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Statistical inferences based on an adaptive progressive type-II censoring from exponentiated exponential distribution

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ABSTRACT

In this paper, the estimation problem (point and interval) is studied under the exponentiated exponential (*EE*) distribution based on an adaptive progressive type-II censoring scheme. In point estimation, the maximum likelihood estimates (*MLE′s*) and Bayes estimates (*BE′s*), based on squared error (*SE*) and linear exponential (*LINEX*) loss functions, are computed. Also, the approximate confidence intervals for the parameters of *EE* distribution are obtained. A comparison study is made between the *BE′s* and *MLE′s* using the estimated risks (*ER′s*) criterion. Finally, point and interval estimations of all parameters are studied based on a generated adaptive type-II progressive censoring sample from a real data set as illustrative example.

1. Introduction

The cumulative distribution function (*CDF*) of a random variable X

$$F(x;\alpha,\beta,\rho)=(1-\rho\, e^{-\beta x})^{\alpha},\quad x>\frac{1}{\beta}\ln\rho,\quad \alpha,\beta,\rho>0,$$

is suggested by Verhulst [15]. Gupta and Kundu [6] used this distribution with $\rho=1$ and called it "generalized exponential"distribution. Other references for the *EE* distribution are Raqab [12], Raqab and Ahsanullah [13], Jaheen [7], Kundu and Gupta [8], Kundu et al. [9], Abdel-Hamid and AL-Hussaini [1], AL-Hussaini [2,3] and Ateya [5] among others. A recent book on exponentiated distributions is that of AL-Hussaini and Ahsanullah [4].

The probability density function (*PDF*) and the *CDF* of *EE* distribution are given, respectively, by

$$f(x;\alpha,\gamma)=\alpha\gamma e^{-\alpha x}(1-e^{-\alpha x})^{\gamma-1},\quad x>0,\quad \alpha,\gamma>0, \tag{1.1}$$

and

$$F(x;\alpha,\gamma)=(1-e^{-\alpha x})^{\gamma}, \tag{1.2}$$

where α is a scale parameter and γ is a shape parameter. Also, the reliability function takes the form

$$S(x;\alpha,\gamma)=1-(1-e^{-\alpha t})^{\gamma}. \tag{1.3}$$

A new censoring scheme which is a mixture of type-I and type-II progressive censoring schemes called adaptive progressive type-II censoring is introduced by Ng et al. [11]. This scheme can be described as follows:

Let n items be placed on a life-test, and the effective sample size $m<n$ be fixed in advance. Moreover, let the progressive censoring scheme $R=(R_1,\dots,R_m)$ be set before starting the experiment. Suppose the experimenter fixes a time T, which represents the time of the experiment, but the test itself may be allowed to run over time T. Let us denote the m completely observed failure times by $X_{i:m:n}, i=1,\dots,m$. If the mth progressively censored failure time occurs before time T, the experiment will be terminated at time $X_{m:\,m:\,n}$. Otherwise, once the experimental time passes time T but the number of observed failures has not reached m, we would want to terminate the experiment as soon as possible.

The paper is organized as follows: The point estimation problem using maximum likelihood and Bayes methods have been studied in Sections 2 and 3, respectively. The interval estimation problem (approximate confidence interval, credibility interval and highest posterior density interval) is studied in Section 4. Simula-

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tion study is carried out in Section 5. Real data is introduced as illustrative example in Section 6. Finally, some concluding remarks are introduced in Section 7.

2. Maximum likelihood estimation

Let $X_i = X^R_{i:m:n}$, $i = 1, 2, \dots, m$, be the adaptive progressively type-II censored sample from *EE* distribution with censored scheme $R = (R_1, R_2, \dots, R_m)$. The realization of the previous censored sample will be denoted by $x^R_{i;m,n,k}$, $i = 1, 2, \dots, m$ which can be written for simplicity as $\boldsymbol{x} = (x_1, \dots, x_m)$.

Suppose that J is the number of failures observed before time T, i.e.

$$X_{J:m:n} < T < X_{J+1:m:n}, J = 0, 1, \dots, m,$$

where $X_{0:m:n} = 0$ and $X_{m+1:m:n} = \infty$, we set $R_{J+1} = \dots = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{J} R_i$. This formulation leads us to terminate the experiment as soon as possible if the $(J+1)$th failure time is greater than T for $J + 1 < m$.

If the failure times of the n items originally on the test are from a continuous population with *CDF* $F(x;\boldsymbol{\theta})$ and *PDF* $f(x;\boldsymbol{\theta})$, for $J = j$, the likelihood function of the vector of parameters $\boldsymbol{\theta}$ given the vector of observations $\boldsymbol{x}$ is given by (see Ng et al. [11])

$$L(\boldsymbol{\theta};\boldsymbol{x}) = d_j \prod_{i=1}^{m} f(x_i;\boldsymbol{\theta}) \prod_{i=1}^{j} [1 - F(x_i;\boldsymbol{\theta})]^{R_i} [1 - F(x_m;\boldsymbol{\theta})]^{n-m-\sum_{i=1}^{j} R_i},$$
$$0 < x_1 < x_2 < \dots < x_m < \infty, \tag{2.1}$$

where

$$d_j = \prod_{i=1}^{m} \left[n - i + 1 - \sum_{k=1}^{\max\{i-1,j\}} R_k \right]. \tag{2.2}$$

By substituting from (1.1) and (1.2) in (2.1), then the likelihood function is given by

$$L(\alpha, \gamma; \boldsymbol{x}) = d_j \alpha^m \gamma^m \left[\prod_{i=1}^{m} e^{-\alpha x_i} Z_i^{\gamma - 1} \right] \left[\prod_{i=1}^{j} (1 - Z_i^{\gamma})^{R_i} \right] (1 - Z_m^{\gamma})^{C_j}, \tag{2.3}$$

where

$$Z_i = 1 - e^{-\alpha x_i}, \quad C_j = n - m - \sum_{i=1}^{j} R_i, \quad i = 1, 2, \dots, m. \tag{2.4}$$

The natural logarithm of the likelihood function (2.3) is given by

$$\ell \equiv \ln L(\alpha, \gamma; \boldsymbol{x}) \propto m \ln \alpha + m \ln \gamma - \alpha \sum_{i=1}^{m} x_i + (\gamma - 1) \sum_{i=1}^{m} \ln Z_i$$
$$+ \sum_{i=1}^{j} R_i \ln(1 - Z_i^{\gamma}) + C_j \ln(1 - Z_m^{\gamma}). \tag{2.5}$$

When the two parameters α and γ are unknown, the likelihood equations for these parameters can be written as

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m} x_i + (\gamma - 1) \sum_{i=1}^{m} \frac{x_i e^{-\alpha x_i}}{Z_i} - \gamma \sum_{i=1}^{j} R_i \frac{Z_i^{\gamma-1} x_i e^{-\alpha x_i}}{(1 - Z_i^{\gamma})}$$
$$- \gamma C_j \frac{Z_m^{\gamma-1} x_m e^{-\alpha x_m}}{(1 - Z_m^{\gamma})} = 0, \tag{2.6}$$

and

$$\frac{\partial \ell}{\partial \gamma} = \frac{m}{\gamma} + \sum_{i=1}^{m} \ln Z_i - \sum_{i=1}^{j} R_i \frac{Z_i^{\gamma} \ln Z_i}{(1 - Z_i^{\gamma})} - C_j \frac{Z_m^{\gamma} \ln Z_m}{(1 - Z_m^{\gamma})} = 0, \tag{2.7}$$

where Z_i and C_j are as given by (2.4). By solving the system of Eqs. (2.6) and (2.7) numerically we obtain the *MLE's* of α and γ.

3. Bayesian estimation

Suppose that the prior *PDF* $\pi(\alpha, \gamma)$ is given by

$$\pi(\alpha, \gamma) = \pi_1(\gamma) \pi_2(\alpha \mid \gamma). \tag{3.1}$$

Suppose that also $\pi_1(\gamma)$ is Gamma(a_1, $1/a_2$) and $\pi_2(\alpha|\gamma)$ is Gamma(b_1, γ/b_2) with respective densities

$$\pi_1(\gamma) \propto \gamma^{a_1 - 1} e^{-\frac{\gamma}{a_2}}, \tag{3.2}$$

and

$$\pi_2(\alpha \mid \gamma) \propto \gamma^{b_1} \alpha^{b_1 - 1} e^{-\frac{\gamma\alpha}{b_2}}. \tag{3.3}$$

By substituting from (3.2) and (3.3) in (3.1), we obtain the joint prior *PDF* of α and γ as follows

$$\pi(\alpha, \gamma) \propto \gamma^{a_1 + b_1 - 1} \alpha^{b_1 - 1} \exp\left\{ -\gamma \left(\frac{\alpha}{b_2} + \frac{1}{a_2} \right) \right\}. \tag{3.4}$$

Therefore, the joint posterior density of the parameters α and γ can be obtained from (2.3) and (3.4), and written as

$$\pi^*(\alpha, \gamma \mid \boldsymbol{x}) \propto \alpha^{m + b_1 - 1} \gamma^{m + a_1 + b_1 - 1}$$
$$\times \exp\left\{ -\gamma \left[\frac{1}{a_2} + \frac{\alpha}{b_2} - \sum_{i=1}^{m} \ln Z_i \right] - \alpha \sum_{i=1}^{m} x_i - \sum_{i=1}^{m} \ln Z_i \right\}$$
$$\times \left[\prod_{i=1}^{j} (1 - Z_i^{\gamma})^{R_i} \right] \left(1 - Z_m^{\gamma} \right)^{C_j}, \tag{3.5}$$

where Z_i and C_j are as given in (2.4).

The *BE* of an unknown parameter depends on the form of the loss function. In this paper, The *BE's* have been considered under two different loss functions, the symmetric *SE* and the asymmetric *LINEX* loss functions. Under the *SE* loss function, the *BE* of the parameter is the posterior mean. The *LINEX* loss function is defined as

$$Ł(\Delta) = e^{\lambda\Delta} - \lambda\Delta - 1, \qquad \lambda \neq 0, \tag{3.6}$$

where $\Delta = \hat{\phi}(\theta) - \phi(\theta)$, the scalar estimation error when $\phi(\theta)$ is estimated by $\hat{\phi}(\theta)$, for more details see Varian [14]. The sign of the constant λ represents the direction and its magnitude represents the degree of asymmetry. For λ close to zero, the *LINEX* is approximately *SE* loss function and therefore almost symmetric. Under the *LINEX* loss function, the Bayes estimate is given by

$$\hat{\phi}_{BL} = -\frac{1}{\lambda} \ln(E_{\phi}(e^{-\lambda\phi(\theta)} \mid \boldsymbol{x})), \tag{3.7}$$

where λ is constant.

Using the Markov Chain Monte Carlo (*MCMC*) technique, the *BE* of the function $\eta \equiv \eta(\alpha, \gamma)$ under *SE* and *LINEX* loss functions can be written, respectively, in the forms

$$\hat{\eta}_{BS} = \frac{1}{N - M} \sum_{i=M+1}^{N} \eta(\alpha_i, \gamma_i), \tag{3.8}$$

and

$$\hat{\eta}_{BL} = -\frac{1}{\lambda} \ln\left[\frac{1}{N - M} \sum_{i=M+1}^{N} \exp(-\lambda \eta(\alpha_i, \gamma_i)) \right], \tag{3.9}$$

where $\eta(\alpha_i, \gamma_i), i = 1, 2, \dots, N$ are generated from the posterior *PDF* (3.5) (using The Gibbs and Metropolis-Hatings algorithms) and M is the burn-in period (that is, a number of iterations before the stationary distribution is achieved).

4. Interval estimation

In this section, we will study the approximate confidence interval, credibility interval (*CI*) and highest posterior density interval (*HPD*) for the two parameters α and γ.

4.1. Approximate confidence interval

Let $x_1 < x_2 < \ldots < x_m$ denote an adaptive progressively type-II censored sample from *EE* distribution with parameters α and γ. In this section, the approximate confidence intervals for the parameters of *EE* distribution are obtained based the previous censored sample. From Eqs. (2.6) and (2.7), we have

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{m}{\alpha^2} - (\gamma - 1)\sum_{i=1}^{m} \frac{x_i^2 e^{-\alpha x_i}}{Z_i^2} - \gamma \sum_{i=1}^{j} R_i x_i^2 e^{-\alpha x_i} \frac{W_{2i}(\alpha, \gamma; x_i)}{(1 - Z_i^{\gamma})^2} - \gamma C_j x_m^2 e^{-\alpha x_m} \frac{W_{2i}(\alpha, \gamma; x_m)}{(1 - Z_m^{\gamma})^2}, \tag{4.1}$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\frac{m}{\gamma^2} - \sum_{i=1}^{j} R_i \frac{Z_i^{\gamma} (\ln Z_i)^2}{(1 - Z_i^{\gamma})^2} - C_j \frac{Z_m^{\gamma} (\ln Z_m)^2}{(1 - Z_m^{\gamma})^2}, \tag{4.2}$$

and

$$\frac{\partial^2 \ell}{\partial \alpha \partial \gamma} = \sum_{i=1}^{m} \frac{x_i e^{-\alpha x_i}}{Z_i} - \sum_{i=1}^{j} R_i x_i e^{-\alpha x_i} \frac{Z_i^{\gamma-1}(1 + \gamma \ln Z_i - Z_i^{\gamma})}{(1 - Z_i^{\gamma})^2} - C_j x_m e^{-\alpha x_m} \frac{Z_m^{\gamma-1}(1 + \gamma \ln Z_m - Z_m^{\gamma})}{(1 - Z_m^{\gamma})^2}, \tag{4.3}$$

where

$$W_{2i}(\alpha, \gamma; x_i) = (1 - Z_i^{\gamma})(-Z_i^{\gamma-1} - Z_i^{\gamma-2} e^{-\alpha x_i}) + \gamma Z_i^{\gamma-2} e^{-\alpha x_i}. \tag{4.4}$$

The Fisher information matrix $I(\alpha, \gamma)$ is obtained by taking the expectation of negative of Eqs. (4.1)–(4.3). In practice, $I^{-1}(\alpha, \gamma)$ can be estimated by $I^{-1}(\hat{\alpha}, \hat{\gamma})$. In case of the large samples, we can use the approximation $(\hat{\alpha}, \hat{\gamma}) \sim N((\alpha, \gamma), I_0^{-1}(\hat{\alpha}, \hat{\gamma}))$, where $I_0(\hat{\alpha}, \hat{\gamma})$ is the observed information matrix given by

$$I_0(\hat{\alpha}, \hat{\gamma}) = \begin{bmatrix} -\dfrac{\partial^2 \ell}{\partial \alpha^2} & -\dfrac{\partial^2 \ell}{\partial \alpha \partial \gamma} \\ -\dfrac{\partial^2 \ell}{\partial \alpha \partial \gamma} & -\dfrac{\partial^2 \ell}{\partial \gamma^2} \end{bmatrix}_{(\hat{\alpha}, \hat{\gamma})}. \tag{4.5}$$

The Approximate confidence intervals for α and γ can be obtained, respectively, by

$$\hat{\alpha} \mp z_{\frac{\tau}{2}} \sqrt{\nu_{11}} \quad \texttt{and} \quad \hat{\gamma} \mp z_{\frac{\tau}{2}} \sqrt{\nu_{22}}, \tag{4.6}$$

where ν_{11} and ν_{22} are the elements on the main diagonal of the covariance matrix $I_0^{-1}(\hat{\alpha}, \hat{\gamma})$ and $z_{\frac{\tau}{2}}$ is the standard normal variate.

4.2. Credibility interval

For a specified value of τ, we define the $(1 - \tau) \times 100\%$ $CI(L_\alpha, U_\alpha)$ for α and $(1 - \tau) \times 100\%$ $CI(L_\gamma, U_\gamma)$ for γ, respectively by

$$\begin{aligned} \int_{L_\alpha}^{\infty} \pi_1^*(\alpha \mid \boldsymbol{x}) d\alpha &= 1 - \frac{\tau}{2}, \quad \int_{U_\alpha}^{\infty} \pi_1^*(\alpha \mid \boldsymbol{x}) d\alpha = \frac{\tau}{2}, \\ \int_{L_\gamma}^{\infty} \pi_2^*(\gamma \mid \boldsymbol{x}) d\gamma &= 1 - \frac{\tau}{2}, \quad \int_{U_\gamma}^{\infty} \pi_2^*(\gamma \mid \boldsymbol{x}) d\gamma = \frac{\tau}{2}, \end{aligned} \tag{4.7}$$

where $\pi_1^*(\alpha \mid \boldsymbol{x})$ and $\pi_2^*(\gamma \mid \boldsymbol{x})$ are the marginal density functions of α and γ, respectively. In many cases it will be very difficult to obtain the marginal *PDF* from the posterior density function. So, we will use Gibbs sampler and Metropolis Hastings algorithms to generate $(\alpha_1, \gamma_1), (\alpha_2, \gamma_2), \ldots, (\alpha_N, \gamma_N)$ from $\pi^*(\alpha, \gamma|\boldsymbol{x})$. Using these generated values of α and γ, we have

$$\pi_1^*(\alpha \mid \boldsymbol{x}) = \frac{1}{N} \sum_{i=1}^{N} \pi^*(\alpha, \gamma_i \mid \boldsymbol{x}), \quad \pi_2^*(\gamma \mid \boldsymbol{x}) = \frac{1}{N} \sum_{i=1}^{N} \pi^*(\gamma, \alpha_i, \mid \boldsymbol{x}). \tag{4.8}$$

Substituting from (4.8) in (4.7), we obtain simple formulas to compute the credibility intervals for α and γ in the following forms

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} \int_{L_\alpha}^{\infty} \pi^*(\alpha, \gamma_i \mid \boldsymbol{x}) d\alpha &= 1 - \frac{\tau}{2}, \\ \frac{1}{N} \sum_{i=1}^{N} \int_{U_\alpha}^{\infty} \pi^*(\alpha, \gamma_i \mid \boldsymbol{x}) d\alpha &= \frac{\tau}{2}, \\ \frac{1}{N} \sum_{i=1}^{N} \int_{L_\gamma}^{\infty} \pi^*(\gamma, \alpha_i, \mid \boldsymbol{x}) d\gamma &= 1 - \frac{\tau}{2}, \\ \frac{1}{N} \sum_{i=1}^{N} \int_{U_\gamma}^{\infty} \pi^*(\gamma, \alpha_i, \mid \boldsymbol{x}) d\gamma &= \frac{\tau}{2}. \end{aligned} \tag{4.9}$$

4.3. Highest posterior density interval

A $(1 - \tau) \times 100\%$ *HPD* interval for α is obtained by solving the following two nonlinear equations

$$\begin{aligned} &\frac{1}{N} \sum_{i=1}^{N} \int_{L_\alpha}^{U_\alpha} \pi^*(\alpha, \gamma_i \mid \boldsymbol{x}) d\alpha = 1 - \tau, \\ &\sum_{i=1}^{N} \pi^*(L_\alpha, \gamma_i \mid \boldsymbol{x}) = \sum_{i=1}^{N} \pi^*(U_\alpha, \gamma_i \mid \boldsymbol{x}). \end{aligned} \tag{4.10}$$

Similarly, the $(1 - \tau) \times 100\%$ *HPD* interval for γ is obtained by solving the following two nonlinear equations

$$\begin{aligned} &\frac{1}{N} \sum_{i=1}^{N} \int_{L_\gamma}^{U_\gamma} \pi^*(\gamma, \alpha_i, \mid \boldsymbol{x}) d\gamma = 1 - \tau, \\ &\sum_{i=1}^{N} \pi^*(L_\gamma, \alpha_i, \mid \boldsymbol{x}) = \sum_{i=1}^{N} \pi^*(U_\gamma, \alpha_i, \mid \boldsymbol{x}). \end{aligned} \tag{4.11}$$

5. Numerical computations

In the following, the *MLE's* and *BE's* are compared based on a Monte Carlo simulation study.

1. For a given vector of prior parameters (a_1, a_2, b_1, b_2) we generate α and γ from the prior densities (3.2) and (3.3).
2. For given α and γ obtained in step (1), we generate progressive adaptive type-II censored samples from the *EE* distribution with PDF (1.1).
3. The *MLE's* of α and γ are computed by solving the nonlinear Eqs. (2.6) and (2.7) by using *FSOLVE* routine from *MATLAB* which solve system of nonlinear equations.
4. The *BE* for the vector of parameters $\eta \equiv (\alpha, \gamma)$ under *SE* and *LINEX* loss functions using *MCMC* method are given, respectively, by (3.8) and (3.9).
5. The squared deviations $(\theta^* - \theta)^2$ are computed for different sample sizes, where $(^*)$ stands for an estimate (*ML* or Bayes) and θ stands for the parameters α or γ.
6. The above steps (2–5) are repeated 10,000 times. The *ER's* are computed by averaging the squared deviations.

In our study, we have used three different censoring schemes and the *ER's*) for each scheme are given in Tables 1 and 2. The studied schemes are

Scheme I $R_m = n - m,\ R_i = 0$ for $i \neq m$.
Scheme II $R_1 = n - m,\ R_i = 0$ for $i \neq 1$.
Scheme III $R_{\frac{m+1}{2}} = n - m,\ R_i = 0$ for $i \neq \frac{m+1}{2}$; if m is odd, and $R_{\frac{m}{2}} = n - m,\ R_i = 0$ for $i \neq \frac{m}{2}$; if m is even.

Also, the coverage probability (with nominal level 0.95) for the approximate confidence intervals has been computed and the results are summarized in Table 3.

Table 1
MLE's, *BE's* and *ER's* of the estimates of α and γ for prior parameters $(a_1 = 2.0, a_2 = 1.0, b_1 = 2.0, b_2 = 1.0)$, $\alpha = 0.7126$ and $\gamma = 3.6683$.

T	(n, m)	Scheme	Method		$\hat{\alpha}$	$\hat{\gamma}$	$ER(\hat{\alpha})$	$ER(\hat{\gamma})$
2	(15,5)		ML		0.7965	4.5343	0.1328	5.4930
			B SEL		0.6170	2.9790	0.0245	0.6032
		I	LINEX	$\lambda = -2$	0.6569	4.1286	0.0224	0.7354
				$\lambda = 0.0001$	0.6170	2.9789	0.0245	0.6032
				$\lambda = 2$	0.5799	2.3395	0.0298	1.8201
			ML		0.9263	4.9865	0.1448	6.4441
			B SEL		0.6913	3.0806	0.0159	0.5282
		II	LINEX	$\lambda = -2$	0.7403	4.2478	0.0210	1.0342
				$\lambda = 0.0001$	0.6913	3.0806	0.0159	0.5283
				$\lambda = 2$	0.6472	2.4552	0.0161	1.5511
			ML		0.9063	4.8963	0.1547	5.9581
			B SEL		0.6568	3.0406	0.0189	0.5273
		III	LINEX	$\lambda = -2$	0.7011	4.1777	0.0202	0.7739
				$\lambda = 0.0001$	0.6568	3.0406	0.0189	0.5274
				$\lambda = 2$	0.6162	2.4064	0.0218	1.6505
1	(15,5)		ML		0.3321	2.2024	0.2309	4.2543
			B SEL		0.3981	2.5815	0.1300	1.3789
		I	LINEX	$\lambda = -2$	0.4197	3.5631	0.1230	0.6748
				$\lambda = 0.0001$	0.3981	2.5814	0.1300	1.3790
				$\lambda = 2$	0.3781	2.0471	0.1380	2.7199
			ML		0.8219	4.4334	0.1184	4.8088
			B SEL		0.6616	3.0293	0.0211	0.5978
		II	LINEX	$\lambda = -2$	0.7079	4.1819	0.0242	0.9388
				$\lambda = 0.0001$	0.6616	3.0292	0.0211	0.5979
				$\lambda = 2$	0.6199	2.4124	0.0229	1.6607
			ML		0.3863	2.4076	0.2836	4.6513
			B SEL		0.4281	2.6201	0.1333	1.3005
		III	LINEX	$\lambda = -2$	0.4535	3.6010	0.1314	0.6501
				$\lambda = 0.0001$	0.4281	2.6200	0.1333	1.3006
				$\lambda = 2$	0.4046	2.0811	0.1376	2.6162
2	(50,25)		ML		0.5494	3.0009	0.0817	1.2940
			B SEL		0.5273	2.7782	0.0179	0.5135
		I	LINEX	$\lambda = -2$	0.5396	3.3912	0.0149	0.4637
				$\lambda = 0.0001$	0.5273	2.7782	0.0179	0.5135
				$\lambda = 2$	0.5151	2.3752	0.0111	1.0705
			ML		0.9740	5.4984	0.0917	1.9821
			B SEL		0.8150	3.8300	0.0181	0.3025
		II	LINEX	$\lambda = -2$	0.8328	4.7537	0.0139	0.9823
				$\lambda = 0.0001$	0.8150	3.8299	0.0181	0.3025
				$\lambda = 2$	0.7975	3.2467	0.0131	0.3363
			ML		0.9808	5.6649	0.1262	1.3940
			B SEL		0.7898	3.8679	0.0123	0.4241
		III	LINEX	$\lambda = -2$	0.8078	4.8226	0.0164	0.4776
				$\lambda = 0.0001$	0.7898	3.8679	0.0123	0.4241
				$\lambda = 2$	0.7720	3.2467	0.0190	0.3950
1	(50,25)		ML		0.2614	1.7355	0.2119	1.8925
			B SEL		0.3038	2.0562	0.1147	1.1719
		I	LINEX	$\lambda = -2$	0.3094	2.3950	0.1705	0.5435
				$\lambda = 0.0001$	0.3038	2.0562	0.1147	1.1719
				$\lambda = 2$	0.2983	1.8201	0.1088	2.4913
			ML		0.8741	4.7113	0.0550	1.8830
			B SEL		0.7580	3.5656	0.0133	0.2319
		II	LINEX	$\lambda = -2$	0.7748	4.3836	0.0160	0.9501
				$\lambda = 0.0001$	0.7580	3.5655	0.0133	0.2319
				$\lambda = 2$	0.7417	3.0426	0.0113	0.5206
			ML		0.2647	1.7526	0.2100	1.8374
			B SEL		0.3064	2.0697	0.1131	1.2789
		III	LINEX	$\lambda = -2$	0.3121	2.4124	0.1289	0.5838
				$\lambda = 0.0001$	0.3064	2.0697	0.1131	1.2789
				$\lambda = 2$	0.3009	1.8308	0.1072	1.4539

6. Application

To illustrate the use of the estimation methods proposed in this paper, a real data set from Lawless [10] has been used. These data represent the breakdown time of an insulating fluid between electrodes at a voltage of 34 kv (min). The 19 times to breakdown are 0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71 and 72.89. These real data are analyzed using $EE(\alpha, \gamma)$. The *K–S*, *AIC*, *BIC* and *P*-value have been computed in Table 4.

From Table 4, under significance level (0.05) and using Kolmogorov–Smirnov table, the critical value for *K–S* test statistic is 0.30143 which is greater than the computed *K–S* test statistics.

Also, we can see that the *P*-value corresponding to the *K–S* test statistics for the introduced distribution is greater than the significance level (0.05) which also means that the introduced model fits the real data set well.

Table 2
MLE's, BE's and *ER's* of the estimates of α and γ for prior parameters $(a_1 = 1.5, a_2 = 1.5, b_1 = 2, b_2 = 2)$, $\alpha = 1.1889$ and $\gamma = 4.3973$.

T	(n, m)	Scheme	Method		$\hat{\alpha}$	$\hat{\gamma}$	$ER(\hat{\alpha})$	$ER(\hat{\gamma})$
2	(15,5)		ML		1.2662	5.0701	0.1828	5.0821
			B SEL		1.0477	3.5834	0.0467	0.8728
		I	LINEX	$\lambda = -2$	1.1466	5.3139	0.0384	1.8305
				$\lambda = 0.0001$	1.0476	3.5833	0.0467	0.8729
				$\lambda = 2$	0.9566	2.6666	0.0733	3.0710
			ML		1.3870	5.2302	0.2187	5.1601
			B SEL		1.1236	3.6916	0.0353	0.8049
		II	LINEX	$\lambda = -2$	1.2356	5.4365	0.0472	2.3315
				$\lambda = 0.0001$	1.1236	3.6915	0.0353	0.8050
				$\lambda = 2$	1.0251	2.7930	0.0486	2.6890
			ML		1.2880	5.0351	0.2157	4.7610
			B SEL		1.0658	3.6241	0.0491	0.8249
		III	LINEX	$\lambda = -2$	1.1681	5.3239	0.0473	1.8775
				$\lambda = 0.0001$	1.0658	3.6240	0.0491	0.8250
				$\lambda = 2$	0.9733	2.7227	0.0710	2.8858
1	(15,5)		ML		0.8609	3.5229	0.4095	4.9142
			B SEL		0.8761	3.2766	0.1874	1.4805
		I	LINEX	$\lambda = -2$	0.9571	4.8655	0.1713	1.2123
				$\lambda = 0.0001$	0.8761	3.2766	0.1874	1.4806
				$\lambda = 2$	0.8021	2.4618	0.2174	3.8355
			ML		1.4122	5.3887	0.2728	5.4420
			B SEL		1.1371	3.7241	0.0453	0.7244
		II	LINEX	$\lambda = -2$	1.2507	5.5018	0.0645	2.3857
				$\lambda = 0.0001$	1.1371	3.7240	0.0453	0.7245
				$\lambda = 2$	1.0370	2.8131	0.0535	2.6074
			ML		1.0699	4.1949	0.5225	6.1090
			B SEL		0.9852	3.4369	0.1591	1.1826
		III	LINEX	$\lambda = -2$	1.0798	5.0581	0.1655	1.4933
				$\lambda = 0.0001$	0.9852	3.4368	0.1591	1.1827
				$\lambda = 2$	0.8993	2.5868	0.1739	3.3876
2	(50,25)		ML		1.2637	5.0398	0.0646	1.9845
			B SEL		1.1138	4.0106	0.0226	0.5187
		I	LINEX	$\lambda = -2$	1.1533	5.2518	0.0196	1.7242
				$\lambda = 0.0001$	1.1138	4.0105	0.0226	0.5187
				$\lambda = 2$	1.0747	3.2461	0.0286	1.5019
			ML		1.4679	5.9694	0.1301	1.9005
			B SEL		1.2794	4.5137	0.0262	0.5051
		II	LINEX	$\lambda = -2$	1.3207	5.8919	0.0372	3.5358
				$\lambda = 0.0001$	1.2794	4.5136	0.0262	0.5051
				$\lambda = 2$	1.2395	3.7031	0.0190	0.7346
			ML		1.4386	5.9253	0.1182	2.0762
			B SEL		1.2260	4.4590	0.0170	0.4144
		III	LINEX	$\lambda = -2$	1.2674	5.8025	0.0232	3.0609
				$\lambda = 0.0001$	1.2260	4.4590	0.0170	0.4144
				$\lambda = 2$	1.1854	3.6224	0.0143	0.7995
1	(50,25)		ML		0.6733	2.6025	0.3229	1.8580
			B SEL		0.7149	2.8277	0.2616	2.7430
		I	LINEX	$\lambda = -2$	0.7389	3.5351	0.2430	1.4204
				$\lambda = 0.0001$	0.7149	2.8277	0.2616	2.7431
				$\lambda = 2$	0.6915	2.3894	0.2811	4.1824
			ML		1.5940	6.4673	0.2496	2.0058
			B SEL		1.3671	4.6840	0.0652	0.5229
		II	LINEX	$\lambda = -2$	1.4125	6.1080	0.0871	4.0433
				$\lambda = 0.0001$	1.3671	4.6839	0.0652	0.5228
				$\lambda = 2$	1.3232	3.8291	0.0483	0.5555
			ML		0.8268	3.2126	0.3912	1.8348
			B SEL		0.8412	3.1505	0.2621	2.4133
		III	LINEX	$\lambda = -2$	0.8700	3.9734	0.2551	1.9228
				$\lambda = 0.0001$	0.8412	3.1504	0.2621	2.4134
				$\lambda = 2$	0.8129	2.6346	0.2710	3.5955

For more illustration, Fig. 1 shows the fitted *CDF* and the empirical *CDF* of *EE* distribution, respectively, computed at the estimated parameters where the dotted curve represents the empirical *CDF* curve. Also, Fig. 2 shows the histogram of the real data and the fitted *PDF* of *EE* distribution, respectively, computed at the estimated parameters where the dotted curve represents the fitted *PDF* curve.

We use $m = 10, T = 12,$ and $R = (3, 0, 0, 2, 0, 0, 1, 2, 1, 0)$. In this case the adaptive progressive censored sample is (0.19, 2.78, 3.16, 4.15, 6.50, 7.35, 8.01, 12.06, 31.75, 32.52). The estimates of the parameters α and γ are obtained in Table 5. Moreover, the result of 95% confidence intervals, *CI* and *HPD* intervals of α and γ are given in Tables 6 and 7.

7. Concluding remarks

In this paper, the estimation problem (point and interval) is studied based on adaptive progressive type-II censoring scheme of *EE* distribution. Also, a real data set is introduced as illustrative

Table 3
Coverage probability for the approximate *ML* confidence intervals, *CI* and *HPD* intervals with prior parameters $(a_1 = 1.0, a_2 = 1.0, b_1 = 3.0, b_2 = 3.0)$, $\alpha = 5.3599$, $\gamma = 2.1419$ and $\tau = 0.05$.

T	n	m	Scheme	Approximate intervals		CI		HPD intervals	
				α_{ML}	γ_{ML}	α_B	γ_B	α_B	γ_B
2	20	10	I	88.97%	87.43%	99.06%	99.88%	99.4%	99.8%
			II	86.72%	87.36%	98.16%	99.89%	98.1%	99.8%
			III	90.79%	91.05%	98.48%	99.81%	98.01%	99.91%
1	20	10	I	88.97%	87.43%	99.06%	99.88%	98.88%	99.92%
			II	86.86%	87.43%	98.16%	99.80%	97.40%	99.85%
			III	90.67%	90.71%	98.48%	99.67%	98.23%	99.86%
2	70	50	I	95.77%	94.66%	96.42%	97.11%	96.52%	97.10%
			II	95.83%	95.34%	96.17%	96.87%	99.99%	98.20%
			III	95.81%	94.03%	96.70%	96.57%	99.78%	83.32%
1	70	50	I	95.77%	94.66%	96.42%	97.11%	96.52%	97.10%
			II	96.29%	95.62%	96.49%	96.97%	99.91%	98.35%
			III	95.86%	94.19%	97.11%	96.88%	99.81%	82.91%

Table 4
MLE's of the parameters, the associated Kolmogorov – Smirnov *K–S*, *AIC* and *BIC* values.

Model	*MLE's*	*K–S*	*P*-value	*AIC*	*BIC*
$EE(\alpha, \gamma)$	$\hat{\alpha} = 0.0534985, \hat{\gamma} = 0.682536$	0.188626	0.433338	170.192	172.081

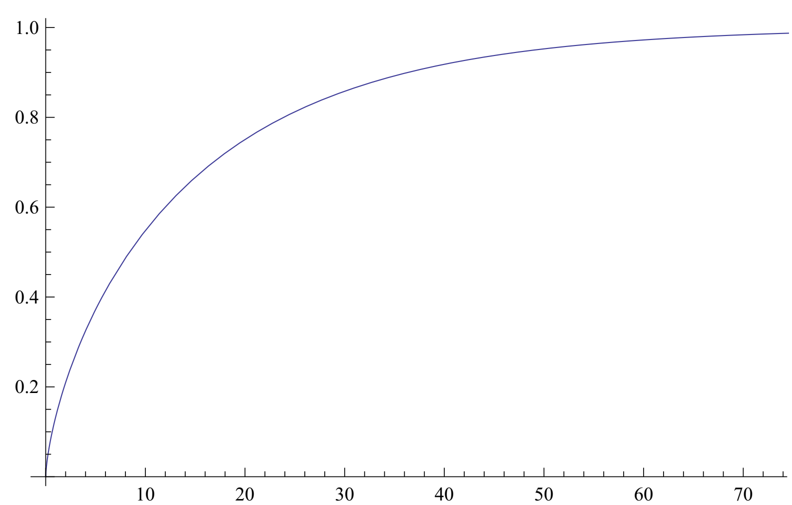

Fig. 1. The fitted *CDF* and the empirical *CDF* of *EE* distribution.

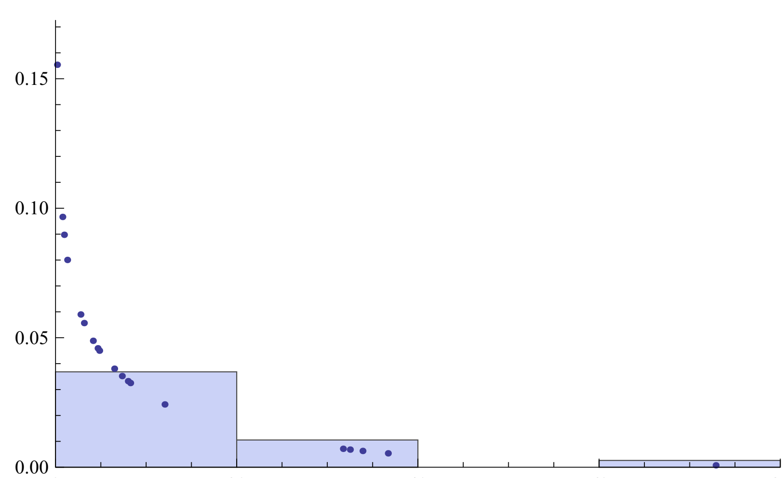

Fig. 2. The histogram of the real data and the fitted *PDF* of *EE* distribution.

Table 5
The *MLE's* of α and γ for prior parameters $(a_1 = 1.0, a_2 = 1.0, b_1 = 1.0, b_2 = 2.0)$.

T	(n, m)	Method		$\hat{\alpha}$	$\hat{\gamma}$
12	(19,10)	ML		0.9339	1.0000
		B SEL		0.0504	1.0172
		LINEX	$\lambda = -2$	0.0509	1.1635
			$\lambda = 0.0001$	0.0504	1.0172
			$\lambda = 2$	0.0499	0.9129

Table 6
The Lower and Upper Bounds for the approximate *ML* confidence interval with prior parameters $(a_1 = 1.0, a_2 = 1.0, b_1 = 1.0, b_2 = 2.0)$.

n	m	α_{ML}			γ_{ML}		
		L	U	Length	L	U	Length
19	10	0.2596	1.6082	1.3487	0.5343	1.4657	0.9313

Table 7
The Lower and Upper Bounds for The *CI* and *HPD* intervals, when $n = 19$ and $m = 10$.

	CI		*HPD* intervals	
	α_B	γ_B	α_B	γ_B
L	0.0081	0.3641	0.0105	0.3216
U	0.5000	1.2876	0.5000	1.2143
Length	0.4919	0.9235	0.4895	0.8927

example. A simulation study is carried out to examine and compare the performance of the proposed methods for different sample sizes and different censoring schemes. From the results, we observe the following.

1. All of the results obtained in this article can be specialized to:
   (a) When $T \to \infty$, the usual progressive type-II censoring scheme has been obtained.
   (b) When $T \to 0$, the usual type-II censoring scheme has been obtained with the complete sample case by taking $(m = n, R_i = 0, i = 1, 2, \dots, m)$.
2. Tables 1 and 2 show that the *BE′s* of all parameters relative to asymmetric loss functions (*LINEX*) are sensitive to the value of the shape parameter λ, also, the *BE′s* based on symmetric and asymmetric loss functions are better than the *MLE′s*.
3. In all cases, as λ tends to zero, the *ER′s* of the *BE′s* using the LINEX loss function are the same using the *SEL* function.
4. From Table 3, we see that the coverage probabilities of the approximate confidence intervals are close to the desired level of 0.95 for α_{ML} and γ_{ML} in case of the large sample size, but not close to the desired level in case of the small sample size.
5. Also, from Table 3, we see that the coverage probabilities of Bayes confidence intervals and *HPD* intervals are nearly close to the desired level of 0.95 for α_B and γ_B in most cases.

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