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Hybrid multistep method for solving second order initial value problems

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ABSTRACT

This paper develops a new parametric hybrid linear multistep method which has off-step points that allow it to be P-stable with high order. The stability analysis of the method is discussed and the stability regions are plotted. The stability regions are extended and the intervals of periodicity are increased with good choices of the parameters. Finally, some numerical tests are solved.

1. Introduction

In the present paper, we will investigate the numerical solutions of special second order initial value or boundary value problems. These problems can be written as:

$$y''(x) = f(x, y(x)), \qquad y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y'_0 \tag{1}$$

with periodical and/or oscillatory solutions, in which the first derivative does not appear explicitly. Multistep phase fitted methods with minimal phase-lag are obtained in [1–4]. The analysis of phase-lag or dispersion error was first introduced by Brusa and Nigro [5]. Several authors such as Thomas [6], Ibrahim [7] studied in detail the phase-lag of numerical methods for solving (1). Several authors [8–11] have developed hybrid methods with the purpose of making the phase-lag of the method smaller in their papers. The development of the new proposed method is based on good choice of the parameters to obtain higher algebraic order for the method and raise phase-lag order and the maximum possible interval of periodicity. Many problems in astronomy, astrophysics, quantum mechanics etc., have mathematical models of the type of interest of the present research. The method constituted is a modification of the dissipative multistep hybrid methods in the sense that multi-free parameters are added to eliminate the phase-lag and amplification error.

In Section 2, a parametric hybrid multistep method and some definitions are introduced. In addition, a theory on the order of convergence of the new method and a theory of the phase-lag analysis of non-symmetric k-step methods together with the direct formula for the computation of the phase-lag of our new methods are presented. In Section 3, special cases of the parametric hybrid multistep methods and their phase-lag order are studied. The stability and interval of periodicity analyses of these special cases are discussed in Section 4. Numerical tests are solved in Section 5. Finally, we present the conclusions and two Appendices A and B.

2. Parametric hybrid multistep methods

The general form of the multistep method is

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f_{n+i}, \tag{2}$$

where k is the number of steps and α_i and β_i are constants.

Consider the multistep methods which can be written in the form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \beta_k (f_{n+k} - \beta_s f_{n+s}). \tag{3}$$

These methods are used for the approximate solution of the initial value problems (IVP) of the form (1). The associated formula

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describing the hybrid point (off-point) is given as:

$$y_{n+s} = h^2 \mu\, f_{n+k} + \sum_{j=0}^{k} \nu_{k-j}\, y_{n+k-j}, \tag{4}$$

where, $x_{n+s} = x_n + s\,h$, μ and ν_i, $i = 0(1)k$ are constants which are chosen so that (3) and (4) have order $p = k+1$ and $p = k-1$, respectively. To get formula (4), Newton's interpolation formula for the double nodes x_{n+k} and for the simple nodes $x_{n+k-1}, \ldots, x_0$ have been used to evaluate the values of y_{n+s} at the off-step points.

Theorem 1. *Let (i) formula (3) be of order $k+1$, (ii) formula (4) be of order $k-1$.*

Then, the convergence of the formulae (3–4) will be satisfied for $k+1$.

Proof. The local truncation error for Eq. (4) of order $k-1$ is

$$y(x_{n+s}) - y_{n+s} = C_1\, h^k y^{(k)}(x_n) + O(h^{k+1}), \tag{5}$$

where $x_{n+s} = x_n + s\,h$, $0 < s < k$ for explicit hybrid formula and $s \geq k$ for the implicit hybrid formula, C_1 is the error constant. And $y(x_{n+s})$, y_{n+s} are the exact and approximate solutions of (1) respectively. Similarly, the local truncation error for Eq. (3) of order $k+1$ is

$$y(x_{n+k}) - y_{n+k} = C_2\, h^{k+2} y^{(k+2)}(x_n) + O(h^{k+3}), \tag{6}$$

where, C_2 is the error constant of method (3). For some η_{n+s} in the interval whose ends are y_{n+s} and $y(x_{n+s})$, we can write

$$f(x_{n+s}, y(x_{n+s})) - f(x_{n+s}, y_{n+s}) = \frac{\partial f}{\partial y}(x_{n+s}, \eta_{n+s}).(y(x_{n+s}) - y_{n+s}) \tag{7}$$

□

Assuming that y_n, $y_{n-1}, \ldots, y_{n+k-1}$ are exact, then from (3) and (4), the difference operator associated with method (3) is

$$\begin{aligned} y(x_{n+k}) - y_{n+k} = {} & h^2 \beta_k [\{ f(x_{n+k}, y(x_{n+k})) - f(x_{n+k}, y_{n+k})\} \\ & - \beta_s \{ f(x_{n+s}, y(x_{n+s})) - f(x_{n+s}, y_{n+s})\}] \\ & + C h^{k+2} y^{(k+2)}(x_n) + O(h^{k+3}) \end{aligned} \tag{8}$$

Now from (5–8) we have,

$$\begin{aligned} & y(x_{n+k}) - y_{n+k} \\ & = h^2\, \beta_k \left(\frac{\partial f}{\partial y}(x_{n+k},\, \eta_{n+k}) \,.(y(x_{n+k}) - y_{n+k}) \right. \\ & \quad \left. - \beta_s \frac{\partial f}{\partial y}(x_{n+s},\, \eta_{n+s}) \,.(y(x_{n+s}) - y_{n+s}) \right) \\ & \quad + C\, h^{k+2}\, y^{(k+2)}(x_n) + O(h^{k+3}), \end{aligned} \tag{9}$$

$\eta_{n+j} \in [y(x_{n+j}), y_{n+j}]$ *for* $j = s, k$. So, the order of the hybrid parametric formulae (3–4) is $k+1$. Two important properties to consider are, the phase-lag and the amplification error, these are actually two different types of truncation errors. The phase-lag is the angle between the analytical solution and the numerical solution while the amplification error is the distance from a standard cyclic solution. Consider the problem

$$y''(t) = -\omega^2 y(t), \qquad \omega \in R, \tag{10}$$

the exact solution is

$$y(t) = e^{\pm i\omega t}. \tag{11}$$

By applying a numerical method for the solution of (10) we obtain a numerical solution of the form

$$y(t) = a(\omega)\, e^{\pm i\theta(\omega t)} \tag{12}$$

Comparing (11 and 12), we obtain the following definitions [12].

Definition 1. The difference $1 - a(\omega)$ is called the dissipation error.

Definition 2. The difference $\theta(\omega) - \omega$ is called the dispersion error or the phase-lag.

Applying method (3–4) to the test Eq. (10) with nontrivial initial conditions on y and y'leads to the following difference equation:

$$\sum_{i=0}^{k} A_i(H)\, y_{n+i} = 0, \tag{13}$$

where $H = \omega h$, h is the step length and $A_j(H)$, $j = 0\,(1)\,m$ are polynomials of H given by

$$\begin{aligned} A_k(H) &= \alpha_k + H^2 \beta_k (1 - \beta_s \nu_k) + H^4 \beta_k \beta_s \mu \\ A_j(H) &= \alpha_j - H^2 \beta_k \beta_s \nu_j, \qquad j = 0\,(1)\,k-1. \end{aligned}$$

The associated characteristic Eq. (13) can be written in the form:

$$\sum_{i=0}^{k} A_i(H)\, \zeta^i = 0 \tag{14}$$

Definition 3. A k-step method with a characteristic equation given by (14) is said to have an interval of periodicity $(0, H_0^2)$ if, for all $H^2 \in (0, H_0^2)$, the roots ζ_i, $i = 1\,(1)\,k$ of Eq. (14) satisfy $\zeta_1 = \exp(i\theta(H))$, $\zeta_2 = \exp(-i\theta(H))$, and $|\zeta_i| \leq 1$, $i = 3(1)k$ where $\theta(H)$ is a real function of H.

Definition 4. [6] For any finite difference method corresponding to the characteristic Eq. (14), the phase-lag is defined as the leading term in the expansion of

$$t = H - \theta(H) = C H^{q+1} + O(H^{q+2}), \tag{15}$$

where C is the phase-lag constant. If the quantity $t = O(H^{q+1})$ *as* $H \to \infty$, the order of phase-lag is q.

By trigonometric expansions, the following equalities are true

$$\begin{aligned} \cos(\theta(H)) &= \cos(H - T) = \cos(H) + C H^{q+2} + O(H^{q+3}), \\ \sin(\theta(H)) &= \sin(H - T) = \sin(H) - C H^{q+1} + O(H^{q+2}). \end{aligned}$$

So,

$$\begin{aligned} \cos(j\theta(H)) &= \cos(jH) + C j^2 H^{q+2} + O(H^{q+3}), \\ \sin(j\theta(H)) &= \sin(jH) - C j^2 H^{q+1} + O(H^{q+2}). \end{aligned} \tag{16}$$

Theorem 2. *The k-step non symmetric finite difference method with characteristic equation given by (14) has phase-lag order q and phase-lag constant C given by*

$$C H^{q+1} + O(H^{q+2}) = \frac{A_k(H)\exp(ikH) + A_{k-1}(H)\exp(i(k-1)H) + \ldots + A_1(H)\exp(iH) + A_0(H)}{kA_k(H) + (k-1)A_{k-1}(H) + \ldots + A_1(H)} \tag{17}$$

Proof. See Appendix A. □

So, we are going to apply the previous theorem on (3–4) with different steps k. The above theorem gives us a direct formula for the computation of the phase-lag of any k-step finite difference methods.

3. The family of parametric hybrid method

3.1. Case 1

Consider the k-step hybrid difference method (3) with $k = 2$, then we get

$$y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h^2 \beta_k \left(f_{n+k} - \beta_s f_{n+s}\right). \tag{18}$$

which is of order three, and

$$y_{n+s} = h^2 \mu\, y''_{n+2} + \nu_2 y_{n+2} + \nu_1 y_{n+1} + \nu_0 y_n, \tag{19}$$

which is of order one. The coefficients of (18) and (19) will be as follows,

$$\alpha_0 = 1, \quad \alpha_1 = -2, \quad \beta_2 = (1-s)/(2-s), \quad \beta_s = 1/(s-1),$$
$$\nu_1 = 2(1-\nu_0) - s \quad and \quad \nu_2 = s - (1-\nu_0)$$

where s, μ and ν_0 are free parameters. The parametric method (18–19) has a local truncation error given by $\text{LTE} = \frac{1}{2}(-5 + 6s)\, h^4 y^{(4)}(\eta)$. The local truncation error LTE = 0 as s = 5/6. The method becomes of order 4 with error constant −5/216.

Another related concept, which is important when solving problems in form (1) is the phase- lag of the method. In phase analysis, one compares the phase of $\exp(\pm iH)$ with the phases of the characteristic Eq. (14). Applying Theorem 2 for k = 2 we obtain,

$$C H^{r+1} + O(H^{r+2}) = \frac{A_0 + A_1 e^{iH} + A_2 e^{2iH}}{A_1 + 2A_2} \tag{20}$$

where,

$$A_0 = 1 - \frac{H^2 \nu_0}{s-2}, \quad A_1 = -2 + H^2\left(1 + \frac{2\nu_0}{s-2}\right),$$
$$A_2 = 1 + \frac{H^4 \mu - H^2 \nu_0}{s-2}. \tag{21}$$

The above hybrid method (18–19) has a phase-lag with order one (r = 1), and the phase lag error constant (C_{pl}) is $\frac{1}{12} + \frac{\mu + \nu_0}{-2+s}$. But if s = 5/6, $\mu = 0$, and $\nu_0 = \frac{2-s}{12} = \frac{7}{72}$, the order of the phase - lag becomes r = 3 and the phase-lag error constant equal 1/240.

3.2. Case 2

Here, we consider the three-step hybrid parametric method (3) with k = 3 which is of order four

$$y_{n+3} + \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h^2 \beta_3 (y''_{n+3} - \beta_s y''_{n+s}), \tag{22}$$

The coefficients in (22) can be easily determined to be

$$\alpha_0 = \frac{11 - 6s}{-11 + 12s}, \quad \alpha_1 = \frac{3(-11 + 8s)}{-11 + 12s},$$
$$\alpha_2 = \frac{-3(11 + 10s)}{-11 + 12s}, \quad \beta_3 = \frac{11 - 18s + 6s^2}{33 - 47s + 12s^2},$$
$$\beta_s = \frac{11}{11 - 18s + 6s^2}. \tag{23}$$

However, the associated formula (4) has order two

$$y_{n+s} = h^2 \mu\, y''_{n+3} + \nu_3 y_{n+3} + \nu_2 y_{n+2} + \nu_1 y_{n+1} + \nu_0 y_n, \tag{24}$$

with the following coefficients:

$$\nu_1 = (s^2 - 5s - 2\mu + 6 - 6\nu_0)/2, \quad \nu_2 = -s^2 + 4s + 2\mu - 3 + 3\nu_0,$$
$$\nu_3 = (s^2 - 3s - 2\mu + 2 - 2\nu_0)/2. \tag{25}$$

where μ, s and ν_0 are free parameters. The method (22–24) with the coefficients given by (23–25) has a local truncation error $\text{LTE} = \frac{(33 - 60s + 22s^2)}{(-132 + 144s)} h^5 y^{(5)}(\eta)$. So, LTE = 0 as $s = \frac{1}{22}(30 \mp \sqrt{174})$. The method becomes of order 5 with an error constant $\frac{-16329 + 211\sqrt{174}}{7590}$.

Similar to Case 1, applying Eq. (14) and Theorem 2 for k = 3, we obtain,

Table 1
The coefficients of formulae (3–4) for k = 4 and k = 5.

k	4	5
α_0	$\frac{(115-104s+22s^2)}{5(23-40s+14s^2)}$	$\frac{3014-2s(1771+s(-649+75))}{-3014+s(6623+s(-3923+675s))}$
α_1	$\frac{-16(25-29s+7s^2)}{5(23-40s+14s^2)}$	$\frac{-11782+s(17551-7315s+915s^2)}{-3014+s(6623+s(-3923+675s))}$
α_2	$\frac{6(95-136s+38s^2)}{5(23-40s+14s^2)}$	$\frac{-4(-5069+s(9101+5s(-859+117s)))}{-3014+s(6623+s(-3923+675s))}$
α_3	$\frac{-16(25-41s+13s^2)}{5(23-40s+14s^2)}$	$\frac{-20276+2s(20473+5s(-2153+321s))}{-3014+s(6623+s(-3923+675s))}$
α_4	0	$\frac{11782-2s(12587+5s(-1429+231s))}{-3014+s(6623+s(-3923+675s))}$
β_s	$\frac{10}{(-10+21s-12s^2+2s^3)}$	$\frac{274}{(274+15(-5+s)s(9+(-5+s)s))}$
β_k	$\frac{12(-10+21s-12s^2+2s^3)}{5(-92+183s-96s^2+14s^3)}$	$\frac{12(274+15(-5+s)s(9+(-5+s)s))}{(-5+s)(-3014+s(6623+s(-3923+675s)))}$
ν_1	$\frac{1}{6(24+6\mu-24\nu_0-26s+9s^2-s^3)}$	$\frac{1}{24(-22\mu-120\nu_0+(-5+s)(-4+s)(-3+s)(-2+s))}$
ν_2	$\frac{1}{2(-12-8\mu+12\nu_0+19s-8s^2+s^3)}$	$\frac{1}{6(28\mu+60\nu_0-(-5+s)(-4+s)(-3+s)(-1+s))}$
ν_3	$\frac{1}{2(8+10\mu-8\nu_0-14s+7s^2-s^3)}$	$\frac{1}{4(-38\mu-40\nu_0+(-5+s)(-4+s)(-2+s)(-1+s))}$
ν_4	$\frac{1}{6(-6-12\mu+6\nu_0+11s-6s^2+s^3)}$	$\frac{1}{6(52\mu+30\nu_0-(-5+s)(-3+s)(-2+s)(-1+s))}$
ν_5	0	$\frac{1}{24(-70\mu-24\nu_0+(-4+s)(-3+s)(-2+s)(-1+s))}$

$$C H^{r+1} + O(H^{r+2}) = \frac{A_0 + A_1 e^{iH} + A_2 e^{2iH} + A_3 e^{3iH}}{A_1 + 2A_2 + 3A_3},$$
$$A_0 = -\frac{33 + 11H^2\nu_0 - 29s + 6s^2}{33 - 47s + 12s^2},$$
$$A_1 = \frac{11H^2(-6 + 2\mu + 6\nu_0 + 5s - s^2) + 6(33 - 35s + 8s^2)}{66 - 94s + 24s^2}, \tag{26}$$
$$A_2 = \frac{-99 + 123s - 30s^2 - 11H^2(-3 + 2\mu + 3\nu_0 + 4s - s^2)}{33 - 47s + 12s^2},$$
$$A_3 = \frac{66 + 22H^4\mu - 94s + 24s^2 + H^2(22\mu + 22\nu_0 + (-3+s)s)}{66 - 94s + 24s^2}.$$

The order of the phase - lag of the method is two, (r = 2), and the phase-lag error constant (C_{pl}) is

$$\frac{33(1 + 4\mu - 4\nu_0) - 29s + 6s^2}{12(-3 + s)(-11 + 12s)}$$

If $s = 121/72$, $\mu = -19/3456$, and $\nu_0 = 19/5184$, the order of the phase-lag becomes 5 and the phase-lag error constant becomes 613/302,400. In the case $k = 4$ and $k = 5$ the coefficients of the methods (3–4) are tabulated in Table 1. In Table 2, we summarized the results of the algebraic and the phase - lag error constants and the orders of methods (3–4) with k = 2–5 for different values of the parameters s, μ, and ν_0 . Based on the above, we obtain the coefficients mentioned in Appendix B.

4. Stability analysis

Here, we will investigate the interval of periodicity and the stability regions of the new obtained methods. It is easy to see that the frequency used in the scalar test equation for the stability analysis (ω) is not equal to the frequency of the scalar test equation for the phase-lag analysis (φ). i.e. $\omega \neq \varphi$. The methods become P-stable in the case of $\omega = \phi$.

Definition 5. [13,14] A method is called P-stable method if its interval of periodicity is equal to $(0, \infty)$.

The associated characteristic Eq. (14) for k = 2 can be written as

$$\pi(\zeta, H^2) = A_2(H^2)\,\zeta^2 + A_1(H^2)\,\zeta + A_0(H^2). \tag{27}$$

The roots of the characteristic polynomial (27) should be complex conjugate pair lying on the unit circle $|\zeta| = 1$. For this requirement we study the relation between the parameters and H^2. Hence, we use Routh–Hurwitz criteria to the characteristic Eq. (27), let $\zeta = (1 + z)/(1 - z)$ which maps the circle $|\zeta| = 1$ into the line

Table 2
The algebraic and phase-lag orders and intervals of periodicity for different steps k and different values of parameters (P = Algebraic order *r* = *Phase-lag order).*

k	*S*	ν_0	μ	*P*	*Algebraic error constant*	*r*	*Phase lag error constant*	*Interval of periodicity*
2	Free	Free	Free	3	$(-5+6s)/12$	1	$\frac{1}{12}+\frac{\mu+\nu_0}{-2+s}$	$(0,\infty)$
2	5/6	Free	Free	4	$-5/216$	1	$\frac{1}{12}+\frac{\mu+\nu_0}{-2+s}$	
2	5/6	$(^{2}$-s)/12	0	4	$-5/216$	3	1/240	
2	s> 2	$\nu_0>0$	$(4-16\nu_0+16\nu_0^2-4s+8\nu_0s+s^2)/(-32+16s)$	3	$(-5+6s)/12$	1	$\frac{1}{12}+\frac{\mu+\nu_0}{-2+s}$	
3	Free	Free	Free	4	$(33-60s+22s^2)/(-132+144s)$	2	$\frac{(11+6(-3+s)s)}{12(-3+s)(-11+18s)}$	
3	$\frac{1}{22}(30\mp\sqrt{174})$	Free	Free	5	$(-16,329+211\sqrt{174})/7590$	2		
3	121/72	19/5184	$-19/3456$	4	$-1343/25,920$	5	613/6, 350, 400	(0,1.0042)
3	$s>3$	$(1452\mu+484\mu^2-99s-1309\mu s+120s^2+275\mu s^2-47s^3+6s^4)/(242-396s+132s^2)$	$(-12s+4s^2)/77$	4	$(33-60s+22s^2)/(-132+144s)$	2	$(11+6(-3+s)s)/12(-3+s)(-11+18s)$	$(0,\infty)$
3	3.5	-.1	-.2	4	$(33-60s+22s^2)/(-132+144s)$	2		
4	Free	Free	Free	5	$\frac{-310+s(698+s(-424+75s))}{75(23+2s(-20+7s))}$	3	$\frac{-(1650\mu+1800\nu_0+(-4+s)(140+s(-127+26s)))}{360(-4+s)(5+4(-2+s)s)}$	
4	$\frac{1,396,7-3,\sqrt{25,531,21,}}{60,92}$	$\frac{,239,264,727,79\ ,+\ 5,651,957,\sqrt{25,531,21,}}{45,217,826,137,60}$	$\frac{3,(-9,0,7,38,,653,\ +\ ,231,,86,9\sqrt{,,255,,31,21,})}{2,260,891,306,88}$	5	$\frac{-310+s(698+s(-424+75s))}{75(23+2s(-20+7s))}$	6	$-\frac{,640,241,5}{1,662,064,911,36}-\frac{31,3\sqrt{,,255,,31,21,}}{43,969,971,20}$	
4	4.5	-0.1	$\mu>237,614$	5				$(0,\infty)$
5	Free	Free	Free	6	$(41100\mu-49320\nu_0+(-5+s)(-3562+5s(845+s(-302+33s))))/(15(-5+s)(-3014+s(6623+s(-3923+675s))))$	4	$\frac{41100\mu-49320\nu_0+(-5+s)(-3562+5s(845+s(-302+33s)))}{180(-5+s)(-274+75s(9+(-5+s)s))}$	
5	Free	$(-5+s)(-6831916+8481775s-3226180s^2+382470s^3)/94595760$	$\frac{-25215s+27043s^2-8800s^3+880s^4}{1051064}$			6	$\frac{-(-1876900+3s(809110+s(-306508+35025s)))}{2071440(-274+75s(9+(-5+s)s))}$	
5	5.5	0.1	0.1			4	$\frac{41100\mu-49320\nu_0+(-5+s)(-3562+5s(845+s(-302+33s)))}{180(-5+s)(-274+75s(9+(-5+s)s))}$	$(6.84013,\infty)$

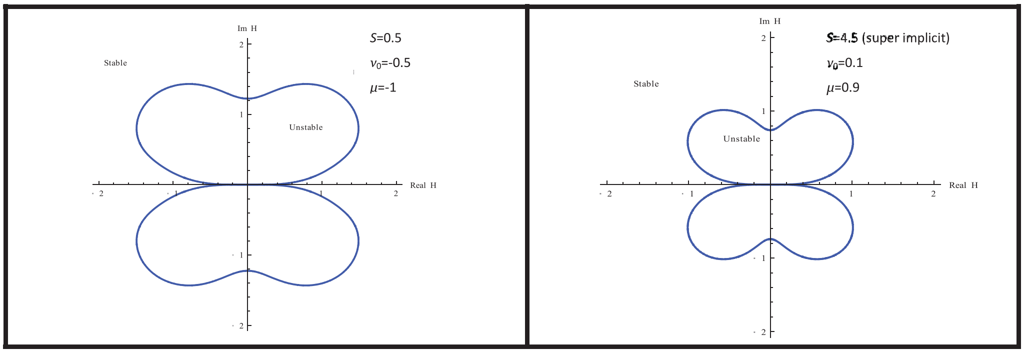

Fig. 1. The absolute stability regions of method (3–4), $k=2$ with different values of the parameters.

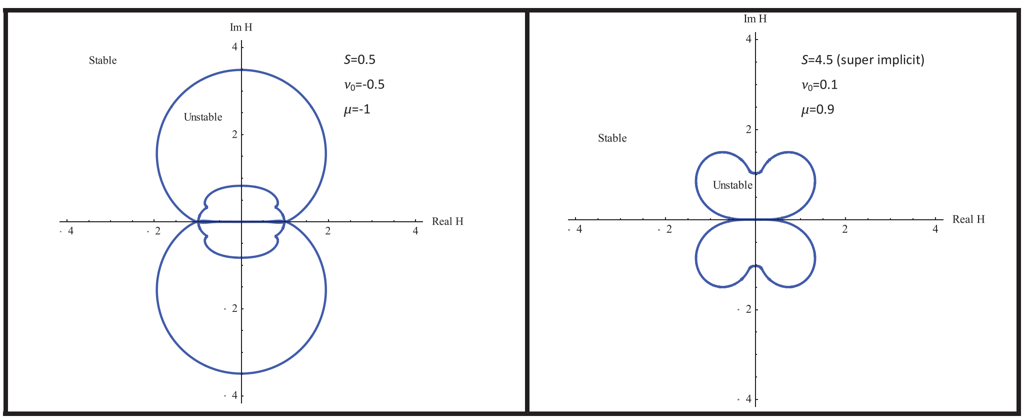

Fig. 2. The absolute stability regions of method (3–4), $k=3$ with different values of the parameters.

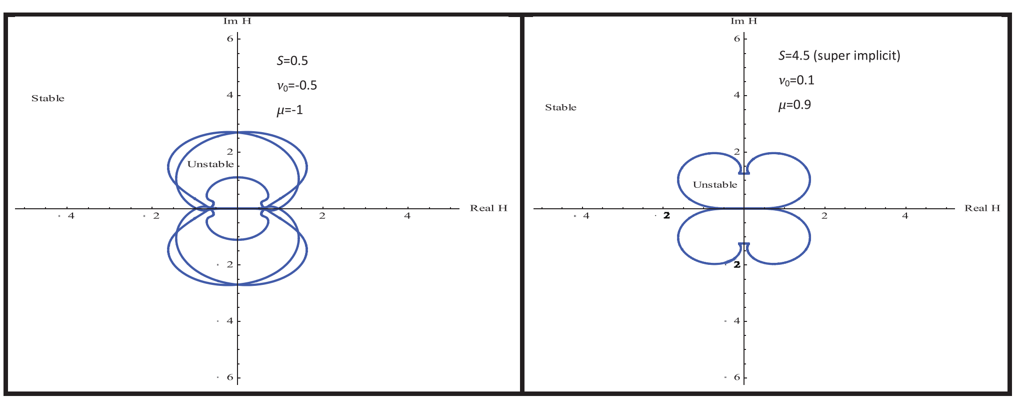

Fig. 3. The absolute stability regions of method (3–4), $k=4$ with different values of the parameters.

$Re(z)=0$ and region $|\zeta|\leq 1$ into $Re(z)\leq 0$, which transforms (27) to

$$\begin{aligned}
&\pi(z,H^2)=d_2(H^2)\,z^2+d_1(H^2)\,z+d_0(H^2)\,,\\
&d_2=\frac{(4(-2+s)+(2-4\nu_0-s)H^2+\mu H^4)}{s-2},\\
&d_1=\frac{2\mu H^4}{s-2},\qquad d_0=\frac{(-2+s)H^2+\mu H^4}{s-2}.
\end{aligned}\tag{28}$$

For $s>2$ (Super implicit off point), $\nu_0>0$ and $\mu>(4-16\nu_0+16{\nu_0}^2-4s+8\nu_0 s+s^2 2)/(-32+16s)$ the interval of periodicity is $(0,\infty)$, so under the previous relations, method (18–19) will be P-stable.

The interval of periodicity of method (3–4) with $k=2$–5 are tabulated in Table 2 for different values of the parameters. To obtain the regions of absolute stability, we apply the boundary locus method on Eq. (14). The absolute stability regions are plotted in Figs. 1–4 for various steps k ($k=2$–5), respectively, with different values of the parameters.

5. Numerical tests

5.1. Test 1

The linear problem with variable coefficient

$$y''=-4x^2y+(4x^2-\omega^2)\sin\omega x-2\sin x^2,$$
$$y(0)=1\ \text{and}\ y'(0)=\omega,\quad x\in[0,\alpha\pi].$$

The analytical solution is: $y(x)=\sin(\omega x)+\cos(x^2),\quad \omega=10.$

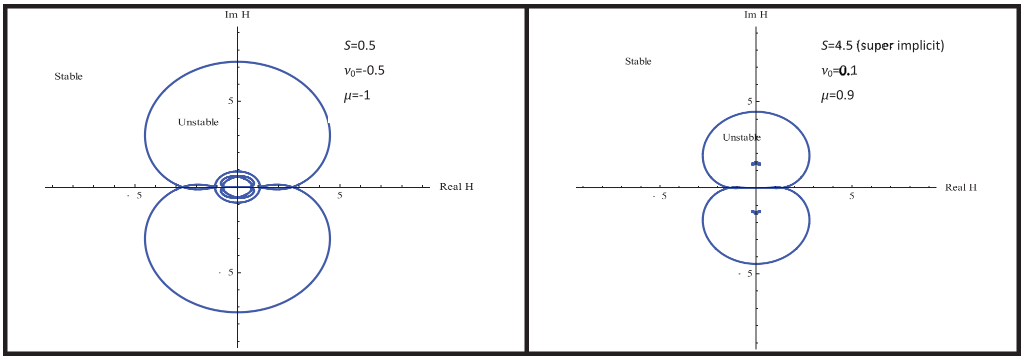

Fig. 4. The absolute stability regions of method (3–4), $k=5$ with different values of the parameters.

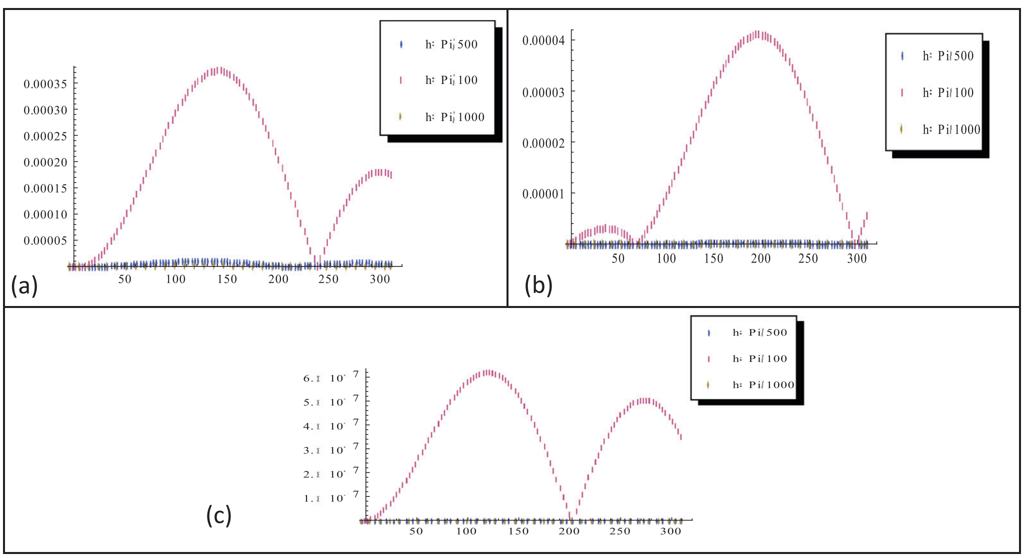

Fig. 5. Absolute errors of Test 4 by method (3–4) for different values of h: (a) k = 2, (b) k = 3, (c) k = 4.

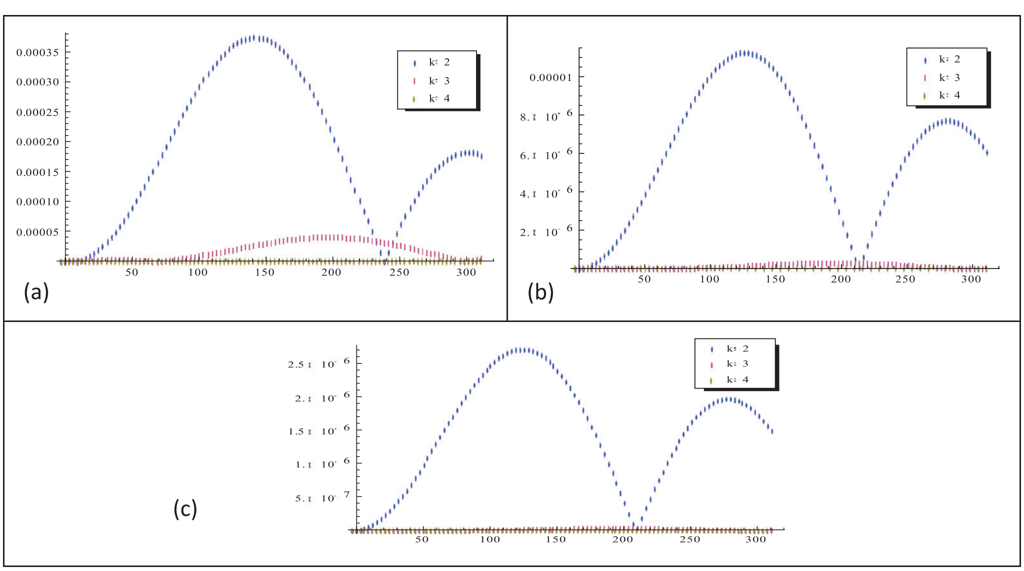

Fig. 6. Absolute errors of Test 4 by method (3,4) for different values of k: (a) h = Pi/100, (b) h = Pi/500 (c) h = Pi/1000.

Table 3
Absolute errors of Test 1.

h	X	s	μ	ν_0	Absolute errors of method (3–4)	Absolute errors of Numerov method (34)
0.01	5	1.9	0.2	0.2	1.28139E−4	2.68222E−5
0.001	5				2.41227E−10	2.6681E−9

5.2. Test 2 (perturbed system)

$$y_1'' = -100\,y_1 - \frac{2y_1\,y_2}{y_1^2+y_2^2} + f_1, \quad y_2'' = -25\,y_2 - \frac{y_1^2-y_2^2}{y_1^2+y_2^2} + f_2$$

$$f_1 = \frac{2\cos(10x)\sin(5x) + 2\,\varepsilon(\sin(5x)\sin(x) - \cos(10x)\cos(x)) - \varepsilon^2\sin(2x))}{(\cos^2(10x) + \sin^2(5x) + 2\varepsilon(\sin(x)\cos(10x) - \cos(x)\sin(5x)) + \varepsilon^2)} + 99\varepsilon\,\sin(x),$$

$$f_2 = \frac{\cos^2(10x)\sin^2(5x) + 2\,\varepsilon(\sin(x)\cos(10x) - \cos(x)\sin(5x)) - \varepsilon^2\sin(2x))}{(\cos^2(10x) + \sin^2(5x) + 2\varepsilon(\sin(x)\cos(10x) - \cos(x)\sin(5x)) + \varepsilon^2)} - 24\,\varepsilon\,\cos(x).$$

The analytical solution is: $y_1(x) = \cos(10x) + \varepsilon\,\sin(x)$, $y_2(x) = \sin(5x) - \varepsilon\,\cos(x)$, $\quad x_0 = 0,\ x_e = 10$.

5.3. Test 3

Consider inhomogeneous equation, [15], $y'' = -\phi^2 y + (\phi^2 - 1)\sin(x)\quad x \geq 0,\ y(0) = 1,\ y'(0) = 11,$ with exact solution $y(x) = \cos(\phi\,x) + \sin(\phi\,x) + \sin(x), \qquad \phi >> 1.$

This is an inhomogeneous equation, whose exact solution consists of a rapidly and slowly oscillating functions; the slowly varying function is due to the inhomogeneous term. In our numerical example we take $\varphi = 10$, $x \in [0, 5]$.

5.4. Test 4

Consider the nonlinear Duffing equation solved on [0, 100 π] , $y''(x) + y(x) + y^3(x) = B\ \cos(\Omega x)$, where $\Omega = 1.01$ and $B = 0.002$. We use the following as the exact solution

$$\begin{aligned} y(x) = {} & 0.200179477536 61852\cos(x) \\ & + 0.246946143255583824 \times 10^{-3}\cos(3x) \\ & + 0.304014985249 \times 10^{-6}\cos(5x) \\ & + 0.374349084378 \times 10^{-9}\cos(7x) \\ & + 0.460964452 \times 10^{-12}\cos(9x) + 0.5676 \times 10^{-15}\cos(1x). \end{aligned}$$

The Tests 1–3 are solved by the multistep method (3–4) with step k = 3 and with fourth order Numerov method which takes the form:

$$y_{n+1} - 2\,y_n + y_{n-1} = h^2(f_{n+1} + 10f_n + f_{n-1})/12 \tag{29}$$

The values of h. In Fig. 5, the absolute errors are plotted with different values of h for the same k (k = 2–4), where in Fig. 6, the absolute errors are plotted with the same h for k = 2–4 (h = Pi/100, Pi/500, Pi/1000).

Table 5
Absolute errors of Test 3.

h	x	s	μ	ν_0	Absolute errors of (3–4)	Absolute errors of (34)
001	10	121/72	-19/3456	19/5184	2.02043 E−3	5.53521−4
0.001	10				2.12927 E−9	5.45108 E−8

6. Conclusion

In this paper, we obtained a new hybrid parametric method with higher order phase-lag. The investigation of the new method consists of: 1. The construction of the method, 2. The study of the truncation error, 3. Using good choices for the values of the free parameters to raise the order of the phase-lag errors, 4. The study of the stability analysis and the intervals of periodicity are tabulated in Table 2 and show that for some values of the parameters the interval of periodicity is (0, ∞) i.e., the methods are P-stable. Finally, test problems are solved and the results are tabulated. The absolute error gets improved using methods (3–4) than using Numerov method (29) as the step size h is decreased for the problems in tests 1 and 3. Despite the fact that in test 2, the perturbed system, the error is comparable with that of the Numerov method (29), still the potential of the current method is obvious when comparing the errors in tests 1 and 3, (Tables 3–5).

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Appendix A

Proof of Theorem 2. Put $\xi = e^{(i\theta H)}$, then, (14) becomes,

$$\begin{aligned} & A_k(H)\,e^{(ik\,\theta(H))} + A_{k-1}(H)\,e^{(i(k-1)\,\theta(H))} \\ & + \ldots + A_1(H)\,e^{(i\,\theta(H))} + A_0(H) = 0, \\ & \text{i.e.}\quad A_k(H)\,\cos(k\,\theta(H)) + A_{k-1}(H)\cos((k-1)\,\theta(H)) \\ & + \ldots + A_1(H)\,\cos(\theta(H)) + A_0(H) \\ & + i(A_k(H)\,\sin(k\,\theta(H)) + A_{k-1}(H)\sin((k-1)\,\theta(H)) \\ & + \ldots + A_1(H)\,\sin(\theta(H))) = 0. \end{aligned}$$

Using (17), the above equation becomes,

$$\begin{aligned} & A_k(H)(\cos(kH) + Ck^2H^{q+2} + O(H^{q+3})) + A_{k-1}(H)(\cos((k-1)H) \\ & + C(k-1)^2H^{q+2} + O(H^{q+3})) + \ldots + A_1(H)(\cos(H) + CH^{q+2} \\ & + O(H^{q+3})) + A_0(H) + i(\,A_k(H)\,(\sin(kH) - CkH^{q+1} + O(H^{q+2})) \\ & + A_{k-1}(H)\,(\sin((k-1)H) - C\,(k-1)H^{q+1} + O(H^{q+2})) \\ & + \ldots + A_1(H)\,(\sin(H) - C\,H^{q+1} + O(H^{q+2}))) = 0. \end{aligned}$$

i.e. $\quad A_k(H)\ \cos(kH) + A_{k-1}(H)\ \cos((k-1)H)$

Table 4
Absolute errors of Test 2.

					Absolute errors of (3–4)		Absolute errors of (34)	
h	x	s	μ	ν_0	Absolute errors of y_1	Absolute errors of y_2	Absolute errors of y_1	Absolute errors of y_2
0.01	10	121/72	-19/3456	19/5184	2.2484 E−8	1.20384E−9	2.09581E−8	1.33617E−9
0.001	10				2.51763 E−4	1.33186E−5	2.30107E−4	1.25844E−5

Table 6
The coefficients $A_i(H)$ in Eq. (13) for $k=4$ and $k=5$.

	k = 4	k = 5
$A_0(H^2)$	$A_0(H^2) = \frac{-120H^2n0+(-4+s)(115+2s(-52+11s))}{5(-4+s)(23+2s(-20+7s))}.$	$-\frac{3288H^2n0}{(-5+s)(-3014+s(6623+s(-3923+675s)))} + \frac{3014-2s(1771+s(-649+75s))}{-3014+s(6623+s(-3923+675s))}.$
$A_1(H^2)$	$-\frac{20H^2(6m-24n0-(-4+s)(-3+s)(-2+s))+16(-4+s)(25+s(-29+7s))}{5(-4+s)(23+2s(-20+7s))},$	$-\frac{137H^2(-22m-120n0+(-5+s)(-4+s)(-3+s)(-2+s))}{(-5+s)(-3014+s(6623+s(-3923+675s)))} + \frac{-11782+s(17551-7315s+915s\wedge^2)}{-3014+s(6623+s(-3923+675s))},$
$A_2(H^2)$	$\frac{6(10H^2(8m-12n0-(-4+s)(-3+s)(-1+s))+(-4+s)(95+2s(-68+19s)))}{5(-4+s)(23+2s(-20+7s))},$	$-\frac{548H^2(28m+60n0-(-5+s)(-4+s)(-3+s)(-1+s))}{(-5+s)(-3014+s(6623+s(-3923+675s)))} - \frac{4(-5069+s(9101+5s(-859+117s)))}{-3014+s(6623+s(-3923+675s))},$
$A_3(H^2)$	$\frac{4(-15H^2(10m-8n0-(-4+s)(-2+s)(-1+s))-4(-4+s)(25+s(-41+13s)))}{5(-4+s)(23+2s(-20+7s))},$	$-\frac{822H^2(-38m-40n0+(-5+s)(-4+s)(-2+s)(-1+s))}{(-5+s)(-3014+s(6623+s(-3923+675s)))} + \frac{-20276+2s(20473+5s(-2153+321s))}{-3014+s(6623+s(-3923+675s))},$
$A_4(H^2)$	$\frac{120H^4m+4H^2(60m-30n0+(-4+s)(-2+s)s)+5(-4+s)(23+2s(-20+7s))}{5(-4+s)(23+2s(-20+7s))},$	$-\frac{548H^2(52m+30n0-(-5+s)(-3+s)(-2+s)(-1+s))}{(-5+s)(-3014+s(6623+s(-3923+675s)))} + \frac{11782-2s(12587+5s(-1429+231s))}{-3014+s(6623+s(-3923+675s))},$
$A_5(H^2)$	0	$1 + \frac{3288H^4m}{(-5+s)(-3014+s(6623+s(-3923+675s)))} - \frac{137H^2(-70m-24n0+(-4+s)(-3+s)(-2+s)(-1+s))}{(-5+s)(-3014+s(6623+s(-3923+675s)))} + \frac{12H^2(274+15(-5+s)s(9+(-5+s)s))}{(-5+s)(-3014+s(6623+s(-3923+675s)))},$

$$
\begin{aligned}
&+ \dots + A_1(H)\ \cos(H) + A_0(H) + i(\ A_k(H)\ \sin(kH) \\
&+ A_{k-1}(H)\ \sin((k-1)H) + \dots + A_1(H)\ \sin(H)) \\
&- iCH^{q+1}(kA_k(H) + (k-1)A_{k-1}(H) + \dots + A_1(H)) \\
&+ H^{q+2}(C\,k^2A_k(H) + C(k-1)^2A_{k-1}(H) + \dots + CA_1(H) \\
&+ i(A_k(H) + A_{k-1}(H) + \dots + A_1(H))) + O(H^{q+3}) = 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{So, } & CH^{q+1} - O(H^{q+2}) - O(H^{q+3}) = (A_k(H)\ \cos(kH) \\
&+ A_{k-1}(H)\ \cos((k-1)H) + \dots + A_1(H)\ \cos(H) + A_0(H) \\
&+ i(\ A_k(H)\ \sin(kH) + A_{k-1}(H)\ \sin((k-1)H) \\
&+ \dots + A_1(H)\ \sin(H))/(kA_k(H) + (k-1)A_{k-1}(H) \\
&+ \dots + A_1(H)).
\end{aligned}
$$

$$
\begin{aligned}
\text{Then } & CH^{q+1} - O(H^{q+2}) = (A_k(H)\ e^{(i\,k\,H)} + A_{k-1}(H)\ e^{(i\,(k-1)\ H)} \\
&+ \dots + A_1(H)\ e^{(i\,H)} + A_0(H))/(kA_k(H) + (k-1)A_{k-1}(H) \\
&+ \dots + A_1(H))
\end{aligned}
$$

Thus the theorem is proved.

Appendix B

Table 6.

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