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Extended inverse Weibull distribution with reliability application

Hassan M. Okasha [a,b], A.H. El-Baz [c,*], A.M.K. Tarabia [c], Abdulkareem M. Basheer [c,d]

[a] Department of Statistics, Faculty of Science, King Abdul Aziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo 11884, Egypt
[c] Department of Mathematics, Faculty of Science, Damietta University, New Damietta, Egypt
[d] Albayda University, Albayda, Yemen

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ABSTRACT

The aim of this paper is to introduce an extension of the inverse Weibull distribution which offers a more flexible distribution for modeling lifetime data. We extend the inverse Weibull distribution by Marshall–Olkin method (MOEIW). Some statistical properties of the MOEIW are explored, such as quantiles, moments and reliability. Moreover, the estimation of the MOEIW parameters is discussed by using Maximum Likelihood Estimation method. In addition, the estimation of the stress-strength parameter is discussed. Finally, the proposed extended model is applied on real data and the results are given which illustrate the superior performance of the MOEIW distribution compared to other models.

1. Introduction

The inverse Weibull (IW) distribution has been used to model, many real life applications for example degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as breakdown of insulating fluid [1]. Inverse Weibull distribution with parameters α (scale parameter) and β (shape parameter) with cumulative distribution function and the probability density function of a random variable X are respectively given by

$$F(x) = e^{-\alpha x^{-\beta}}, \quad x \geq 0, \quad \alpha > 0, \quad \beta > 0 \tag{1}$$

$$f(x) = \alpha\beta x^{-(\beta+1)} e^{-\alpha x^{-\beta}}, \quad x \geq 0, \quad \alpha > 0, \quad \beta > 0. \tag{2}$$

Keller et al. [2] obtained the inverse Weibull model by investigating failures of mechanical components subject to degradation. Calabria and Pulcini [3] computed the maximum likelihood and least squares estimates of the parameters of the inverse Weibull distribution. They also obtained the Bayes estimator of the model parameters as well as confidence limits for reliability and tolerance limits, see Calabria and Pulcini [4,5] and Johnson et al. [6] for additional details. Khan et al. [7] presented some important theoretical properties of the inverse Weibull distribution. The generalizations of the inverse Weibull and related distributions with applications are given by Oluyede and Yang [8].

On the other hand, Marshall and Olkin [9] proposed a transformation of the baseline (cdf) by adding a new parameter to obtain a family of distributions

$$G(x;\theta) = \frac{F(x)}{1-\overline{\theta}\,(1-F(x))}$$

$$-\infty < x < \infty, \quad \theta > 0, \quad \overline{\theta} = 1-\theta. \tag{3}$$

Moreover, Marshall–Olkin method is used to obtain new distributions and their properties are studied e.g., Alice and Jose [10] introduced Marshall–Olkin logistic processes, Gui [11] introduced Marshall–Olkin power lognormal distribution and studied its statistical properties of the new distribution. Cordeiro and Lemonte [12] studied some mathematical properties of Marshall–Olkin extended Weibull distribution. Jose and Krishna [13] studied the Marshall–Olkin extended Uniform distribution. Marshall–Olkin Extended Lomax distribution was introduced by Ghitany et al.[14]. Okasha et al. [15] introduced Marshall–Olkin extended

* Corresponding author.
E-mail addresses: hassanokasha@yahoo.com (H.M. Okasha), ali_elbaz@yahoo.com, elbaz@du.edu.eg (A.H. El-Baz), a_tarabia@yahoo.com (A.M.K. Tarabia), abdulkareem_basheer@yahoo.com (A.M. Basheer).

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generalized linear exponential distribution. Marshall–Olkin extended Pareto distribution was introduced by Ghitany [16], Ghitany et al. [17] conducted a detailed study of Marshall–Olkin extended Weibull distribution, that can be obtained as a compound distribution mixing with exponential distribution, and apply it to model censored data.

The rest of the paper is organized as follows: In Section 2, we define our proposed model namely the Marshall–Olkin extended inverse Weibull and its special cases are presented. In Section 3, its reliability analysis is given. In Section 4, its statistical properties are given. In Section 5, the parameters of this model are estimated using Maximum Likelihood Estimation method. The estimation of the stress-strength parameters are discussed in Section 6. Finally, the proposed model is applied on real data and the results are given in Section 7.

2. New model

In this section we will give the Marshall–Olkin extended inverse Weibull (MOEIW) distribution and some of its sub-models.

2.1. MOEIW specification

Let $\Theta = (\alpha, \beta, \theta)$ and by substitution the cumulative function of inverse Weibull given by (1) in Marshall–Olkin given by (3) we get a new distribution denoted as MOEIW (x, Θ) distribution with cdf given by

$$G(x;\Theta) = \frac{e^{-\alpha x^{-\beta}}}{1-\bar{\theta}(1-e^{-\alpha x^{-\beta}})}, \quad x \geq 0, \quad \Theta > 0 \tag{4}$$

which is equivalent to

$$G(x;\Theta) = \frac{e^{-\alpha x^{-\beta}}}{\theta-(\theta-1)e^{-\alpha x^{-\beta}}}, \quad x \geq 0, \quad \Theta > 0. \tag{5}$$

its corresponding probability density function (pdf) is given by

$$g(x;\Theta) = \frac{\alpha\beta\theta x^{-(\beta+1)}e^{-\alpha x^{-\beta}}}{[\theta-(\theta-1)e^{-\alpha x^{-\beta}}]^2}, \quad x \geq 0, \quad \Theta > 0 \tag{6}$$

Fig. 1 gives graphical representation of pdf for different values of α, β and θ.

- Expansion for the density function

For $|z| < 1$ and $\rho > 0$, we have

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(\rho+j)}{\Gamma(\rho)j!}z^j \tag{7}$$

where $\Gamma(.)$ is the gamma function.

By using (7) the denominator in (6) can be expressed as

$$(\theta-(\theta-1)e^{-\alpha x^{-\beta}})^{-2} = \frac{1}{\theta^2}\sum_{j=0}^{\infty}(j+1)(1-\frac{1}{\theta})^j e^{-\alpha j x^{-\beta}}$$

Then

$$g(x) = \sum_{j=0}^{\infty}\frac{1}{\theta}(1-\frac{1}{\theta})^j\alpha(j+1)\beta x^{-(\beta+1)}e^{-\alpha(j+1)x^{-\beta}}$$

2.2. MOEIW Sub-models

Some of the sub-models of the MOEIW distribution are listed below:

(i) When θ =1, we have the inverse Weibull (IW) distribution.
(ii) When θ =1 and α=1, we have the Fréchet (F) distribution.
(iii) When θ =1 and β=2, we have the inverse Rayleigh (IR) distribution.
(iv) When θ =1 and β=1, we have the inverse exponential (IE) distribution.

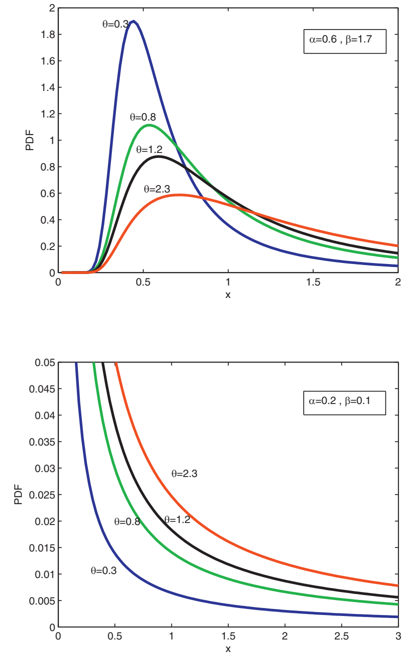

Fig. 1. plots of the PDF of the MOEIW distribution.

3. Reliability analysis

The reliability function (survival function) of MOEIW distribution is given by

$$\bar{G}(x;\Theta) = \frac{\theta(1-e^{-\alpha x^{-\beta}})}{\theta-(\theta-1)e^{-\alpha x^{-\beta}}}, \quad x \geq 0, \quad \Theta > 0. \tag{8}$$

3.1. Hazard rate function

The hazard rate function(failure rate) of a lifetime random variable X with MOEIW distribution is given by

$$h(x;\Theta) = \frac{\alpha\beta x^{-(\beta+1)}e^{-\alpha x^{-\beta}}}{(\theta-(\theta-1)e^{-\alpha x^{-\beta}})(1-e^{-\alpha x^{-\beta}})}, \quad x \geq 0 \tag{9}$$

Fig. 2 gives graphical representations of HRF for different values of α, β and θ.

3.2. Mean residual life

The mean residual life(MRL) function describes the aging process so, it is very important in reliability and survival analysis. The mean residual life(MRL) function of a lifetime random variable X is given by

$$\mu(x) = \frac{1}{\bar{G}(x)}\int_x^{\infty} tg(t)dt - x, \quad x > 0$$

Theorem 3.1. *The MRL function of a lifetime random variable X with MOEIW is given by*

$$\mu(x) = \frac{1}{\bar{G}(x)}\frac{1}{\theta} \times \sum_{j=0}^{\infty}\left(1-\frac{1}{\theta}\right)^j(\alpha(j+1))^{\frac{1}{\beta}}\gamma \times \left(1-\frac{1}{\beta}, \alpha(j+1)x^{-\beta}\right) - x, \quad \beta > 1 \tag{10}$$

Proof. From definition of MRL, we get

$$\mu(x) = \frac{1}{\bar{G}(x)}\int_x^{\infty}\frac{1}{\theta}\sum_{j=0}^{\infty}\left(1-\frac{1}{\theta}\right)^j\alpha(j+1)\beta t^{-\beta}e^{-\alpha(j+1)t^{-\beta}}dt - x.$$

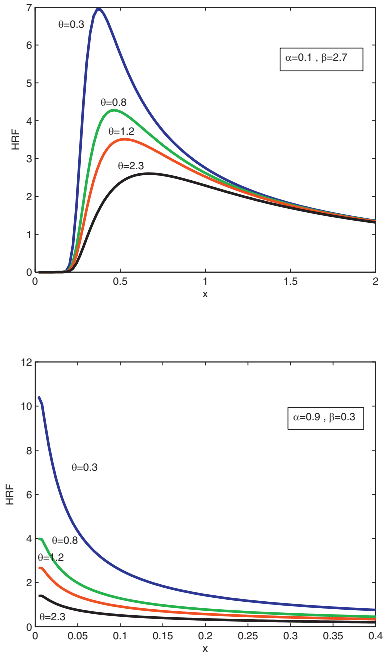

Fig. 2. Plots of the HRF of the MOEIW distribution.

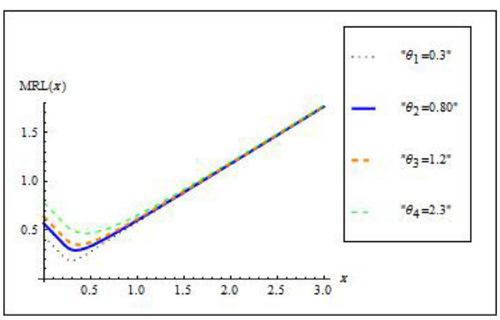

Fig. 3. Plot of the MRL of the MOEIW distribution where $\alpha = 0.1$, $\beta = 2.7$.

Table 1
Some properties of MOEIW for selected values of α=0.6 and β=3 at x=0.8.

parameter θ	HRF	MRL
θ=0.3	3.81615	0.376321
θ=0.8	2.28825	0.507389
θ=1.2	1.73313	0.593127
θ=2.3	1.03958	0.781758

Thus

$$\mu(x) = \frac{1}{\bar{G}(x)}\frac{1}{\theta}\sum_{j=0}^{\infty}(1-\frac{1}{\theta})^j(\alpha(j+1))^{\frac{1}{\beta}}\gamma$$
$$\times\left(1-\frac{1}{\beta},\alpha(j+1)x^{-\beta}\right)-x$$

where $\gamma(c,x) = \int_0^x t^{c-1}e^{-t}dt$, $c > 0$.

Fig. 3 gives graphical representations of MRL for different values of θ. Table 1gives the values of HRF and MRL for the selected value of $\alpha = 0.6$, $\beta = 3$ and x=0.8 and for different value of the parameter θ. One can observe that the values of HRF is decreasing and values of MRL is increasing. □

Table 2
Median and Moments of MOEIW for selected values of α=0.5 and β=4.5.

parameter θ	Median	Mean	Variance	Skewness	Kurtosis
θ=0.3	0.787	0.851	0.066	5.06	137.42
θ=0.8	0.898	0.983	0.118	4.36	104.23
θ=1.2	0.958	1.05	0.150	4.13	94.83
θ=2.3	1.07	1.18	0.215	3.83	83.59

4. Statistical properties

In this section, we study the statistical properties of the MOEIW, specially quantiles, moments, and moment generating function.

4.1. Quantiles

The quantile of any distribution is given by solving the equation

$$G(x_p) = p, \qquad 0 < p < 1 \tag{11}$$

the follows theorem gives the quantile of MOEIW distribution.

Theorem 4.1. *If X has MOEIW* (α, β, θ) *distribution, then The quantile of a random variable X, is given by*

$$x_p = G^{-1}(p) = \left[\frac{1}{\alpha}\log\left(1-\frac{p-1}{p\theta}\right)\right]^{-1/\beta} \tag{12}$$

Proof. By assume $y = e^{-\alpha x^{-\beta}}$ the cdf of MOEIW can be written as $G(x) = \frac{e^y}{\theta-(\theta-1)e^y}$.

The pth quantile function is obtained by solving $G(x) = p$and the obtain result in $y = e^{-\alpha x^{-\beta}}$ by solving for x we get $x_p = G^{-1}(p) = [\frac{1}{\alpha}\log(1-\frac{p-1}{p\theta})]^{-1/\beta}$. □

4.2. The moments

In this subsubsection we will present the rth moments of MOEIW distribution. Moments are important in any statistical analysis.

Theorem 4.2. *If X has MOEIW* (α, β, θ) *distribution, then the rth moments of a random variable X, is given by*

$$E(X^r) = \sum_{j=0}^{\infty}\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^j(\alpha(j+1))^{\frac{r}{\beta}}\Gamma(1-\frac{r}{\beta}), \qquad \beta > r \tag{13}$$

Proof. From definition of moments, we get

$$E(X^r) = \int_0^{\infty} x^r\frac{1}{\theta}\sum_{j=0}^{\infty}\left(1-\frac{1}{\theta}\right)^j\alpha(j+1)\beta x^{-(\beta+1)}e^{-\alpha(j+1)x^{-\beta}}dx$$
$$= \sum_{j=0}^{\infty}\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^j\int_0^{\infty}\alpha(j+1)\beta x^r x^{-(\beta+1)}e^{-\alpha(j+1)x^{-\beta}}dx$$

put $y = \alpha(j+1)x^{-\beta}$. Thus

$$E(X^r) = \sum_{j=0}^{\infty}\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^j(\alpha(j+1))^{\frac{r}{\beta}}\int_0^{\infty}y^{-\frac{r}{\beta}}e^{-y}dy$$
$$= \sum_{j=0}^{\infty}\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^j(\alpha(j+1))^{\frac{r}{\beta}}\Gamma(1-\frac{r}{\beta}), \qquad \beta > r$$

we can easy obtain the mean and the variance by substituting r=1 and r=2 in (13) respectively. □

Table 2 gives the median and moments of MOEIW for selected values of $\alpha = 0.5$ and $\beta = 4.5$ and for different values of parameter θ.

4.3. Moment generating function

The moment generating function of a random variable X provides the basis of an alternative route to analytic results compared with working directly with the cumulative distribution function or probability density function of X.

Theorem 4.3. *If X has MOEIW (α, β, θ) distribution, then the moment generating function of a random variable X, is given by*

$$M_X(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^k}{k!}\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^j(\alpha(j+1))^{\frac{k}{\beta}}\Gamma\left(1-\frac{k}{\beta}\right), \quad \beta > k \tag{14}$$

Proof. We have

$$M_X(t) = \int_0^{\infty} e^{tx}g(x)dx$$

by using the Taylor's series expansion of the function e^{tx}, we obtain

$$M_X(t) = \int_0^{\infty}\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}\left(\sum_{j=0}^{\infty}\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^j\alpha(j+1)\beta x^{-(\beta+1)}e^{-\alpha(j+1)x^{-\beta}}dx\right)$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^k}{k!}\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^j\int_0^{\infty}\alpha(j+1)\beta x^{k-\beta-1}e^{-\alpha(j+1)x^{-\beta}}dx$$

put $y=\alpha(j+1)x^{-\beta}$, we get the above result. □

5. Maximum likelihood estimation

Let $X_1, \ldots, X_n$ be a random sample from MOEIW $\Theta = (\alpha, \beta, \theta)$ distribution; then the likelihood function is given by:

$$\ell(x_1, \ldots, x_n|\Theta) = \prod_{i=1}^{n} g(x_i) = \prod_{i=1}^{n}\frac{\alpha\beta x_i^{-(\beta+1)}e^{-\alpha x_i^{-\beta}}}{\theta[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}]^2}$$

The logarithm of the likelihood function is then

$$L(x_1, \ldots, x_n|\Theta) = n\log(\alpha) + n\log(\beta) - n\log(\theta) - (\beta+1) \times \sum_{i=1}^{n}\log(x_i) - \alpha\sum_{i=1}^{n}x_i^{-\beta} - 2\sum_{i=1}^{n}\log[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}] \tag{15}$$

Normal equations can be obtained by taking the first partial derivatives of the log-likelihood function with respect to the three parameters in Θ, we obtain the components of the score vector $U_\Theta = (U_\alpha, U_\beta, U_\theta)^\top$

$$\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}x_i^{-\beta} - 2\sum_{i=1}^{n}\frac{x_i^{-\beta}(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}}{[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}]} \tag{16}$$

$$\frac{\partial L}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log(x_i) + \alpha\sum_{i=1}^{n}x_i^{-\beta}\log(x_i) + 2\sum_{i=1}^{n}\frac{\alpha x_i^{-\beta}\log(x_i)(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}}{[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}]} \tag{17}$$

$$\frac{\partial L}{\partial\theta} = -\frac{n}{\theta} + 2\sum_{i=1}^{n}\frac{\frac{1}{\theta^2}e^{-\alpha x_i^{-\beta}}}{[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}]} \tag{18}$$

The maximum likelihood estimates $\hat{\Theta}$ of $\Theta = (\alpha, \beta, \theta)$ are obtained by solving the nonlinear equations $\frac{\partial L}{\partial\alpha} = 0$, $\frac{\partial L}{\partial\beta} = 0$ and $\frac{\partial L}{\partial\theta} = 0$. These equations are not in closed form and the values of the parameters α, β and θ must be found by using iterative methods. For a given known scale parameter (α=0.5) and known shape parameter (β=0.7), 1000 different random samples are simulated from MOEIW models with different sizes and different values of the scale parameter θ by using Mathematica. We studied the behavior of the MLEs from unknown scale parameter α and shape parameter β. The values of θ are 0.8 and 1. Tables 3–5 represent MLEs of parameters α, β and θ respectively. From these tables, it is observed that the mean square error (MSE) and the value of bias for the estimates of α, β and θ are decreasing when the sample size n is increasing.

Table 3
MLE of parameter α.

MOEIW(α, β, θ)	n	Estimate	Bias	MSE
MOEIW(0.5,0.7,0.8)	30	0.539	0.0396	0.295
	70	0.578	0.078	0.244
	100	0.560	0.0605	0.196
	150	0.553	0.0533	0.144
MOEIW(0.5,0.7,1)	30	0.555	0.0556	0.31
	70	0.586	0.0861	0.243
	100	0.562	0.0629	0.195
	150	0.5505	0.0505	0.130

Table 4
MLE of parameter β.

MOEIW(α, β, θ)	n	Estimate	Bias	MSE
MOEIW(0.5,0.7,0.8)	30	0.80	0.102	0.0713
	70	0.738	0.0387	0.0392
	100	0.724	0.0247	0.029
	150	0.713	0.0131	0.0214
MOEIW(0.5,0.7,1)	30	0.789	0.0896	0.0664
	70	0.728	0.0281	0.0340
	100	0.718	0.0180	0.0270
	150	0.701	0.00107	0.0205

Table 5
MLE of parameter θ.

MOEIW(α, β, θ)	n	Estimate	Bias	MSE
MOEIW(0.5,0.7,0.8)	30	1.829	1.029	5.369
	70	1.432	0.6325	3.07
	100	1.306	0.5065	2.019
	150	1.155	0.355	1.288
MOEIW(0.5,0.7,1)	30	1.967	0.9672	5.710
	70	1.641	0.6415	3.603
	100	1.533	0.533	2.475
	150	1.368	0.368	1.411

The second derivatives of the log likelihood function of MOEIW with respect to α, β and θ are given by

$$\frac{\partial^2 L}{\partial\alpha^2} = \frac{-n}{\alpha^2} + 2\sum_{i=1}^{n}\frac{x_i^{-2\beta}(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}}{[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}]^2} \tag{19}$$

$$\frac{\partial^2 L}{\partial\beta\partial\alpha} = \sum_{i=1}^{n}x_i^{-\beta}\log(x_i) + 2\sum_{i=1}^{n} \times \frac{x_i^{-\beta}\log(x_i)(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}} - \alpha x_i^{-\beta}]}{[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}]^2} \tag{20}$$

$$\frac{\partial^2 L}{\partial\theta\partial\alpha} = -2\sum_{i=1}^{n}\frac{x_i^{-\beta}\frac{1}{\theta^2}e^{-\alpha x_i^{-\beta}}}{[1-(1-\frac{1}{\theta})e^{-\alpha x_i^{-\beta}}]^2} \tag{21}$$

$$\frac{\partial^2 L}{\partial \beta^2} = \frac{-n}{\beta^2} - \alpha \sum_{i=1}^{n} x_i^{-\beta} (\log(x_i))^2 - 2 \sum_{i=1}^{n} \times \frac{\alpha x_i^{-\beta} (\log(x_i))^2 (1-\frac{1}{\theta}) e^{-\alpha x_i^{-\beta}} [1-(1-\frac{1}{\theta}) e^{-\alpha x_i^{-\beta}} - \alpha x_i^{-\beta}]}{[1-(1-\frac{1}{\theta}) e^{-\alpha x_i^{-\beta}}]^2} \tag{22}$$

$$\frac{\partial^2 L}{\partial \theta \partial \beta} = 2 \sum_{i=1}^{n} \frac{\frac{\alpha}{\theta^2} x_i^{-\beta} \log(x_i) e^{-\alpha x_i^{-\beta}}}{[1-(1-\frac{1}{\theta}) e^{-\alpha x_i^{-\beta}}]^2} \tag{23}$$

$$\frac{\partial^2 L}{\partial \theta^2} = \frac{n}{\theta^2} - 2 \sum_{i=1}^{n} \frac{[2-2e^{-\alpha x_i^{-\beta}} + \frac{1}{\theta} e^{-\alpha x_i^{-\beta}}] \frac{1}{\theta^3} e^{-\alpha x_i^{-\beta}}}{[1-(1-\frac{1}{\theta}) e^{-\alpha x_i^{-\beta}}]^2}. \tag{24}$$

If we denote the MLE of $\Theta = (\alpha, \beta, \theta)$ by $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta})$, then the observed information matrix is given by

$$I(\Theta) = \begin{bmatrix} -\frac{\partial^2 L}{\partial \alpha^2} & -\frac{\partial^2 L}{\partial \beta \partial \alpha} & -\frac{\partial^2 L}{\partial \theta \partial \alpha} \\ -\frac{\partial^2 L}{\partial \alpha \partial \beta} & -\frac{\partial^2 L}{\partial \beta^2} & -\frac{\partial^2 L}{\partial \theta \partial \beta} \\ -\frac{\partial^2 L}{\partial \alpha \partial \theta} & -\frac{\partial^2 L}{\partial \beta \partial \theta} & -\frac{\partial^2 L}{\partial \theta^2} \end{bmatrix}$$

The approximate $(1-\gamma)100\%$ confidence intervals (CIs) for the parameters α, β and θ are determined, respectively as

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\frac{V(\hat{\alpha})}{n}}, \quad \hat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{\frac{V(\hat{\beta})}{n}} \quad \text{and} \quad \hat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{\frac{V(\hat{\theta})}{n}}$$

where $V(\hat{\alpha})$, $V(\hat{\beta})$ and $V(\hat{\theta})$ are the variance of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$, which are given by the diagonal elements of $I^{-1}(\Theta)$ and $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ percentile of the standard normal distribution.

6. Estimation of the stress-strength parameter

In the reliability, the stress-strength (supply-demand) model describes the life of a component which has a random strength X that is subjected to a random stress Z. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X > Z$. Hence, $R = Pr(X > Z)$ is a measure of component reliability. It has many applications in several areas of engineering and science. We now study two cases.

Case one : $\theta_1 \neq \theta_2$

We now derive the reliability R when X and Z have independent MOEIW(α, β, θ_1) and MOEIW(α, β, θ_2) distributions with the same scale parameter α and the same shape parameter β. The pdf of X and the cdf of Z can be expressed from respectively as:

$$g_1(x) = \frac{1}{\theta_1} \sum_{j=0}^{\infty} \left(1 - \frac{1}{\theta_1}\right)^j \alpha (j+1) \beta x^{-(\beta+1)} e^{-\alpha (j+1) x^{-\beta}}$$

$$G_2(x) = \frac{1}{\theta_2} \sum_{k=0}^{\infty} \left(1 - \frac{1}{\theta_2}\right)^k e^{-\alpha (k+1) x^{-\beta}}.$$

We have

$$R = \int_0^{\infty} g_1(x) G_2(x) dx = \frac{1}{\theta_1 \theta_2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(1 - \frac{1}{\theta_1}\right)^j \left(1 - \frac{1}{\theta_2}\right)^k \times \alpha (j+1) \int_0^{\infty} \beta x^{-(\beta+1)} e^{-\alpha (j+k+2) x^{-\beta}} dx.$$

Then

$$R = \frac{1}{\theta_1 \theta_2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(1 - \frac{1}{\theta_1}\right)^j \left(1 - \frac{1}{\theta_2}\right)^k (j+1)[j+k+2]^{-1}. \tag{25}$$

We estimate the stress-strength parameter for the MOEIW distribution, assuming that $X \sim$ MOEIW(α, β, θ_1) and $Z \sim$ MOEIW(α, β, θ_2). Let $x_1, \ldots, x_n$ and $z_1, \ldots, z_m$ be independent observations from MOEIW(α, β, θ_1) and MOEIW(α, β, θ_2) respectively. The total log-likelihood is given by

$$\ell_R(\Theta^*) = (n+m) \log(\alpha\beta) - n \log(\theta_1) - m \log(\theta_2) - (\beta+1) \times \left(\sum_{i=1}^{n} \log(x_i) + \sum_{j=1}^{m} \log(z_j) \right) - \alpha \left(\sum_{i=1}^{n} x_i^{-\beta} + \sum_{j=1}^{m} z_j^{-\beta} \right) - 2 \sum_{i=1}^{n} \log\left[1 - (1 - \frac{1}{\theta_1}) e^{-\alpha x_i^{-\beta}}\right] - 2 \sum_{j=1}^{m} \log\left[1 - \left(1 - \frac{1}{\theta_2}\right) e^{-\alpha z_j^{-\beta}}\right] \tag{26}$$

By taking the first partial derivatives of the total log-likelihood with respect to the four parameters in Θ^*, we obtain the components of score vector $U_{\Theta^*} = (U_\alpha, U_\beta, U_{\theta_1}, U_{\theta_2})^\top$

$$\frac{\partial \ell_R}{\partial \alpha} = \frac{n+m}{\alpha} - \left(\sum_{i=1}^{n} x_i^{-\beta} + \sum_{j=1}^{m} z_j^{-\beta} \right) - 2 \sum_{i=1}^{n} \frac{x_i^{-\beta} (1 - \frac{1}{\theta_1}) e^{-\alpha x_i^{-\beta}}}{[1 - (1 - \frac{1}{\theta_1}) e^{-\alpha x_i^{-\beta}}]} - 2 \sum_{j=1}^{m} \frac{z_j^{-\beta} (1 - \frac{1}{\theta_2}) e^{-\alpha z_j^{-\beta}}}{[1 - (1 - \frac{1}{\theta_2}) e^{-\alpha z_j^{-\beta}}]} \tag{27}$$

$$\frac{\partial \ell_R}{\partial \beta} = \frac{n+m}{\beta} - \left(\sum_{i=1}^{n} \log(x_i) + \sum_{j=1}^{m} \log(z_j) \right) + \alpha \sum_{i=1}^{n} x_i^{-\beta} \log(x_i) + \alpha \sum_{j=1}^{m} z_j^{-\beta} \log(z_j) + 2 \sum_{i=1}^{n} \frac{\alpha x_i^{-\beta} \log(x_i) (1 - \frac{1}{\theta_1}) e^{-\alpha x_i^{-\beta}}}{[1 - (1 - \frac{1}{\theta_1}) e^{-\alpha x_i^{-\beta}}]} + 2 \sum_{j=1}^{m} \frac{\alpha z_j^{-\beta} \log(z_j) (1 - \frac{1}{\theta_2}) e^{-\alpha z_j^{-\beta}}}{[1 - (1 - \frac{1}{\theta_2}) e^{-\alpha z_j^{-\beta}}]} \tag{28}$$

$$\frac{\partial \ell_R}{\partial \theta_1} = -\frac{n}{\theta_1} + 2 \sum_{i=1}^{n} \frac{\frac{1}{\theta_1^2} e^{-\alpha x_i^{-\beta}}}{[1 - (1 - \frac{1}{\theta_1}) e^{-\alpha x_i^{-\beta}}]} \tag{29}$$

$$\frac{\partial \ell_R}{\partial \theta_2} = -\frac{m}{\theta_2} + 2 \sum_{j=1}^{m} \frac{\frac{1}{\theta_2^2} e^{-\alpha z_i^{-\beta}}}{[1 - (1 - \frac{1}{\theta_2}) e^{-\alpha z_i^{-\beta}}]} \tag{30}$$

The MLEs $\hat{\Theta}^*$ of Θ^* is obtained by solving the system of nonlinear equations $U_{\Theta^*} = 0$ numerically. From the solution of these equations, we can estimate R by inserting the estimate $\hat{\Theta}^*$ in Eq. (25).

Case two : $\theta_1 \neq \theta_2$, $\alpha_1 \neq \alpha_2$

We derive the reliability R when X and Z have independent MOEIW(α_1, β, θ_1) and MOEIW(α_2, β, θ_2) distributions with the same shape parameter β. The pdf of X and the cdf of Z can be expressed from respectively as:

$$g_1(x) = \frac{1}{\theta_1} \sum_{j=0}^{\infty} \left(1 - \frac{1}{\theta_1}\right)^j \alpha_1 (j+1) \beta x^{-(\beta+1)} e^{-\alpha_1 (j+1) x^{-\beta}}$$

$$G_2(x) = \frac{1}{\theta_2} \sum_{k=0}^{\infty} \left(1 - \frac{1}{\theta_2}\right)^k e^{-\alpha_2 (k+1) x^{-\beta}}.$$

We have

$$R=\int_0^\infty g_1(x)G_2(x)dx=\frac{1}{\theta_1\theta_2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(1-\frac{1}{\theta_1})^j(1-\frac{1}{\theta_2})^k \times\alpha_1(j+1)\int_0^\infty \beta x^{-(\beta+1)}e^{-(\alpha_1(j+1)+\alpha_2(k+1))x^{-\beta}}dx.$$

Then

$$R=\frac{\alpha_1}{\theta_1\theta_2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(1-\frac{1}{\theta_1}\right)^j\left(1-\frac{1}{\theta_2}\right)^k(j+1) \times[\alpha_1(j+1)+\alpha_2(k+1)]^{-1}. \tag{31}$$

We estimate the stress-strength parameter for the MOEIW distribution, assuming that $X\sim$ MOEIW$(\alpha_1,\ \beta,\ \theta_1)$ and $Z\sim$ MOEIW$(\alpha_2,\ \beta,\ \theta_2)$. Let $x_1,\ldots,x_n$ and $z_1,\ldots,z_m$ be independent observations from MOEIW$(\alpha_1,\ \beta,\ \theta_1)$ and MOEIW$(\alpha_2,\ \beta,\ \theta_2)$respectively. The total log-likelihood is given by

$$\begin{aligned}\ell_R(\Theta^*)&=n\log(\alpha_1\beta)+m\log(\alpha_2\beta)-n\log(\theta_1)-m\log(\theta_2)\\&-(\beta+1)\left(\sum_{i=1}^{n}\log(x_i)+\sum_{j=1}^{m}\log(z_j)\right)-\alpha_1\sum_{i=1}^{n}x_i^{-\beta}\\&-\alpha_2\sum_{j=1}^{m}z_j^{-\beta}-2\sum_{i=1}^{n}\log\left[1-\left(1-\frac{1}{\theta_1}\right)e^{-\alpha_1x_i^{-\beta}}\right]\\&-2\sum_{j=1}^{m}\log\left[1-\left(1-\frac{1}{\theta_2}\right)e^{-\alpha_2z_j^{-\beta}}\right].\end{aligned}\tag{32}$$

By taking the first partial derivatives of the total log-likelihood with respect to the four parameters in Θ^*, we obtain the components of score vector $U_{\Theta^*}=(U_{\alpha_1},U_{\alpha_2},U_\beta,U_{\theta_1},U_{\theta_2})^\top$

$$\frac{\partial\ell_R}{\partial\alpha_1}=\frac{n}{\alpha_1}-\sum_{i=1}^{n}x_i^{-\beta}-2\sum_{i=1}^{n}\frac{x_i^{-\beta}(1-\frac{1}{\theta_1})e^{-\alpha_1x_i^{-\beta}}}{[1-(1-\frac{1}{\theta_1})e^{-\alpha_1x_i^{-\beta}}]}\tag{33}$$

$$\frac{\partial\ell_R}{\partial\alpha_2}=\frac{m}{\alpha_2}-\sum_{j=2}^{m}z_j^{-\beta}-2\sum_{j=1}^{m}\frac{z_j^{-\beta}(1-\frac{1}{\theta_1})e^{-\alpha_1z_j^{-\beta}}}{[1-(1-\frac{1}{\theta_2})e^{-\alpha_2z_j^{-\beta}}]}\tag{34}$$

$$\begin{aligned}\frac{\partial\ell_R}{\partial\beta}&=\frac{n+m}{\beta}-\left(\sum_{i=1}^{n}\log(x_i)+\sum_{j=1}^{m}\log(z_j)\right)+\alpha_1\sum_{i=1}^{n}x_i^{-\beta}\log(x_i)\\&+\alpha_2\sum_{j=1}^{m}z_j^{-\beta}\log(z_j)+2\sum_{i=1}^{n}\frac{\alpha_1x_i^{-\beta}\log(x_i)(1-\frac{1}{\theta_1})e^{-\alpha_1x_i^{-\beta}}}{[1-(1-\frac{1}{\theta_1})e^{-\alpha_1x_i^{-\beta}}]}\\&+2\sum_{j=1}^{m}\frac{\alpha_2z_j^{-\beta}\log(z_j)(1-\frac{1}{\theta_2})e^{-\alpha_2z_j^{-\beta}}}{[1-(1-\frac{1}{\theta_2})e^{-\alpha_2z_j^{-\beta}}]}\end{aligned}\tag{35}$$

$$\frac{\partial\ell_R}{\partial\theta_1}=-\frac{n}{\theta_1}+2\sum_{i=1}^{n}\frac{\frac{1}{\theta_1^2}e^{-\alpha_1x_i^{-\beta}}}{[1-(1-\frac{1}{\theta_1})e^{-\alpha_1x_i^{-\beta}}]}\tag{36}$$

$$\frac{\partial\ell_R}{\partial\theta_2}=-\frac{m}{\theta_2}+2\sum_{j=1}^{m}\frac{\frac{1}{\theta_2^2}e^{-\alpha_2z_j^{-\beta}}}{[1-(1-\frac{1}{\theta_2})e^{-\alpha_2z_j^{-\beta}}]}\tag{37}$$

The MLEs $\hat{\Theta}^*$ of Θ^* is obtained by solving the system of nonlinear equations $U_{\Theta^*}=0$ numerically. From the solution of these equations, we can estimate R by inserting the estimate $\hat{\Theta}^*$ in Eq. (31).

Table 6
Data set.

12	15	22	24	24	32	32	33	34
38	38	43	44	48	52	53	54	54
55	56	57	58	58	59	60	60	60
60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84
85	87	91	95	96	98	99	109	110
121	127	129	131	143	146	146	175	175
211	233	258	258	263	297	341	341	376

Table 7
Descriptive statistics.

Median	Mean	Variance	Skewness	Kurtosis
70	99.81	6580.121	1.796	5.61

Table 8
MLE for data set.

Distributions	parameters		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
MOEIW(α, β, θ)	546.261	2.479	81.335
IW(α, β, 1)	284.08	1.415	–
F(1, β, 1)	–	0.31	–
IR(α, 2, 1)	2187.9	–	–
IE(α, 1, 1)	60.0975	–	–

Table 9
Some measures for the fitted models.

Distributions	-2L	AIC	CAIC	BIC	HQIC
MOEIW	779.4	785.4	785.7	792.2	788.1
IW	791.2	795.2	795.46	799.85	797.1
F	1026.5	1028.5	1028.6	1030.8	1029
IR	813.47	815.47	815.52	817.75	816.3
IE	805.343	807.34	807.4	809.62	808.2

7. Fitting reliability data

In this section, We analysis real data to illustrate that the MOEIW can be a good lifetime model comparing with many known distributions such as inverse Weibull (IW), Fréchet (F), inverse Rayleigh (IR) and inverse exponential (IE) distributions.

The data set in Table 6 have been obtained from Bjerkedal [18]. The data set consists 72 observations.

Table 7 gives a descriptive summary for these data.

The parameter of the sample is estimated numerically. We use Eqs. (16)–(18) to obtain MLEs estimate, Table 8 lists the MLEs of the parameters of the MOEIW and their sub-models (i.e. IW, F, IR and IE) distributions.

Table 9 presents the log-likelihood values (L), Akaike information criterion (AIC), consistent Akaike information criterion(CAIC), Bayesian information criterion(BIC) and Hannan–Quinn information criterion (HQCI) statistics for the fitted MOEIW and their sub-models (i.e. IW, F, IR and IE) distributions. From this Table, we can conclude that the MOEIW model provides a better fit to the current data than the other models.

The observed information of the data and the variance covariance matrices are respectively

$$I_0(\hat{\alpha},\hat{\beta},\hat{\theta})=\begin{bmatrix}0.849\times10^{-4} & -0.195 & 0.556\times10^{-3}\\ -0.195 & 4.721\times10^{2} & -1.302\\ 0.556\times10^{-3} & -1.302 & 0.369\times10^{-2}\end{bmatrix}$$

and

$$I_0^{-1}(\hat{\alpha},\hat{\beta},\hat{\theta})=\begin{bmatrix}8.096\times10^{5} & 4.175 & -1.233\times10^{5}\\ 4.175 & 0.0751 & 27.115\\ -1.233\times10^{5} & 27.115 & 0.283\times10^{5}\end{bmatrix}$$

Table 10
Goodness-of-fit statistics corresponds to data set.

Distributions	K-S	p-value	A	W
MOEIW	0.07627	0.8445	0.6341	0.10522
IW	0.1381	0.1381	1.5177	0.2538
F	0.6398	8.95×10^{-27}	44.042	9.6086
IR	0.2369	4.88×10^{-4}	6.5639	1.2661
IE	0.1846	0.0128	4.6425	0.8379

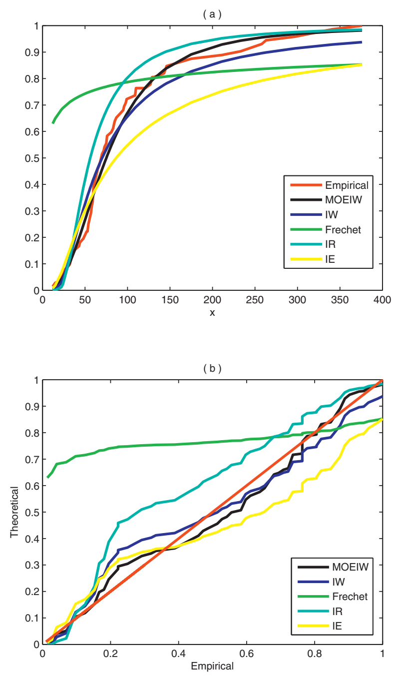

Fig. 4. (a)The empirical and fitted cumulative functions of selected models. (b) P-P plot for the MOEIW and sub-models distributions.

The approximate 95% two sided confidence intervals of the unknown parameters α, β and θ are [338.4 , 754.1], [2.41 , 2.54] and [42.4 , 120.1] respectively.

From Table 10 the small K-S distance and the large p-value for the test indicate that these data fit the MOEIW quite well. In addition, the values of the following statistics: A, W and corresponding p-value for the model MOEIW and its sub-model(i.e. IW, F, IR and IE) can provide good fits for lifetime data. We conclude that this data fits the MOEIW much better the IW distribution.

To test how well the MOEIW distribution fits these data, the hypotheses is $H_0 : G = G_{MOEIW}$ versus $H_1 \colon G \neq G_{MOEIW}$. We use the Kolmogorov–Smirnov (K–S) distances between the empirical distribution function and the fitted distribution function to determine the appropriateness of the model. The K–S value and corresponding p-value are respectively 0.07627 and 0.8445.

Fig. 4(a) gives the empirical and fitted cumulative functions of selected models, while Fig. 4(b) gives the graphical between empirical distribution and theoretical MOEIW distribution. It's clear from these two figures that MOEIW distribution fit well these data.

8. Conclusion

In this paper, we introduced our proposed model, named Marshall–Olkin extended inverse Weibull. Many properties of our proposed model were investigated, including mean residual life, mean inactivity time and moments. The estimation of parameters are obtained by maximum likelihood method. Moreover, estimation of the stress-strength parameter is discussed. The proposed extended model is applied on real data and the results are given which illustrate the superior performance of the MOEIW distribution compared to other models.

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