\varepsilon^4+5\sqrt{3}\sqrt{-s^4\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^8}\right)^2}}{\varepsilon^2\sqrt[3]{\left(9s^2\varepsilon^4+\sqrt{3}\sqrt{-s^4\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^8}\right)}}}}{2\sqrt{3}} -$$

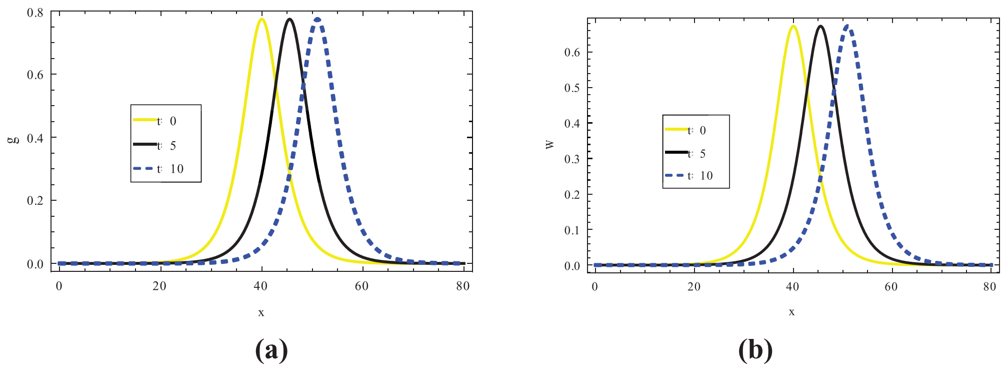


Fig. 2. Absolute values of the exact solutions for the CMRLWE at different time levels.

$$\sqrt{\frac{1}{2}\begin{matrix}-\frac{\sqrt[3]{\left(\frac{5}{3}\right)^{2}\left(18s^{2}\varepsilon^{4}+2\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}{\varepsilon^{2}}\\-\frac{8\sqrt[6]{2}\sqrt[6]{5^{5}}s\sqrt{\frac{s}{\varepsilon}}\varepsilon}{\sqrt[3]{\left(27s^{2}\varepsilon^{4}+3\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}\\-\frac{20\sqrt{3}s}{\varepsilon\sqrt{\frac{8\sqrt[3]{3^{2}}\sqrt[6]{2}\sqrt[6]{5^{5}}\left(\frac{s}{\varepsilon}\right)^{\frac{3}{2}}\varepsilon^{4}+\sqrt[3]{6\left(45s^{2}\varepsilon^{4}+5\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}^{2}}{\varepsilon^{2}\sqrt[3]{\left(9s^{2}\varepsilon^{4}+\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}}}\end{matrix}},$$

$$a_2=-\frac{\sqrt{\frac{8\sqrt[6]{2}\sqrt[3]{3^{2}}\sqrt[6]{5^{5}}\sqrt{\left(\frac{s}{\varepsilon}\right)^{2}}\varepsilon^{4}+\sqrt[3]{6\left(45s^{2}\varepsilon^{4}+5\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}^{2}}{\varepsilon^{2}\sqrt[3]{\left(9s^{2}\varepsilon^{4}+\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}}}{2\sqrt{3}}$$

$$+\frac{1}{2}\sqrt{\begin{matrix}-\frac{\sqrt[3]{\left(\frac{5}{3}\right)\left(18s^{2}\varepsilon^{4}+2\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}{\varepsilon^{2}}\\-\frac{8\sqrt[6]{2}\sqrt[6]{5^{5}}s\sqrt{\frac{s}{\varepsilon}}\varepsilon}{\sqrt[3]{\left(27s^{2}\varepsilon^{4}+3\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}\\-\frac{20\sqrt{3}s}{\varepsilon\sqrt{\frac{8\sqrt[6]{2}\sqrt[3]{3^{2}}\sqrt[6]{5^{5}}\sqrt{\left(\frac{s}{\varepsilon}\right)^{2}}\varepsilon^{4}+\sqrt[3]{6\left(45s^{2}\varepsilon^{4}+5\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}^{2}}{\varepsilon^{2}\sqrt[3]{\left(9s^{2}\varepsilon^{4}+\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}}}\end{matrix}}$$

$$a_3=\frac{\sqrt{\frac{8\sqrt[6]{2}\sqrt[3]{3^{2}}\sqrt[6]{5^{5}}\sqrt{\left(\frac{s}{\varepsilon}\right)^{2}}\varepsilon^{4}+\sqrt[3]{6\left(45s^{2}\varepsilon^{4}+5\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}^{2}}{\varepsilon^{2}\sqrt[3]{\left(9s^{2}\varepsilon^{4}+\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}}}{2\sqrt{3}}$$

$$-\frac{1}{2}\sqrt{\begin{matrix}-\frac{\sqrt[3]{\left(\frac{5}{3}\right)^{2}\left(18s^{2}\varepsilon^{4}+2\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}{\varepsilon^{2}}\\-\frac{8\sqrt[6]{2}\sqrt[6]{5^{5}}s\sqrt{\frac{s}{\varepsilon}}\varepsilon}{\sqrt[3]{\left(27s^{2}\varepsilon^{4}+3\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}\\+\frac{20\sqrt{3}s}{\varepsilon\sqrt{\frac{8\sqrt[6]{2}\sqrt[3]{3^{2}}\sqrt[6]{5^{5}}\sqrt{\left(\frac{s}{\varepsilon}\right)^{2}}\varepsilon^{4}+\sqrt[3]{6\left(45s^{2}\varepsilon^{4}+5\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}^{2}}{\varepsilon^{2}\sqrt[3]{\left(9s^{2}\varepsilon^{4}+\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}}}\end{matrix}},$$

$$a_4=\frac{\sqrt{\frac{8\sqrt[6]{2}\sqrt[3]{3^{2}}\sqrt[6]{5^{5}}\sqrt{\left(\frac{s}{\varepsilon}\right)^{2}}\varepsilon^{4}+\sqrt[3]{6\left(45s^{2}\varepsilon^{4}+5\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}^{2}}{\varepsilon^{2}\sqrt[3]{\left(9s^{2}\varepsilon^{4}+\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}}}{2\sqrt{3}}+$$

$$\sqrt{\frac{1}{2}\begin{matrix}-\frac{\sqrt[3]{\left(\frac{5}{3}\right)^{2}\left(18s^{2}\varepsilon^{4}+2\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}{\varepsilon^{2}}\\-\frac{8\sqrt[6]{2}\sqrt[6]{5^{5}}s\sqrt{\frac{s}{\varepsilon}}\varepsilon}{\left(27s^{2}\varepsilon^{4}+3\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)^{\frac{1}{3}}}\\+\frac{20\sqrt{3}s}{\varepsilon\sqrt{\frac{8\sqrt[6]{2}\sqrt[3]{3^{2}}\sqrt[6]{5^{5}}\sqrt{\left(\frac{s}{\varepsilon}\right)^{2}}\varepsilon^{4}+\sqrt[3]{6\left(45s^{2}\varepsilon^{4}+5\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}^{2}}{\varepsilon^{2}\sqrt[3]{\left(9s^{2}\varepsilon^{4}+\sqrt{3}\sqrt{-s^{4}\left(-27+256\sqrt{10}\sqrt{\frac{s}{\varepsilon}}\right)\varepsilon^{8}}\right)}}}}\end{matrix}} \tag{16}$$

By back substitution (16) into (5) and (6) with (2) and setting $s=s+1, s>0$, we obtain the exact solution of the CGRLWE in the form

$$g(x,t)=\sqrt[3]{\frac{10s}{\varepsilon}\sec h^{2}\left[\pm\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right]},$$

$$g(x,t)=\sqrt[3]{(-i)^{2}}\sqrt[3]{\frac{10s}{\varepsilon}\csc h^{2}\left[\pm\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t)\right]},$$

$$w_1(x,t)=a_1\sqrt[3]{\sec h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))},$$

$$w_1(x,t)=\sqrt[3]{(-i)^{2}}a_1\sqrt[3]{\csc h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))},$$

$$w_2(x,t)=a_2\sqrt[3]{\sec h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))},$$

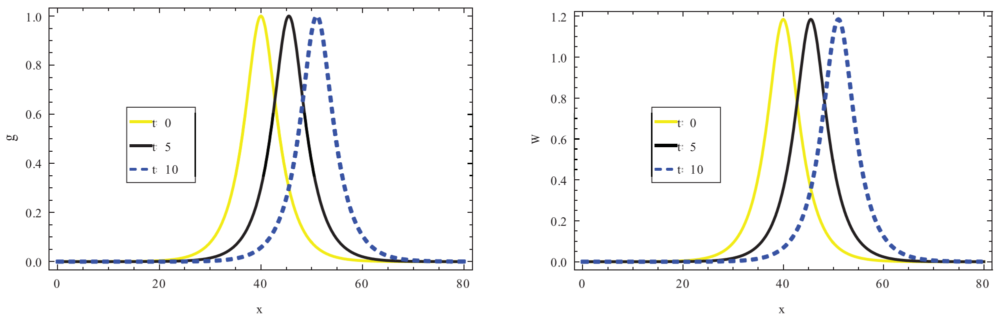
$$w_2(x,t)=\sqrt[3]{(-i)^{2}}a_2\sqrt[3]{\csc h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))},$$

$$w_3(x,t)=a_3\sqrt[3]{\sec h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))},$$

$$w_3(x,t)=\sqrt[3]{(-i)^{2}}a_3\sqrt[3]{\csc h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))},$$

$$w_4(x,t)=a_4\sqrt[3]{\sec h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))},$$

$$w_4(x,t)=\sqrt[3]{(-i)^{2}}a_4\sqrt[3]{\csc h^{2}(\frac{3}{2}\sqrt{\frac{s}{\mu(s+1)}}(x-(s+1)t))}.$$



(a) **(b)**

Fig. 3. Absolute values of the exact solutions for the CGRLWE at different time levels.

Now we can plot these solutions at different time levels and we can show the motion of solitary waves on Fig. 3.

4. The Kudryashov method

In this section, we can solve CGRLWE by using the Kudryashov method as follows form [13]

$$w(\zeta) = \sum_{m=0}^{k} \frac{a_m}{(1+\exp(\zeta))^m}, \tag{17}$$

where $a_0, a_1, \dots, a_k$ are constants to be determined.

We may choose the solutions of (10) in the form

$$\begin{aligned} h(\zeta) &= a_0 + \frac{a_1}{1+\exp(\zeta)} + \frac{a_2}{(1+\exp(\zeta))^2}, \\ l(\zeta) &= b_0 + \frac{b_1}{1+\exp(\zeta)} + \frac{b_2}{(1+\exp(\zeta))^2}, \end{aligned} \tag{18}$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are arbitrary constants to be determined.

Substituting from (18) into (10) at $p = 1$, and setting the coefficients of the same power of $\exp(\zeta)$ equal to zero, we obtain the following algebraic equations.

$$\begin{aligned} &\varepsilon a_1^2 + 2a_2 - 2a_2 s + 2a_2 s\mu + 2\varepsilon a_2 a_0 + 2\varepsilon a_2^2 + a_1 - s a_1 + s\mu a_1 \\ &\quad + \varepsilon a_1 a_0 + 3\varepsilon a_2 a_1 + \varepsilon b_1 b_0 + \varepsilon b_1^2 + \varepsilon b_1 b_2 + 2\varepsilon b_2 b_0 + 2\varepsilon b_1 b_2 \\ &\quad + \varepsilon b_2^2 = 0, \\ &2\varepsilon a_1^2 + 4a_2 - 4a_2 s - 14a_2 s\mu + 4\varepsilon a_2 a_0 + 3a_1 - 3s a_1 - 3s\mu a_1 \\ &\quad + 3\varepsilon a_1 a_0 + 3\varepsilon a_1 a_2 + 3\varepsilon b_1 b_0 + 2\varepsilon b_1^2 + 3\varepsilon b_1 b_2 + 4\varepsilon b_2 b_0 = 0, \\ &\varepsilon a_1^2 + 2a_2 - 2a_2 s + 8a_2 s\mu + 2\varepsilon a_2 a_0 + 3a_1 - 3s a_1 - 3s\mu a_1 \\ &\quad + 3\varepsilon a_1 a_0 + 3\varepsilon b_1 b_0 + \varepsilon b_1^2 + 2\varepsilon b_2 b_0 = 0, \\ &a_1 - a_1 s + a_1 s\mu + \varepsilon a_1 a_0 + \varepsilon b_1 b_0 = 0, \\ &\varepsilon b_1^2 + 2b_2 - 2b_2 s + 2b_2 s\mu + 2\varepsilon b_2 b_0 + 2\varepsilon b_2^2 + b_1 - s b_1 + s\mu b_1 \\ &\quad + \varepsilon b_1 b_0 + 3\varepsilon b_2 b_1 = 0, \\ &2\varepsilon b_1^2 + 4b_2 - 4b_2 s - 14b_2 s\mu + 4\varepsilon b_2 b_0 + 3b_1 - 3s b_1 - 3s\mu b_1 \\ &\quad + 3\varepsilon b_1 b_0 + 3\varepsilon b_1 b_2 = 0, \\ &\varepsilon b_1^2 + 2b_2 - 2b_2 s + 8b_2 s\mu + 2\varepsilon b_2 b_0 + 3b_1 - 3s b_1 - 3s\mu b_1 \\ &\quad + 3\varepsilon b_1 b_0 = 0, b_1 - b_1 s + b_1 s\mu + \varepsilon b_1 b_0 = 0. \end{aligned} \tag{19}$$

Solving the system of algebraic Eqs. (18) with by MATHEMATICA program, we obtain two cases of solutions for our system (10).

4.1. Case (1)

$$\begin{aligned} a_0 &= \frac{(1+i\sqrt{3})(-1+s-s\mu)}{2\varepsilon}, \; a_1 = \frac{6s\mu(1+i\sqrt{3})}{\varepsilon} = -a_2, \\ b_0 &= \frac{(-1+s-s\mu)}{2\varepsilon}, b_1 = \frac{6s\mu}{\varepsilon} = -b_2, \end{aligned} \tag{20}$$

Substitution (20) and (18) into (2), we get new exact solution for the coupled generalized regularized long wave is obtained

$$\begin{aligned} w(x,t) &= \frac{(1+i\sqrt{3})(-1+s-s\mu)}{2\varepsilon} + \frac{6s\mu(1+i\sqrt{3})\exp(x-st)}{\varepsilon(1+\exp(x-st))^2}, \\ g(x,t) &= \frac{(-1+s-s\mu)}{2\varepsilon} + \frac{12s\mu\exp(x-st)}{\varepsilon(1+\exp(x-st))^2}, \end{aligned}$$

4.2. Case (2)

$$\begin{aligned} a_0 &= \frac{(1-i\sqrt{3})(-1+s-s\mu)}{2\varepsilon}, \quad a_1 = \frac{6s\mu(1-i\sqrt{3})}{\varepsilon} = -a_2, \\ b_0 &= \frac{(-1+s-s\mu)}{2\varepsilon}, \quad b_1 = \frac{6s\mu}{\varepsilon} = -b_2, \end{aligned} \tag{21}$$

Substitution (21) and (17) into (2), we get new exact solution for the coupled generalized regularized long wave is obtained

$$\begin{aligned} w(x,t) &= \frac{(1-i\sqrt{3})(-1+s-s\mu)}{2\varepsilon} + \frac{6s\mu(1-i\sqrt{3})\exp(x-st)}{\varepsilon(1+\exp(x-st))^2}, \\ g(x,t) &= \frac{(-1+s-s\mu)}{2\varepsilon} + \frac{12s\mu\exp(x-st)}{\varepsilon(1+\exp(x-st))^2}, \end{aligned}$$

5. Conclusion

The sine-cosine functions method and the Kudryashov method have been successfully applied to find the exact solution for different nonlinear partial differential equations such as CGRLW, CRLW and CMRLW equations. The sine-cosine functions method is used to find a new exact solution. Also we applied the Kudryashov method to obtain the exact solutions of CRLWE. The proposed methods can be extended to solve the problems of nonlinear partial differential equations arising in the theory of solitons and other areas.

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