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# Characterization of the generalized Pareto distribution by general progressively Type-II right censored order statistics

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ABSTRACT

In this article, we establish recurrence relations for single and product moments based on general progressively Type-II right censored order statistics (GPTIICOS). Characterization for generalized Pareto distribution (GPD) using relation between probability density function and distribution function and using recurrence relations of single and product moments of GPTIICOS are also obtained. Further, the results are specialized to the progressively Type-II right censored order statistics (PTIICOS) and specialized to the Pareto, uniform and exponential distributions.

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## 1. Introduction

In failure data analysis, it is common that some individuals cannot be observed for the full failure times. GPTIICOS is a useful and more general scheme in which a specific fraction of individuals at risk may be removed from the study at each of several ordered failure times. Progressively censored samples have been considered, among others, by Davis and Feldstein [9], Balakrishnan et al. [6], and Guilbaud [10]. This scheme of censoring was generalized by Balakrishnan and Sandhu [5] as follows: at time $X_0 \equiv 0$, $n$ units are placed on test; the first $r$ failure times, $X_1, \ldots, X_r$, are not observed; at time $X_i + 0$, where $X_i$ is the $i$th ordered failure time $(i = r+1, \ldots, m-1)$, $R_i$ units are removed from the test randomly, so prior to the (i+1)th failure there are $n_i = n - - i - \sum_{j=r+1}^{i} R_j$ units on test; finally, at the time of the $m$th failure, $X_m$, the experiment is terminated, i.e., the remaining $R_m$ units are removed from the test. The $R_i$'s, $m$ and $r$ are prespecified integers which must satisfy the conditions: $0 \le r < m \le n$, $0 \le R_i \le n_{i-1}$ $for$ $i = r+1, \ldots, m-1$ with $n_r = n - r$ and $R_m = n_{m-1} - 1$, (see Fernandez [4]).

If the failure times are based on an absolutely continuos distribution function (cdf) $F$ with probability density function (pdf) $f$, the joint probability density function of the GPTIICOS failure times $X_{r+1:m:,n}, X_{r+2:m:,n}, \ldots, X_{m:m:,n}$, is given by

$$f_{X_{r+1:m:,n},X_{r+2:m:,n},\ldots,X_{m:m:,n}}(x_{r+1}, x_{r+2}, \ldots, x_m) = K_{(n,m-1)}[F(x_{r+1}, \theta)]^r$$

$$\times \prod_{i=r+1}^{m} f(x_i, \theta)[1 - F(x_i, \theta)]^{R_i}, x_{r+1} < x_{r+2} < \ldots < x_m, \quad (1.1)$$

where,

$$K_{(n,m-1)} = \frac{n!}{r!(n-r)!}\left(\prod_{j=r}^{m-1} n_j\right), \ n_i = n - i - \sum_{j=r+1}^{i} R_j,$$

$i = r+1, \ldots, m-1$.

Mohie El-Din et al. [16] derived recurrence relations for expectations of functions of order statistics for doubly truncated distributions and their applications. Aggarwala and Balakrishnan [3] derived recurrence relations for single and product moments of PTIICOS from exponential, Pareto and power function distributions and their truncated forms. Hashemi and Asadi [11] derived some characterization results on generalized Pareto distribution based on progressive Type-II right censoring via conditional moments. Abd El-Aty and Mohie El-Din [1] derived recurrence relations for single and double moments of generalized order statistics from the inverted linear exponential distribution and any continuous function. Bermudez, and Kotz [7,8] derived Parameter estimation of the generalized Pareto distribution. Mohie El-Din and Kotb [15] derived recurrence relation for product moments and characteriza-

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tion of generalized order statistics based on a general class of doubly truncated Marshall–Olkin extended distributions. Mohie El-Din et al. [14] discussed estimation for parameters of Feller–Pareto distribution from PTIICOS and some characterizations. Abdel-Hamid and Al-Hussaini [2] derived Inference and optimal design based on step-partially accelerated life tests for the generalized Pareto distribution under progressive Type-I censoring.

Imtiyaz et al. [13] introduced some characterization results for another form of generalized Pareto distribution using generalized order statistics.

Throughout this paper, we introduce recurrence relations among single and product moments based on GPTIICOS. Moreover, we specialized the results to the PTIICOS and to the Pareto, uniform and exponential distribution. Also characterization for GPD using recurrence relations of single and product moments based on GPTIICOS, are obtained.

Let $X_{r+1:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)} < X_{r+2:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)} < \ldots < X_{m:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)}$ be the $m$ ordered observed failure times in a sample of size $(n-r)$ under GPTIICOS scheme from the GPD with probability density function (pdf) given by

$$f(x,\gamma,\theta,\sigma)=\begin{cases}\frac{1}{\sigma}\left[1-\frac{\gamma}{\sigma}(x-\theta)\right]^{\left(\frac{1}{\gamma}-1\right)} & \sigma>0, \quad \theta\in R,\ \gamma\neq 0,\\ \frac{1}{\sigma}e^{\frac{-(x-\theta)}{\sigma}} & \sigma>0, \quad \theta\in R,\ \gamma=0.\end{cases} \tag{1.2}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x,\gamma,\theta,\sigma)=\begin{cases}1-\left[1-\frac{\gamma}{\sigma}(x-\theta)\right]^{\left(\frac{1}{\gamma}\right)} & \gamma\neq 0,\\ 1-e^{\frac{-(x-\theta)}{\sigma}} & \gamma=0.\end{cases} \tag{1.3}$$

For $\gamma\le 0$ we have $\theta\le x<\infty$ and for $\gamma>0$ we have $\theta\le x<\theta+\frac{\sigma}{\gamma}$.

The distribution function given by Imtiyaz et al. [13] is defined as

$$F(x)=\left[1-(1+m)x^{-\alpha}\right]^{\frac{1}{1+m}}, \quad (1+m)^{\frac{1}{\alpha}}\langle x\langle\infty,\ \alpha\rangle 0,\ m\rangle -1,$$

can be transferred to survival function of the first part of Eq. (1.3) where $\gamma>0$, using the following transformation,

Let $x^{-\alpha}=\frac{(y-\theta)}{\sigma}$ *and* $(1+m)=\gamma$ *then*

$$F(y)=\left[1-\gamma\frac{(y-\theta)}{\sigma}\right]^{\frac{1}{\gamma}},\ \theta<y\left\langle\theta+\frac{\sigma}{\gamma},\ \gamma\right\rangle 0.$$

which is called generalized Pareto distribution given by (Bermudez, and Kotz [7,8] and Abd El-Hamid et al.[2]).

**Note (1)**

We can obtain some distributions as particular cases from generalized Pareto distribution with pdf given by (1.2) as follows:

1- For $\gamma<0$, Eq. (1.2) reduces to the case of Pareto distribution.
2- For $\gamma<0$ *and* $\gamma\theta=-\sigma$, Eq. (1.2) reduces to the case of ordinary Pareto distribution.
3- For $\gamma=1$, Eq. (1.2) reduces to the case of uniform distribution.
4- For $\gamma=0$, Eq. (1.2) reduces to the case of exponential distribution.

It may be noticed that from (1.2) and (1.3) the relation between pdf and cdf is given by,

$$[1-F(x)]=\begin{cases}[\sigma+\gamma\theta-\gamma x]f(x) & \gamma\neq 0,\\ \sigma f(x) & \gamma=0.\end{cases} \tag{1.4}$$

For any continuous distribution, we denote the $i$th single moments of the GPTIICOS in view of Eq. (1.1) as

$$\begin{aligned}\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i)}}&=E\left[X_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)}\right]^i=K_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}<\ldots<x_m<\infty}x_q^i[F(x_{r+1})]^r\\&\times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}f(x_{r+2})[1-F(x_{r+2})]^{R_{r+2}}\ldots f(x_m)\\&\times[1-F(x_m)]^{R_m}dx_{r+1}\ldots dx_m,\end{aligned} \tag{1.5}$$

when $i=1$, the superscript in the notation of the mean of the GPTIICOS may be omitted without any confusion, and the $i$th and $j$th product moments as

$$\begin{aligned}\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j)}}&=E\left[X_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^i}\ X_{s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^j}\right]\\&=K_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}<\ldots<x_m<\infty}x_q^ix_s^j[F(x_{r+1})]^r\\&\times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}f(x_{r+2})\\&\times[1-F(x_{r+2})]^{R_{r+2}}\ldots f(x_m)[1-F(x_m)]^{R_m}\\&\times dx_{r+1}\ldots dx_m.\end{aligned} \tag{1.6}$$

## 2. Recurrence relations of single and product moments

In this section, we introduce the recurrence relations for single and product moments based on GPTIICOS.

In the next theorem we introduce the recurrence relations for single moments based on GPTIICOS.

**Theorem 2.1.** *If* $X_{r+1:n}\le X_{r+2:n}\le\ldots\le X_{n:n}$ *be the order stasistics of a random sample of size* $(n-r)$ *following GPD, for* $r+2\le q\le m-1$, $m\le n$ *and* $i\ge 0$, *then*

$$\begin{aligned}&[(R_q+1)+\gamma(i+1)]\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}}\\&=(\sigma+\gamma\theta)(i+1)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i)}}+\left(n-R_{r+1}-\ldots-R_{q-1}-q+1\right)\\&\times\mu_{q-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+1)}}-(n-R_{r+1}-\ldots-R_q-q)\\&\times\mu_{q:m-1:n}^{(R_{r+1},R_{r+2},\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+1)}}.\end{aligned} \tag{2.1}$$

**Proof.**

From Eqs. (1.4) and (1.5), we get

$$\begin{aligned}&(\sigma+\gamma\theta)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i)}}-\gamma\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}}\\&=K_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}<\ldots<x_{q-1}<x_{q+1}<\ldots<x_m<\infty}Z_1\left(x_{q-1},x_{q+1}\right)\\&\times[F(x_{r+1})]^rf(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f\left(x_{q-1}\right)\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}}\\&\times f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}}\ldots f(x_m)[1-F(x_m)]^{R_m}\\&\times dx_{r+1}\ldots dx_{q-1}dx_{q+1}\ldots dx_m,\end{aligned} \tag{2.2}$$

where

$$Z_1\left(x_{q-1},x_{q+1}\right)=\int_{x_{q-1}}^{x_{q+1}}x_q^i[1-F(x_q)]^{R_q+1}\ dx_q. \tag{2.3}$$

Upon integrating the integral in (2.3) by parts, we get

$$\begin{aligned}Z_1\left(x_{q-1},x_{q+1}\right)&=\frac{x_{q+1}^{i+1}\left[1-F\left(x_{q+1}\right)\right]^{R_q+1}-x_{q-1}^{i+1}\left[1-F\left(x_{q-1}\right)\right]^{R_q+1}}{i+1}\\&+\left(\frac{R_q+1}{i+1}\right)\int_{x_{q-1}}^{x_{q+1}}x_q^{i+1}f(x_q)[1-F(x_q)]^{R_q}dx_q.\end{aligned} \tag{2.4}$$

Substituting by Eq. (2.4) in Eq. (2.2) and simplifying, yields Eq. (2.1).

This completes the proof. □

**Special cases**

1- Theorem (2.1) will be valid for the PTIICOS as a special case from the GPTIICOS when $r=0$,

$$[(R_q+1)+\gamma(i+1)]\mu_{q:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+1)}}$$
$$=(\sigma+\gamma\theta)(i+1)\mu_{q:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}}+\left(n-R_1-\ldots-R_{q-1}-q+1\right)$$
$$\times\mu_{q-1:m-1:n}^{\left(R_1,R_2,\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m\right)^{(i+1)}}-(n-R_1-\ldots-R_q-q)$$
$$\times\mu_{q:m-1:n}^{\left(R_1,R_2,\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m\right)^{(i+1)}}.$$

2- For $r=0$ *and* $q=m$

$$[(R_m+1)+\gamma(i+1)]\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+1)}}$$
$$=(\sigma+\gamma\theta)(i+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}}+(n-R_1-\ldots-R_{m-1}-m+1)$$
$$\times\mu_{m-1:m-1:n}^{\left(R_1,R_2,\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m\right)^{(i+1)}}.$$

3- For $r=0$ *and* $2\le m\le n$

$$[(R_1+1)+\gamma(i+1)]\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+1)}}$$
$$=(\sigma+\gamma\theta)(i+1)\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(i)}}$$
$$-(n-R_1-1)\mu_{q:m-1:n}^{((R_1+R_2+1),R_3,\ldots,R_m)^{(i+1)}},$$

and for $r=0,\ m=1$ *and* $n=1,2,\ldots$

$$[1+\gamma(i+1)]\mu_{1:1:n}^{(n-1)^{(i+1)}}=(\sigma+\gamma\theta)(i+1)\mu_{1:1:n}^{(n-1)^{(i)}}.$$

4- For $r=0,\ m=1,\ n=1$ *and* $R_1=\ldots=R_m=0$,

$$\mu^{(i+1)}=\frac{(\sigma+\gamma\theta)(i+1)}{[1+\gamma(i+1)]}\mu^{(i)}.$$

(i) For $i=0$ we obtained the expected value, $\mu^{(1)}=E[X]=\mu=\frac{(\sigma+\gamma\theta)}{[1+\gamma]}$.

(ii) For $i=1$ we obtained the second moment, $\mu^{(2)}=E[X^2]=\frac{2(\sigma+\gamma\theta)}{(1+2\gamma)}\mu$.

(iii) From (i) and (ii), we obtained the variance, $Var(X)=\frac{(\sigma+\gamma\theta)^2}{(1+2\gamma)(1+\gamma)^2}$.

(iv) For $i=2$ we obtained the third moment, $\mu^{(3)}=\frac{3(\sigma+\gamma\theta)}{(1+3\gamma)}\mu^{(2)}$, and we also obtained the skewness, $\gamma_1=\frac{\mu^{(3)}-3\mu[\ \mu^{(2)}-\mu^2]-\mu^3}{[[\ \mu^{(2)}-\mu^2]]^{\frac{3}{2}}}$.

(v) For $i=3$ we obtained the fourth moment, $\mu^{(4)}=\frac{4(\sigma+\gamma\theta)}{(1+4\gamma)}\mu^{(3)}$, and the kurtosis, $Kurt[X]=\frac{\mu^{(4)}-3\mu^4+6\mu^2\ \mu^{(2)}-4\mu\ \mu^{(3)}}{[\ \mu^{(2)}-\mu^2]^2}$.

In the next two theorems, we introduce recurrence relations for product moments based on GPTIICOS.

**Theorem 2.2.** *If* $X_{r+1:n}\le\ldots\le X_{n:n}$ *be the order stasistics of a random sample of size* $(n-r)$ *following GPD, for* $r+1\le q<s\le m-1,\ m\le n$ *and* $i,j\ge 0$, *then*

$$[(R_q+1)+\gamma(i+1)]\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1,j)}}$$
$$=(\sigma+\gamma\theta)(i+1)\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j)}}+\left(n-R_{r+1}-\ldots-R_{q-1}-q+1\right)$$
$$\times\mu_{q-1,s-1:m-1:n}^{\left(R_{r+1},R_{r+2},\ldots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m\right)^{(i+1,j)}}-(n-R_{r+1}-\ldots-R_q-q)$$
$$\times\mu_{q,s-1:m-1:n}^{\left(R_{r+1},R_{r+2},\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m\right)^{(i+1,j)}}.\tag{2.5}$$

**Proof.**

Similarly as proved in Theorem 2.1. □

**Special case**

This theorem will be valid for the PTIICOS as a special case from the GPTIICOS when $r=0$,

$$[(R_q+1)+\gamma(i+1)]\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i+1,j)}}$$
$$=(\sigma+\gamma\theta)(i+1)\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}}+\left(n-R_1-\ldots-R_{q-1}-q+1\right)$$
$$\times\mu_{q-1,s-1:m-1:n}^{\left(R_1,R_2,\ldots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m\right)^{(i+1,j)}}-(n-R_1-\ldots-R_q-q)$$
$$\times\mu_{q,s-1:m-1:n}^{\left(R_1,R_2,\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m\right)^{(i+1,j)}}.$$

**Theorem 2.3.** *If* $X_{r+1:n}\le\ldots\le X_{n:n}$ *be the order stasistics of a random sample of size* $(n-r)$ *following GPD, for* $r+1\le q<s\le m-1,\ m\le n$ *and* $i,j\ge 0$, *then*

$$[(R_s+1)+\gamma(j+1)]\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j+1)}}$$
$$=(\sigma+\gamma\theta)(j+1)\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j)}}+(n-R_{r+1}-\ldots-R_{s-1}-s+1)$$
$$\times\mu_{q,s-1:m-1:n}^{(R_{r+1},\ldots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+1)}}-(n-R_{r+1}-\ldots-R_s-s)$$
$$\times\mu_{q,s:m-1:n}^{(R_{r+1},\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+1)}}.\tag{2.6}$$

**Proof.**

Similarly as proved in Theorem 2.1. □

**Special case**

For $r=0$, we obtain the recurrence relation of PTIICOS.

$$[(R_s+1)+\gamma(j+1)]\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j+1)}}$$
$$=(\sigma+\gamma\theta)(j+1)\mu_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(i,j)}}+(n-R_1-\ldots-R_{s-1}-s+1)$$
$$\times\mu_{q,s-1:m-1:n}^{(R_1,\ldots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+1)}}-(n-R_1-\ldots-R_s-s)$$
$$\times\mu_{q,s:m-1:n}^{(R_1,\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+1)}},$$

and for $s=m$

$$[(R_m+1)+\gamma(j+1)]\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j+1)}}$$
$$=(\sigma+\gamma\theta)(j+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j)}}+(n-R_1-\ldots-R_{m-1}-m+1)$$
$$\times\mu_{m-1:m-1:n}^{(R_1,\ldots,R_{m-2},(R_{m-1}+R_m+1))^{(j+1)}}.$$

## 3. The characterization

In this section, we introduce the characterization of the GPD using the relation between pdf and cdf and using recurrence relations for single and product moments based on GPTIICOS.

### 3.1. Characterization via differential equation for the GPD

In the next theorem, we introduce the characterization of the GPD using relation between pdf and cdf.

**Theorem 3.1.** *Let X be a continuous random variable with pdf* $f(\cdot)$, *cdf* $F(\cdot)$ *and survival function* $[1-F(\cdot)]$. *Then X has GPD iff*

$$[1-F(x)]=\begin{cases}[\sigma-\gamma\mu+\gamma x]f(x) & \gamma\neq 0,\\ \sigma f(x) & \gamma=0.\end{cases}\tag{3.1}$$

**Proof.**

**Necessity:**
From Eqs. (1.2) and (1.3) we can easily obtain Eq. (3.1).
**Sufficiency:**
Suppose that Eq. (3.1) is true. Then we have:

$$\frac{-d[1-F(x)]}{1-F(x)}=\begin{cases}\frac{1}{\sigma-\gamma\theta+\gamma x}dx & \gamma\neq 0\\ \frac{1}{\sigma}dx & \gamma=0.\end{cases}$$

By integrating, we get

$$-\ln|1-F(x)| = \begin{cases} \dfrac{1}{\gamma}\ln|\sigma-\gamma\theta+\gamma x| + C & \gamma\neq 0 \\ \dfrac{x}{\sigma}+C & \gamma=0, \end{cases} \tag{3.2}$$

where C is an arbitrary constant.

Now, since $[1-F(\mu)]=1$, then putting $x=\theta$ in Eq. (3.2), we get $C=\begin{cases}\frac{-1}{\gamma}\ln|\sigma| & \gamma\neq 0\\ \frac{-\theta}{\sigma} & \gamma=0.\end{cases}$ therefore,

$$\ln|[1-F(x)]| = \begin{cases} \dfrac{-1}{\gamma}\ln\left|\dfrac{\sigma-\gamma\theta+\gamma x}{\sigma}\right| & \gamma\neq 0 \\ -\dfrac{x-\theta}{\sigma} & \gamma=0, \end{cases}$$

hence,

$$F(x) = \begin{cases} 1-\left[\dfrac{\sigma-\gamma\theta+\gamma x}{\sigma}\right]^{\frac{-1}{\gamma}} & \gamma\neq 0 \\ 1-e^{\dfrac{-(x-\theta)}{\sigma}}. & \gamma=0 \end{cases}$$

That is the distribution function of GPD. This completes the proof. □

### 3.2. Characterization via recurrence relation for single moments

In the next theorem, we will introduce the characterization of the GPD using recurrence relations for single moments based on GPTIICOS.

**Theorem 3.2.** *Let* $X_{r+1:n}\leq X_{r+2:n}\leq\ldots\leq X_{n:n}$ *be the order stasistics of a random sample of size* $(n-r)$*. Then X has GPD iff, for* $r+2\leq q\leq m-1,\ m\leq n$ *and* $i\geq 0$,

$$\begin{aligned}
&[(R_q+1)+\gamma(i+1)]\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}} \\
&= (\sigma+\gamma\theta)(i+1)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i)}} + (n-R_{r+1}-\ldots-R_{q-1}-q+1) \\
&\quad\times\mu_{q-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+1)}} - (n-R_{r+1}-\ldots-R_q-q) \\
&\quad\times\mu_{q:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+1)}}.
\end{aligned} \tag{3.3}$$

**Proof.**

**Necessity:**
Theorem 2.1 proved the necessary part of this theorem.
**Sufficiency:**
Assuming that Eq. (3.3) holds, then we have:

$$\begin{aligned}
&(\sigma+\gamma\theta)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i)}} - (\gamma)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}} \\
&= \frac{(R_q+1)}{(i+1)}\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}} - \frac{(n-R_{r+1}-\ldots-R_{q-1}-q+1)}{(i+1)} \\
&\quad\times\mu_{q-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+1)}} + \frac{(n-R_{r+1}-\ldots-R_q-q)}{(i+1)} \\
&\quad\times\mu_{q:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+1)}},
\end{aligned} \tag{3.4}$$

where,

$$\begin{aligned}
&\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}} \\
&= K_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}\ldots<x_{q-1}<x_{q+1}<\ldots<x_m<\infty} Z_3\left(x_{q-1},x_{q+1}\right) \\
&\quad\times[F(x_{r+1})]^r f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f\left(x_{q-1}\right)\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}} \\
&\quad\times f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}}\ldots f(x_m)[1-F(x_m)]^{R_m} \\
&\quad\times dx_{r+1}\ldots dx_{q-1}dx_{q+1}\ldots dx_m,
\end{aligned} \tag{3.5}$$

where,

$$Z_3\left(x_{q-1},x_{q+1}\right) = \int_{x_{q-1}}^{x_{q+1}} x_q^{i+1}f(x_q)[1-F(x_q)]^{R_q}dx_q. \tag{3.6}$$

Upon integrating the integral in (3.6) by parts, we get

$$\begin{aligned}
&Z_3\left(x_{q-1},x_{q+1}\right) \\
&= \frac{-1}{R_q+1}x_{q+1}^{i+1}\left[1-F\left(x_{q+1}\right)\right]^{R_q+1} + \frac{1}{R_q+1}x_{q-1}^{i+1}\left[1-F\left(x_{q-1}\right)\right]^{R_q+1} \\
&\quad+\frac{i+1}{R_q+1}\int_{x_{q-1}}^{x_{q+1}} x_q^i[1-F(x_q)]^{R_q+1}dx_q.
\end{aligned} \tag{3.7}$$

Substituting in Eq. (3.5), we get

$$\begin{aligned}
&\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}} \\
&= \frac{i+1}{R_q+1}K_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}\ldots<x_{q-1}<x_{q+1}<\ldots<x_m<\infty}[F(x_{r+1})]^r \\
&\quad\times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f\left(x_{q-1}\right)\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}}\int_{x_{q-1}}^{x_{q+1}} \\
&\quad\times x_q^i[1-F(x_q)]^{R_q+1}dx_q f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}}\ldots f(x_m) \\
&\quad\times[1-F(x_m)]^{R_m}dx_{r+1}\ldots dx_{q-1}dx_{q+1}\ldots dx_m \\
&\quad+\frac{K_{(n,m-1)}}{R_q+1}\iint\cdots\int_{0<x_{r+1}\ldots<x_{q-1}<x_{q+1}<\ldots<x_m<\infty} x_{q-1}^{i+1}[F(x_{r+1})]^r \\
&\quad\times f(x_{r+1})\times[1-F(x_{r+1})]^{R_{r+1}}\ldots f\left(x_{q-1}\right)\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}+R_q+1} \\
&\quad\times f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}}\ldots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\ldots \\
&\quad\times dx_{q-1}dx_{q+1}\ldots dx_m - \frac{K_{(n,m-1)}}{R_q+1} \\
&\quad\times\iint\cdots\int_{0<x_{r+1}\ldots<x_{q-1}<x_{q+1}<\ldots<x_m<\infty} x_{q+1}^{i+1}[F(x_{r+1})]^r \\
&\quad\times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_q+R_{q+1}+1} \\
&\quad\times f\left(x_{q+2}\right)\left[1-F\left(x_{q+2}\right)\right]^{R_{q+2}}\ldots f(x_m)[1-F(x_m)]^{R_m} \\
&\quad\times dx_{r+1}\ldots dx_{q-1}dx_{q+1}\ldots dx_m \\
&= K_{(n,m-1)}\frac{i+1}{R_q+1}\iint\cdots\int_{0<x_{r+1}\ldots<x_{q-1}<x_{q+1}<\ldots<x_m<\infty}[F(x_{r+1})]^r\ldots \\
&\quad\times\int_{x_{q-1}}^{x_{q+1}} x_q^i[1-F(x_q)]^{R_q+1}dx_q f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f\left(x_{q-1}\right) \\
&\quad\times\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}}\times f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}} \\
&\quad\ldots f(x_m)[1-F(x_m)]^{R_m}dx_{r+1}\ldots dx_{q-1}dx_{q+1}\ldots dx_m \\
&\quad-\frac{\left(n-R_{r+1}-\ldots-R_{q-1}-q+1\right)}{R_q+1} \\
&\quad\times\mu_{q-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m)^{(i+1)}} \\
&\quad+\frac{(n-R_{r+1}-\ldots-R_q-q)}{R_q+1} \\
&\quad\times\mu_{q:m-1:n}^{(R_{r+1},R_{r+2},\ldots,(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m)^{(i+1)}}.
\end{aligned} \tag{3.8}$$

Substituting for $\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}}$ from Eq. (3.8) in Eq. (3.4), we get

$$\begin{aligned}
&(\sigma+\gamma\theta)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i)}} - (\gamma)\mu_{q:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1)}} \\
&= K_{(n,m-1)}\iint\cdots\int_{0<x_{r+1}<\ldots<x_m<\infty} x_q^i[F(x_{r+1})]^r[1-F(x_q)]^{R_q+1} \\
&\quad\times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}}\ldots f\left(x_{q-1}\right)
\end{aligned}$$

$$\times\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}} f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}} \times \ldots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1}\ldots dx_m. \tag{3.9}$$

We get

$$K_{(n,m-1)} \iint \cdots \int_{0<x_{r+1}<\ldots<x_m<\infty} x_q^i(\sigma+\gamma\theta-\gamma x_q) \times[F(x_{r+1})]^r f(x_q)[1-F(x_q)]^{R_q} \times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \ldots f\left(x_{q-1}\right)\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}} f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}} \ldots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1}\ldots dx_m$$
$$= K_{(n,m-1)} \iint \cdots \int_{0<x_{r+1}<\ldots<x_m<\infty} x_q^i[F(x_{r+1})]^r[1-F(x_q)]^{R_q+1}\ldots \times f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} f\left(x_{q-1}\right)\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}} \times f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}} \ldots \times f(x_m)[1-F(x_m)]^{R_m} \times dx_{r+1}\ldots dx_m. \tag{3.10}$$

We get

$$K_{(n,m-1)} \iint \cdots \int_{0<x_{r+1}<\ldots<x_m<\infty} x_q^i \times[(\sigma+\gamma\theta-\gamma x_q)f(x_q)-[1-F(x_q)]][F(x_{r+1})]^r \times[1-F(x_q)]^{R_q} f(x_{r+1})[1-F(x_{r+1})]^{R_{r+1}} \ldots f\left(x_{q-1}\right)\left[1-F\left(x_{q-1}\right)\right]^{R_{q-1}} f\left(x_{q+1}\right)\left[1-F\left(x_{q+1}\right)\right]^{R_{q+1}} \ldots f(x_m)[1-F(x_m)]^{R_m} dx_{r+1}\ldots dx_m = 0. \tag{3.11}$$

Using Muntz–Szasz theorem, [See, Hwang and Lin [12]], we get

$$[1-F(x_q)] = [\sigma+\gamma\theta-\gamma x_q]f(x_q)\ .$$

Using Theorem 3.1, we get

$$F(x)=\begin{cases}1-\left[\frac{\sigma+\gamma\theta-\gamma x}{\sigma}\right]^{\frac{1}{\gamma}} & \gamma\neq 0\\ 1-e^{\frac{-(x-\theta)}{\sigma}} & \gamma=0.\end{cases}$$

That is the distribution function of GPD. This completes the proof. □

### 3.3. Characterization via recurrence relation for product moments

In the next two theorems, we will introduce the characterization of GPD using recurrence relations for product moments based on GPTIICOS.

**Theorem 3.3.** *Let* $X_{r+1:n} \le X_{r+2:n} \le \ldots \le X_{n:n}$ *be the order stasistics of a random sample of size* $(n-r)$. *Then X has GPD iff, for* $r+1 \le q < s \le m-1$, $m \le n$ *and* $i, j \ge 0$,

$$[(R_q+1)+\gamma(i+1)]\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i+1,j)}} = (\sigma+\gamma\theta)(i+1)\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j)}} +\left(n-R_{r+1}-\ldots-R_{q-1}-q+1\right) \times\mu_{q-1,s-1:m-1:n}^{\left(R_{r+1},R_{r+2},\ldots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\ldots,R_m\right)^{(i+1,j)}} -(n-R_{r+1}-\ldots-R_q-q) \times\mu_{q,s-1:m-1:n}^{\left(R_{r+1},R_{r+2},\ldots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\ldots,R_m\right)^{(i+1,j)}}. \tag{3.12}$$

**Proof.**

**Necessity:**
Theorem 2.2 proved the necessary part of this theorem.
**Sufficiency:**
Similarly as proved in Theorem 3.2 we obtain the distribution function of GPD given by

$$F(x)=\begin{cases}1-\left[\frac{\sigma+\gamma\theta-\gamma x}{\sigma}\right]^{\frac{1}{\gamma}} & \gamma\neq 0\\ 1-e^{\frac{-(x-\theta)}{\sigma}} & \gamma=0.\end{cases}$$

This completes the proof. □

**Theorem 3.4.** *Let* $X_{r+1:n} \le \ldots \le X_{n:n}$ *be the order stasistics of a random sample of size* $(n-r)$. *Then X has GPD iff, for* $r+1 \le q < s \le m-1$, $m \le n$ *and* $i, j \ge 0$,

$$[(R_s+1)+\gamma(j+1)]\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j+1)}} = (\sigma+\gamma\theta)(j+1)\mu_{q,s:m:n}^{(R_{r+1},R_{r+2},\ldots,R_m)^{(i,j)}} +(n-R_{r+1}-\ldots-R_{s-1}-s+1) \times\mu_{q,s-1:m-1:n}^{(R_{r+1},R_{r+2},\ldots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)^{(i,j+1)}} -(n-R_{r+1}-\ldots-R_s-s) \times\mu_{q,s:m-1:n}^{(R_{r+1},R_{r+2},\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)^{(i,j+1)}}. \tag{3.13}$$

**Proof.**

**Necessity:**
Theorem 2.3 proved the necessary part of this theorem.
**Sufficiency:**
Similarly as proved in Theorem 3.2 we obtain the distribution function of GPD given by

$$F(x)=\begin{cases}1-\left[\frac{\sigma+\gamma\theta-\gamma x}{\sigma}\right]^{\frac{1}{\gamma}} & \gamma\neq 0\\ 1-e^{\frac{-(x-\theta)}{\sigma}} & \gamma=0.\end{cases}$$

This completes the proof. □

**Note (2)**
The same theorems can be obtained as a special cases for some distributions presented in Note (1).

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