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Original Article

A remark on the stability and boundedness criteria in retarded Volterra integro-differential equations

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ABSTRACT

In this article, the authors obtain some clear assumptions for the asymptotic stability (AS) and boundedness (B) of solutions of non-linear retarded Volterra integro-differential equations (VIDEs) of first order by constructing a new Lyapunov functional (LF). The results obtained are new and differ from those found in the literature, and they also contain and improve a result found in the literature under more less restrictive conditions. We establish an example and give a discussion to indicate the applicability of the weaker conditions obtained. We also employ MATLAB-Simulink to display the behaviors of the orbits of the (VIDEs) considered.

1. Introduction

In the last years, a lot of interesting results related to the qualitative behaviors of solutions; stability (S), instability (I), (AS), exponential stability (ES), etc., of (VIDEs) have been obtained in the literature. For a comprehensive study on qualitative properties of (VIDEs), we refer the readers to the works of [1–39] and theirs references.

We know that qualitative behaviors of solutions have many important roles in the subjects and applications of ordinary differential equations (ODEs), integral equations (IEs) and integro-differential equations (IDEs) with or without retardations (see, the books [1] and [2] and their references). Therefore, it is worth examining the qualitative properties of the solutions of the retarded (IDEs).

We would now like to summarize some related works on the subject below.

In 1983, the authors in [3] investigated the (S) of solutions of (VIDE) of the form

$$x'(t) = A(t)x + \int_0^t C(t,s)x(s)ds, \quad t \geq t_0, \tag{1}$$

where $A(t)$ and $C(t,s)$ are $n \times n-$ matrices and the solution x is given on the interval $(0, t_0)$.

In [3], various kinds of (S) for (VIDE) (1) are defined and equivalence relations between them are established. Several criteria for (S) of solutions are given and the basic idea in [3] is that $A(t)$ is "negative" and dominates the integral term involving $C(t,s)$. The proofs in [3] rely to a very large extent on (LFs).

After that, in 1985, the authors of [4] dealt with a (VIDE) given by

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds + F(t), \tag{2}$$

where $t \geq 0$, $x(t_0) = x_0$, x is an $n-$vector, $n \geq 1$, $A(t)$ and $K(t,s)$ are $n \times n-$ matrices defined and continuous on $0 \leq t < \infty$ and $0 \leq s \leq t < \infty$, respectively, and the function $F : \Re^+ \to \Re^n$ is continuous for all $t \in \Re^+$, $\Re^+ = [0, \infty)$. The authors in [4] commented the asymptotic behaviors (ABs) of solutions of (VIDE) (2) in which $A(t)$ is not necessarily a stable matrix. An equivalent equation which involves an arbitrary function is derived. Hence, a proper choice of this function would pave a way for the new coefficient matrix $B(t)$ (corresponding$A(t)$) to be stable. The main approach in [4] is by way of deriving an equivalence theorem (Lemma 1, which is given

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below) which has the potential to supply them a stable matrix $B(t)$ corresponding to $A(t)$.

Later, in 1987, the authors in [5] considered (VIDE) of the form

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds + f(t, x(t)), \tag{3}$$

where $t \geq 0,\ x(t_0) = x_0,\ x$ is an $n-$vector, $n \geq 1,\ n \times n-$ matrices $A(t)$ and $K(t,s)$ satisfy the properties mentioned in (VIDE) (2) for the existence and uniqueness of the solutions, respectively, and $f : \Re^+ \times \Re^n \to \Re^n$ is a continuous function with $f(t,0) = 0$ on $\Re^+$. The authors benefited from the following lemma in [4] to obtain some stronger specific conditions for certain (ABs) of solutions of (VIDE) (3).

Lemma 1 [4]**.** *Let $\Phi(t,s)$ be an $n \times n$ continuously differentiable matrix function on $0 \leq s \leq t < \infty$. Then* (VIDE) (3) *corresponds to*

$$y'(t) = B(t)y(t) + \int_0^t L(t,s)y(s)ds + G(t,y), y(0) = x_0,$$

in which

$$B(t) = A(t) - \Phi(t,t),$$

$$L(t,s) = K(t,s) + \frac{\partial}{\partial s}\Phi(t,s) + \Phi(t,s)A(s) + \int_s^t \Phi(t,u)K(u,s)ds$$

and

$$G(t,y) = f(t,y) + \Phi(t,0)x_0 + \int_0^t \Phi(t,s)f(s,x(s))ds.$$

Obviously, it is noticeable that Lemma 1 shows that (VIDE) given by (3) can be transformed to an equivalent (VIDE) with $A(t)$ and $K(t,s)$ replaced by other matrices, and which as a special case introduces the resolvent kernel corresponding to $K(t,s)$. A (LF), similar to that one used by the authors in [3], is then used to obtain specific conditions on the (ABs) of solutions of (VIDE) (3). Besides, certain special cases of (VIDE) (3), where $f(t,x) = 0$ and $K(t,s) = K(t-s)$, are also considered and some comments are also done on the (ABs) of the solutions of the equations considered.

In this paper, we consider (VIDEs) in the following form with a constant retardation

$$z'(t) = B(t)z(t) + \int_{t-\tau}^t D(t,s)z(s)ds + F(t, z(t), z(t-\tau)) + P(t), \tag{4}$$

where $t \geq 0,\ \tau > 0,\ \tau \in \Re$, fixed constant retardation with $t - \tau \geq 0,\ z(0) = z_0,\ z$ is an $n-$vector, $n \geq 1,\ B(t)$ and $D(t,s)$ are continuously differentiable $n \times n-$matrix functions on $0 \leq t < \infty$ and $0 \leq s \leq t < \infty$, respectively, $F : \Re^+ \times \Re^n \times \Re^n \to \Re^n$ and $P : \Re^+ \to \Re^n$ are continuous functions on their receptive domains with $F(t,0,0) = 0,\ \Re^+ = [0, \infty)$.

In particular, the motivation of this paper has been inspired by the results of ([3–5]), the papers and books in the references of this article and those in the literature.

To the best of our information, in view of the results of [5], it follows that instead of (VIDE) (3), the construction of an equivalent integro-differential system which involves an $n \times n-$ matrix $\Phi(t,s)$ continuously differentiable for $0 \leq s \leq t < \infty$ does not give an advantage to study the stability of the solutions of (VIDE) (3), while we convert this (VIDE) to the another equivalent (VIDE) (see Rama Mohana Rao and Raghavendra [5] and the assumptions of this paper given below). Here, we will show this case for the problems to be taken under consideration. That is, we mean that instead of the transform done in [5], if we construct a suitable (LF) for (VIDE) (3), it can be obtained the stability and boundedness results under suitable and weaker conditions. This fact illustrate the advantage of the proposed method. The first important reason for motivation of this paper is the former fact.

Next, the investigation of the qualitative properties of solutions of (VIDEs) with time-lag has many important attractions and places in theory and applications of these equations in sciences and engineering. We would not like to give the details of these facts here.

It is also notable that any investigation of the (S) and (B) in a (VIDE) by (LF) technique, first demands the description or construction of a suitable (LF), which allows to validity of the results obtained. In reality, this investigation can be an arduous task and the state becomes more difficult when we substitute an (ODE) with a retarded (IDE). However, once a viable (LF) is defined or constructed, researchers can continue with working with it for a long time, getting more information about some qualitative behaviors of solutions.

That is, it is difficult to construct suitable (LFs) and via those (LFs) to discuss the qualitative properties of that kind of functional integro-differential equations. In this paper, we carry the result of [5, Theorem 4.1]) to a more general (VIDE) with constant time-lag. In addition, we give an additional result, boundedness of solutions, for (VIDE) (4). The second important reason for motivation of this paper is this fact.

Briefly, our intention in this paper is to investigate and find new specific conditions for the (AS), and (B) of solutions of (VIDE) (4) by constructing a new and suitable (LF), when $P(.) \equiv 0$ and $P(.) \neq 0$, respectively. In reality, if we replace 0 (zero) in place of the term $t - \tau$ in (VIDE) (4), then it seems that (VIDEs) (1)-(3) are special cases of (VIDE) (4). This fact and constructing a new and suitable auxiliary (LF) are the contributions of this work to the topic and the newness of this article. In addition, carrying out the topic and problems in the literature from the case of without retardation to a general case of with retardation shows another benefaction, improvement and newness of this paper.

In view of the information already given, it follows that the (VIDE) discussed by ([3–5]) are without a retardation. However, in this paper, the (VIDE) to be discussed here is with a constant retardation.

The results to be obtained here are also differ from that obtained in the literature (see, ([1–4,6–39] and theirs references).

We obtain here the result of [5, Theorem 4.1] under weaker conditions. Actually, this is another originality and newness of this paper. Finally, investigating the (B) of solutions of (VIDE) (4) in the case $P(.) \neq 0$gives a new and additional result to that of [5]. In view of all the mentioned information, it can be checked the new and novel properties of the present paper.

2. Asymptotic stability

Using the perturbations theory and the Lyapunov's stability theorem, the authors [5] proved the following theorem on the (AS) of the solution $x(t) \equiv 0$ of (VIDE) (3).

In [5], it is accepted the existence of the following conditions for (VIDE) (3).

A. Assumptions:

$(A1)$ There exists a positive continuous function $\lambda(t)$ for $0 \leq t < \infty$ such that

$\|f(t,x)\| \leq \lambda(t)\|x\|$ with $\lambda(t)$ tends to zero as $t \to \infty$.

$(A2)$ Let $H(t)$ be an $n \times n-$ real, symmetric bounded and continuously differentiable matrix for $0 \leq t < \infty$. It is also assumed that

$$y^T[B^T(t)H(t) + H(t)B(t) + H'(t)]y \leq -\gamma\|y\|^2$$

for $y \in \Re^n,\ y \neq 0$, in which $\gamma > 0$, and $\gamma \in \Re$.

$(A3)$ There exists an $n \times n-$ matrix $\Phi(t,s)$ continuously differentiable for $0 \leq s \leq t < \infty$ satisfying the conditions

$$\int_0^t \|\Phi(u,t)\|du \text{ is defined for all } t \in \Re^+, \quad \Re^+ = [0, \infty),$$

$$\|\Phi(t,0)\| \to 0 \text{ as } t \to \infty, \quad \text{and}$$

$$\int_0^\infty \|\Phi(t,0)\|dt < \infty.$$

$(A4)$ $\|B(t)\|$ is bounded and $\int_0^\infty \|L(u,t)\|du$ is defined for all $t \in \Re^+$,

where

$$B(t) = A(t) - \Phi(t,t),$$

$$L(t,s) = K(t,s) + \frac{\partial}{\partial s}\Phi(t,s) + \Phi(t,s)A(s) + \int_s^t \Phi(t,u)K(u,s)ds.$$

$(A5)$ $H_0[\|\Phi(t,0)\| + \int_0^t \|L(t,s)\|ds + \int_t^\infty \|L(u,t)\|du]$

$$+\lambda_0 H_0\left[\int_t^\infty \|\Phi(t,s)\|ds + \int_t^\infty \|\Phi(u,t)\|du\right] \le \alpha_0,$$

where $\alpha_0 \in \Re$, $\alpha_0 > 0$, $\lambda_0 = \sup_{t\ge 0}\lambda(t)$ and $H_0 = \sup_{t\ge 0} H(t)$.

Theorem A. [5]. *Let assumptions* $(A1)-(A5)$ *hold. If* $B(t)$ *is bounded and* $\gamma > \alpha_0 + 2\lambda_0 H_0$ *holds, then the trivial solution of* (VIDE) (3) *is* (AS).

We are now ready to introduce one of the main results of this paper.

Let

$$P(t) \equiv 0.$$

B. Assumptions

We assume the following conditions are true:

$(H1)$ $\|F(t,z,z(t-\tau))\| \le f(t)\min\{\|z\|, \|z(t-\tau)\|\}$ with $f(t) \to 0$ as $t \to \infty$, in which $f(t)$ is a positive and continuous function for $t \in \Re^+$, $\Re^+ = [0,\infty)$, and $f_0 = \sup_{t\ge 0} f(t)$.

$(H2)$ Let $H(t)$ be an $n\times n-$ real, symmetric, bounded and continuously differentiable matrix for all $t \in \Re^+$. It is assumed that

$$z^T[B^T(t)H(t) + H(t)B(t) + H'(t)]z \le -\phi_0\|z\|^2 \quad \text{for all } z \in \Re^n, \quad z \ne 0,$$

in which $\phi_0 > 0$, and $\phi_0 \in \Re$.

$(H3)$ $\|B(t)\|$ is bounded and $\int_0^\infty \|D(u,t)\|du$ is defined for all $t \in \Re^+$.

$(H4)$ $H_0\int_{t-\tau}^t \|D(t,s)\|ds + H_0\int_{t-\tau}^\infty \|D(u+\tau,t)\|du \le K_0$,

where $K_0 \in \Re$, $K_0 > 0$, and $H_0 = \sup_{t\ge 0}\|H(t)\|$.

Theorem 1. *Let assumptions* $(H1)-(H4)$ *hold. If* $[\phi_0 - 2f_0H_0 - K_0] \ge \varepsilon > 0$,*then the trivial solution of* (VIDE) (4) *is* (AS).

Proof. We construct a (LF) $W(.) = W(t,z(t))$ communicated by

$$W(.) = z^T(t)H(t)z(t) + \lambda\int_0^t\int_{t-\tau}^\infty \|D(u+\tau,s)\|du\|z(s))\|^2ds, \tag{5}$$

where $\lambda > 0$, $\lambda \in \Re$, and we choose this constant later in the proof.

Since the assumptions of Theorem 1 hold, then it can be clear that $W(.)$ is positive definite.

Differentiating $W(.)$ with respect to t, we can obtain that

$$\begin{aligned}
W'(.) &= (z')^T H(t)z + z^T H'(t)z + z^T H(t)z' \\
&\quad + \lambda\int_{t-\tau}^\infty \|D(u+\tau,t)\|du\|z(t)\|^2 - \lambda\int_0^t \|D(t,s)\|\|z(s)\|^2ds \\
&= z^T\big(H'(t) + B^T(t)H(t) + H(t)B(t)\big)z \\
&\quad + z^T H(t)\int_{t-\tau}^t D(t,s)z(s)ds + z^T H(t)F(t,z,z(t-\tau)) \\
&\quad + z^T H^T(t)F(t,z,z(t-\tau)) + z^T H^T(t)\int_{t-\tau}^t D(t,s)z(s)ds \\
&\quad + \lambda\int_{t-\tau}^\infty \|D(u+\tau,t)\|du\|z(t)\|^2 - \lambda\int_0^t \|D(t,s)\|\|z(s)\|^2ds \\
&\le z^T\big(H'(t) + B^T(t)H(t) + H(t)B(t)\big)z \\
&\quad + 2H_0f_0\|z\|^2 + 2H_0\int_{t-\tau}^t \|D(t,s)\|\|z(s)\|\|z(t)\|ds \\
&\quad + \lambda\int_{t-\tau}^\infty \|D(u+\tau,t)\|du\|z(t)\|^2 - \lambda\int_0^t \|D(t,s)\|\|z(s)\|^2ds.
\end{aligned}$$

Benefited from assumptions$(H1)-(H4)$ and the estimate $|ab| \le 2^{-1}(a^2+b^2)$, it can be followed that

$$\begin{aligned}
W'(.) &\le -\phi_0\|z\|^2 + 2f_0H_0\|z\|^2 \\
&\quad + H_0\int_{t-\tau}^t \|D(t,s)\|(\|z(s)\|^2 + \|z(t)\|^2)ds \\
&\quad + \lambda\int_{t-\tau}^\infty \|D(u+\tau,t)\|du\|z(t)\|^2 - \lambda\int_0^t \|D(t,s)\|\|z(s)\|^2ds \\
&= -\phi_0\|z\|^2 + 2f_0H_0\|z\|^2 + H_0\int_{t-\tau}^t \|D(t,s)\|\|z(s)\|^2ds \\
&\quad + H_0\int_{t-\tau}^t \|D(t,s)\|ds\|z(t)\|^2 + \lambda\int_{t-\tau}^\infty \|D(u+\tau,t)\|du\|z(t)\|^2 \\
&\quad - \lambda\int_0^t \|D(t,s)\|\|z(s)\|^2ds.
\end{aligned}$$

Let $\lambda = H_0$. Then, we have

$$\begin{aligned}
W'(.) &\le -\left[\phi_0 - 2f_0H_0 - H_0\int_{t-\tau}^t \|D(t,s)\|ds\right]\|z(t)\|^2 \\
&\quad + \left[H_0\int_{t-\tau}^\infty \|D(u+\tau,t)\|du\right]\|z(t)\|^2.
\end{aligned}$$

By the assumptions $(H1)-(H4)$ and $\phi_0 - 2f_0H_0 - K_0 \ge \varepsilon > 0$, we can conclude that

$$W'(.) \le -\varepsilon\|z(t)\|^2.$$

Thus, in view of the above examination, we can reach the desired result, that is, the trivial solution of (VIDE) (4) is (AS). Hence, we arrive at the conclusion of Theorem 1.

3. Boundedness

Let $P(t) \ne 0$.

C. Assumptions

$(H5)$ $\int_0^\infty \|P(s)\|ds < \infty$ and $\|P(t)\| \to 0$ as $t \to \infty$.

Theorem 2. *We suppose that assumptions* $(H1)-(H5)$ *hold. Then, all solutions of* (VIDE) (4) *are bounded.*

Proof. In this proof, we use the (LF) $W(.) = W(t,z(t))$ given by (5). In the case of the consideration of assumptions of $(H1)-(H5)$ and (5), we can reach the following:

$$\begin{aligned}
W'(.) &\le -\varepsilon\|z\|^2 + 2\|H(t)\|\|P(t)\|\|z\| \\
&\le 2\|P(t)\|H_0(1+\|z\|^2) \\
&= 2H_0\|P(t)\| + 2\|P(t)\|H_0\|z\|^2 \\
&\le 2H_0\|P(t)\| + 2\|P(t)\|W(.).
\end{aligned} \tag{6}$$

Integrating estimate (6) from zero 0 to t, we get

$$\begin{aligned}
W(t,z(t)) &\le W(t_0,z(t_0)) + 2H_0\int_0^t \|P(s)\|ds \\
&\quad + 2\int_0^t \|P(s)\|W(s,z(s))ds.
\end{aligned}$$

Then, the utilization of the Gronwall's inequality shows that $W(.)$ has an upper positive constant bound. That is,

$$W(t,z(t)) \le \Gamma_0\exp\left(\int_0^\infty \|P(s)\|\right)ds,$$

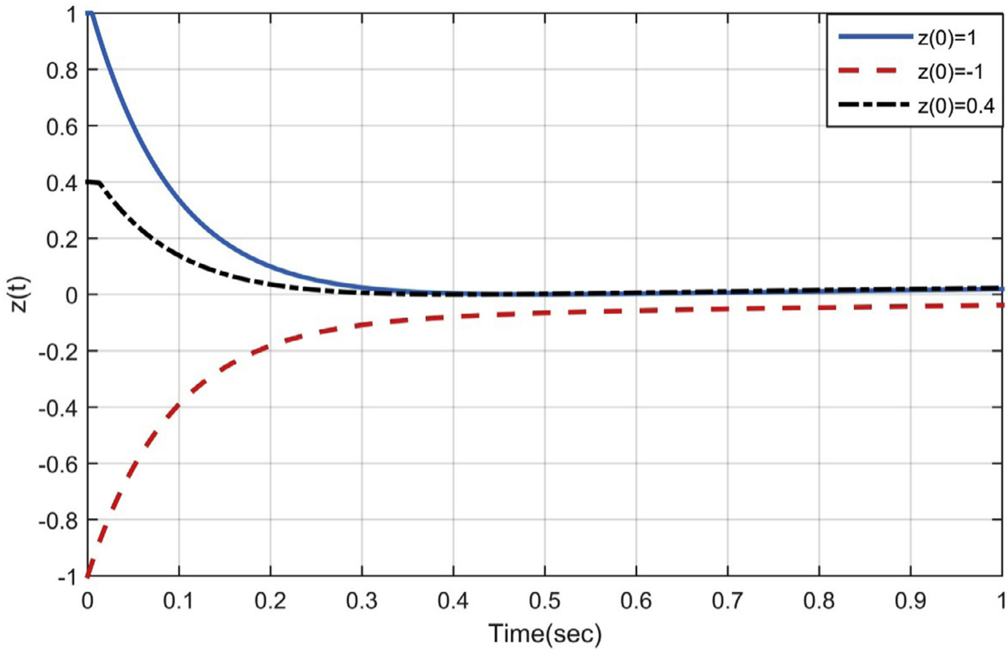

Fig. 1. Trajectory of z(t) for Example 1.

where

$$\Gamma_0 = W(t_0, z(t_0)) + 2H_0 \int_0^{\infty} \|P(s)\| ds.$$

Hence, it now is notable that

$$H_0\|z\|^2 \le z^T(t)H(t)z(t) \le \ W(t, z(t)) \le \Gamma_0 \exp\left(\int_0^{\infty} \|P(s)\|\right) ds.$$

Thus, by assumption $(H5)$,we can come the desired result, that is, the solutions of (VIDE) (4) are bounded. Hence, we can reach the conclusion of Theorem 2.

Example 1. For the case $n = 1$, as a specific subcase of (VIDE) (4), we consider non-linear (VIDE) with a constant retardation,

$$z' = -\left(10 + \frac{1}{1+t}\right)z - \int_{t-1}^{t} \exp(-t+s)z(s)ds$$
$$+ \frac{1}{10}\exp(-t)(|z|\exp(-|z| - |z(t-1)|) + \frac{1}{1+t^2},$$

for $t - 1 \ge 0, x \in \Re$.

When we compare this equation with (VIDE) (4) and consider the assumptions of Theorems 1 and 2, it follows the existence of the relations below:

$$B(t) = -10 - \frac{1}{1+t}, \quad t \ge 0,$$

$$D(t, s) = -\exp(-t+s),$$

$$F(t, z, z(t-\tau)) = \frac{1}{10}\exp(-t)(|z|\exp(-|z| - |z(t-1)|)),$$

$$P(t) = \frac{1}{1+t^2},$$

$$|B(t)| = \left|-10 - \frac{1}{1+t}\right| \le 11,$$

that is,

$|B(t)|$ is bounded,

$$|F(t, z, z(t-\tau))| \le \frac{1}{10}|z|\exp(-t)$$

$$f(t) = \frac{1}{10}\exp(-t), \quad f(t) \to 0 \text{ as } t \to \infty,$$

$$f_0 = \frac{1}{10}.$$

Let $H(t) = 1$. Then $H_0 = 1, H'(t) = 0$.

$$z^T[H'(t) + H(t)B(t) + B^T(t)H(t)]z \le -20|z|^2 \text{ for all } z \in \Re, \quad z \ne 0,$$

$$\phi_0 = 20,$$

$$\int_0^{\infty} |D(u,t)|du = \int_0^{\infty} \exp(-u+t)du \text{ is defined},$$

$$H_0\int_{t-1}^{t} |D(t,s)|ds + H_0\int_{t-1}^{\infty} |D(u+\tau, t)|du$$
$$= \int_{t-1}^{t} \exp(-t+s)ds + \int_{t-1}^{\infty} \exp(-u-1+t)du$$
$$= 2 - \frac{1}{e} < 1,7 = K_0,$$

$$\phi_0 - 2f_0H_0 - K_0 = 20 - \frac{1}{5} - 1,7 = 18,01 = \varepsilon > 0.$$

The desired result, that is, the (AS) of the trivial solution, for the (VIDE) considered is shown by the following graph (see Fig. 1).

Over and above, the (B) of the solutions for the (VIDE) considered is shown by the following graph (see Fig. 2).

Hence, all the assumptions, $(H1)-(H4)$and $(H1)-(H5)$ of Theorems 1 and 2, respectively, can be held. Thus, we can conclude that the trivial solution is (AS) and all solutions are bounded for the homogenous case and non-homogeneous case of the equation considered, respectively.

4. Discussion

We pay our attention to a type of non-linear (VIDEs) of first order with constant retardation. The (AS) and (B) behaviors of solutions (VIDEs) are examined by the aid of the Lyapunov's functional approach, when $P(t) \equiv 0$and $P(t) \ne 0$in (VIDE)(4), respectively. The claim made by the authors is illustrated as the following:

1^0) The results obtained, Theorem 1 has an extension and improvement and Theorem 2 gives an additional result to that of [5, Theorem 4.1], respectively.

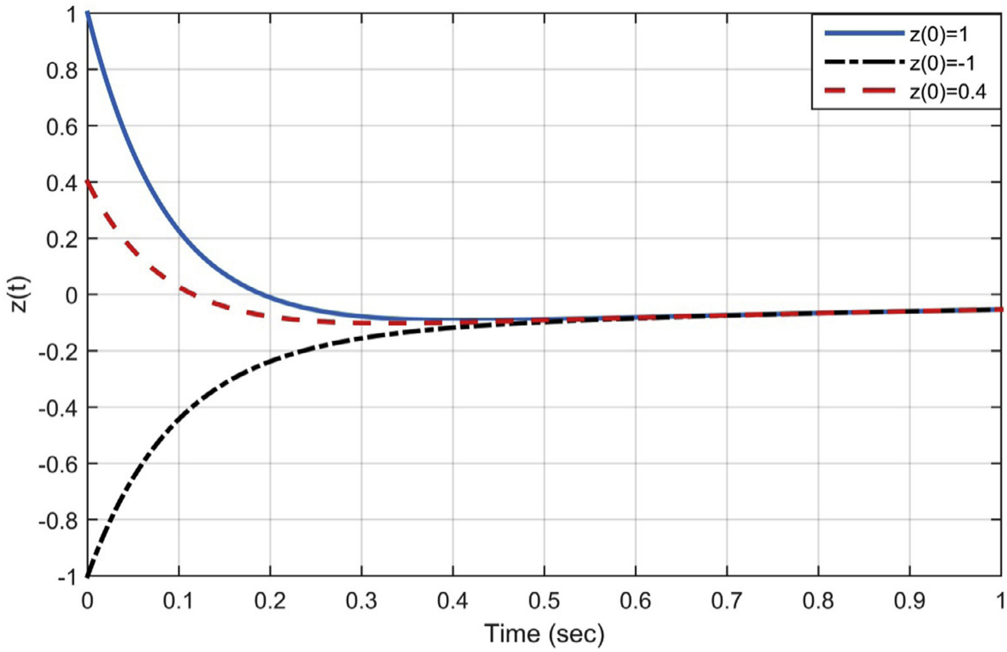

Fig. 2. Trajectory of z(t) for Example 1.

2^0) It is clear that our equation, (VIDE) (4), includes the (VIDE) investigated by the authors in [3–5] if we take zero "$0''$ instead of neglect the delay term $t-\tau$. In addition, we carry out the result of [5, Theorem 4.1] from case of without a retardation to the case of with a retardation. This case is an improvement, extension and contribution to the works of the authors in [3–5].

3^0) It can be seen that the conditions of Theorem 1 are less restrictive than those obtained in [5, Theorem 4.1]). In fact, when we compare the assumptions of Theorem 1 with that of ([5, *Theorem* 4.1], we see that Theorem 1 does not include assumption ($A3$)in [5], that is, the assumption,

($A3$) There exists an $n \times n-$ matrix $\Phi(t,s)$ which is continuously differentiable for $0 \le s \le t < \infty$ such that satisfying the conditions

$$\int_0^t \|\Phi(u,t)\| du \text{ is defined for all } t \in \Re^+,\ \Re^+ = [0,\infty),$$

$$\|\Phi(t,0)\| \to 0 \text{ as } t \to \infty, \text{ and}$$

$$\int_0^\infty \|\Phi(t,0)\| dt < \infty.$$

In addition, our assumption($H4$)of Theorem 1 is less restrictive than that ($A5$)of [5, Theorem 4.1]. In fact, if we take zero "$0''$ instead of the delay term $t-\tau$, this idea can be easily seen when we compare the assumption

$$H_0 \int_{t-\tau}^t \|D(t,s)\| ds + H_0 \int_{t-\tau}^\infty \|D(u+\tau,t)\| du \le K_0$$

of Theorem 1 with the assumption

$$H_0 \left[\|\Phi(t,0)\| + \int_{t-\tau}^t \|L(t,s)\| ds + \int_{t-\tau}^t \|L(u,t)\| du \right] + \lambda_0 H_0 \left[\int_{t-\tau}^\infty \|\Phi(t,s)\| ds + \phi_0 \int_{t-\tau}^\infty \|\Phi(u,t)\| du \right] \le \alpha_0,$$

$$L(t,s) = K(t,s) + \frac{\partial}{\partial s}\Phi(t,s) + \Phi(t,s)A(s) + \int_s^t \Phi(t,u)K(u,s) ds,$$

of [5, Theorem 4.1], Theorem A.

4^0) Assumption $B(t) = A(t) - \Phi(t,t)$ of [5, Theorem 4.1] leads that $B(t) = A(t)$ in Theorem 1. Because we do not need the term $\Phi(t,t)$. This case shows the advantage of the method of used in this paper without using perturbation theory.

5^0) On the other hand, the assumptions of Theorem 1 are very clear, elegant and comprehensible. That is, the assumptions of Theorem 1 have very simple forms and the applicability and correctness of them can be easily checked and verified. In spite of this fact, probably, it may be difficult to say the same statements for the assumptions of [5, Theorem 4.1].

6^0) We arrive at items $1^0) - 5^0$)since we use a different (LF) here than that used in [5, Theorem 4.1].

As a result of the investigation of this paper, when the above information is taken into consideration as a whole, it can be clearly seen that the importance of the work done, its quality and its contribution to the literature. Furthermore, in the literature, there are no examples in most of the scientific investigations related to this paper (see, the references of this paper), and if any result is given with an example in the literature, the graphics of the solutions are not included therein (see, [1-39]). These facts indicate the attractiveness and awareness of this work.

5. Conclusion

In this paper, we consider a class of non-linear (VIDE) of first order. We study the (AS) and (B) of solutions of the (VIDEs) considered via construction of a new (LF). By the way, we prove two new theorems relative to the (AS) and (B) of solutions. The first one includes weaker conditions than that in [5, Theorem 4.1] and improves that result for a more general case with constant retardation. The next one has a new contribution to the results of [5] and that in the literature since the (B) of solutions did not discuss by the authors in [5] and those found in the literature. We also give an example which satisfies the assumptions of both of the theorems in the special cases. We also draw the orbits of the solutions for the example considered by MATLAB-Simulink so that the verifications of the results hold (see Fig. 1 and Fig. 2). The results obtained can be useful for the researchers investigating the specific properties of solutions of functional differential equations models in applications, and they may have contributions to science and engineering.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

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