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Original Article

BV structure on the Hochschild cohomology of Sullivan algebras☆

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ABSTRACT

Let X be a closed, simply connected manifold of dimension m and LX the space of free loops on X. If $(\wedge V, d)$ is the minimal Sullivan model of X where V is finite dimensional, then there is a Gerstenhaber algebra $(\wedge V \otimes \wedge s^{-1}V^{\#}, d_0)$, where $V^{\#}$ is the graded dual of V, and its homology is isomorphic to the loop space homology $\mathbb{H}_*(LX)$. In this paper we define a BV structure on $(\wedge V \otimes \wedge s^{-1}V^{\#}, d_0)$ which extends the Gerstenhaber bracket.

Introduction

Let X be a closed and oriented manifold of dimension m and LX the space of free loops on X. In their seminal paper [1], Chas and Sullivan showed that the shifted homology $\mathbb{H}_*(LX) = H_{*+m}(LX)$ is a Batalin–Vilkovsky algebra (BV-algebra for short). In particular $\mathbb{H}_*(LX)$ is a Gerstenhaber algebra. On the other hand, given a differential graded algebra, the Hochschild cohomology of A, $HH^*(A; A)$, is a Gerstenhaber algebra [2].

When coefficients are taken in a field there is an isomorphism of graded vector spaces [3]

$$\Phi : \mathbb{H}_*(LX) \to HH^*(C^*X; C^*X).$$

If X is simply connected and $\Bbbk = \mathbb{Q}$, then Φ is an isomorphism of Gerstenhaber algebras [4]. However Tradler defined a BV structure on $HH^*(A; A)$ when A is an A_∞-algebra with symmetric, non-degenerate ∞-inner product [5] and Menichi constructed a BV structure on $HH^*(A; A)$ when A is a symmetric algebra [6]. Félix and Thomas showed that there an isomorphism of BV-algebras

$$\mathbb{H}_*(LX) \to HH^*(C^*X; C^*X)$$

when X is simply connected and coefficients are in $\mathbb{Q}$ [7].

Moreover the Gerstenhaber structure of $HH^*(A; A)$ can be computed in terms of derivations of A, when $A = \wedge V$ is a Sullivan algebra [8]. Furthermore if V is finite dimensional, then $HH^*(A; A)$ is the homology of a complex of the form $(\wedge V \otimes \wedge s^{-1}V^{\#}, d_0)$ where $V^{\#}$ is the graded dual of V [9]. In this paper, we define a BV structure on $\wedge V \otimes \wedge s^{-1}V^{\#}$ which extends the Gerstenhaber structure. We do not know if that structure is isomorphic to the one defined by Chas–Sullivan on $\mathbb{H}_*(LX)$ in [1].

The paper is organized as follows. In §1, we give the classic definition of the Hochschild cohomology $HH^*(A; A)$ and derive its computation when $A = \wedge V$ is a Sullivan algebra. In §2 we define a BV structure on $\wedge V \otimes \wedge s^{-1}V^{\#}$ and show that it extends the Gerstenhaber structure.

1. Hochschild cohomology

For a graded vector space V, we use the grading convention $V^n = V_{-n}$. The kth suspension of V is defined by $(s^kV)_n = M_{n-k}$. In the same way $(s^kV)^n = V^{n+k}$.

Let (A, d) be an augmented differential graded cochain algebra over a field $\Bbbk$ of characteristic 0 and $\bar{A} = \ker(A \xrightarrow{\epsilon} \Bbbk)$. If (M, d) and (N, d) are differential graded A-modules, respectively right and left A-modules, the two-sided bar construction $\mathbb{B}(M; A; N)$ is defined by

$$\mathbb{B}_k(M; A; N) = M \otimes T^k(s\bar{A}) \otimes N.$$

An element $m[a_1|a_2|\cdots a_k]n$ is of degree $|m| + |n| + \sum_{i=1}^{k} |sa_i|$. The differential decomposes into $d = d_0 + d_1$, where,

$$d_0 : \mathbb{B}_k(M; A; N) \to \mathbb{B}_k(M; A; N),$$
$$d_1 : \mathbb{B}_k(M; A; N) \to \mathbb{B}_{k-1}(M; A; N),$$

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are defined as follows (see for instance [10]).

$$\begin{aligned} d_0(m[a_1|a_2|\cdots a_k]n) &= (dm)[a_1|a_2|\cdots a_k]n \\ &\quad - \sum_{i=1}^{k}(-1)^{\epsilon(i)}m[a_1|\cdots|da_i|\cdots|a_k]n \\ &\quad + (-1)^{\epsilon(k+1)}m[a_1|a_2|\cdots a_k](dn), \\ d_1(m[a_1|a_2|\cdots a_k]n) &= (-1)^{|m|}(ma_1)[a_2|\cdots a_k]n \\ &\quad + \sum_{i=2}^{k}(-1)^{\epsilon(i)}m[a_1|\cdots|a_{i-1}a_i|\cdots|a_k]n \\ &\quad - (-1)^{\epsilon(k+1)}m[a_1|a_2|\cdots a_{k-1}](a_kn), \end{aligned}$$

where $\epsilon(i) = |m| + \sum_{j=1}^{i-1}|sa_j|$.

The canonical projection

$$\mathbb{B}(A;A;A) \to A$$

is a semi-free resolution of A as an $A \otimes A^{op}$-module [11]. Therefore the normalized Hochschild cochain complex is given by

$$\begin{aligned} (C^*(A;A), D) &= Hom_{A\otimes A^{op}}(\mathbb{B}(A;A;A), A) \\ &\cong (Hom(T(s\bar{A}), A), D_0 + D_1), \end{aligned}$$

where $Df = df - (-1)^{|f|}fd$. It is explicitly given by [10].

$$\begin{aligned} (D_0f)([a_1|a_2|\cdots|a_k]) &= d(f([a_1|a_2|\cdots|a_k])) \\ &\quad + \sum_{i=1}^{k}(-1)^{\eta(i)}f([a_1|\cdots|da_i|\cdots|a_k]) \end{aligned} \quad (1)$$

and

$$\begin{aligned} (D_1f)([a_1|a_2|\cdots|a_k]) &= -(-1)^{|sa_1||f|}a_1f([a_2|\cdots|a_k]) \\ &\quad + (-1)^{\eta(k)}f([a_1|\cdots|a_{k-1}])a_k \\ &\quad - \sum_{i=2}^{k}(-1)^{\eta(i)}f([a_1|\cdots|a_{i-1}a_i|\cdots|a_k]), \end{aligned} \quad (2)$$

where $\eta(i) = |f| + |sa_1| + \cdots + |sa_{i-1}|$.

Definition 1. A Gerstenhaber algebra is a graded commutative algebra $G = \oplus_i G_i$ together with a bracket

$$G_i \otimes G_j \to G_{i+j+1}, \quad x \otimes y \mapsto \{x, y\},$$

such that sG is a graded Lie algebra and the bracket acts like a derivation of algebras. That is, for $x, y, z \in G$,

1. $\{x, y\} = -(1)^{(|x|+1)(|y|+1)}\{y, x\}$,
2. $\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x, z\}\}$,
3. $\{x, yz\} = \{x, y\}z + (-1)^{|y|(|x|+1)}y\{x, z\}$.

The last property can also be written in two other ways:

$$\{x, yz\} = \{x, y\}z + (-1)^{|y||z|}\{x, z\}y, \quad (3)$$

or

$$\{xy, z\} = x\{y, z\} + (-1)^{|x||y|}y\{x, z\}. \quad (4)$$

Note that Eq. (3) generalizes to

$$\{a, b_1\cdots b_n\} = \sum_i (-1)^{\epsilon_i}\{a, b_i\}b_1\cdots\hat{b}_i\cdots b_n, \quad (5)$$

where $\epsilon_i = |b_i|(|b_1| + \cdots + |b_{i-1}|)$.

Let (A, d) be a cochain algebra. As $T(s\bar{A})$ is a coalgebra, the complex $C^*(A; A)$ is endowed with a product which induces a graded commutative algebra structure on $HH^*(A; A)$. Moreover, there is a bracket on $C^*(A; A)$, inducing a Gerstenhaber algebra structure on $HH^*(A, A)$ [2].

A derivation θ of degree k is a linear mapping $A^n \to A^{n-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{k|a|}a\theta(b)$. Let $Der_k\ A$ denote the vector space of all derivations of degree k and $DerA = \oplus_k Der_k A$. With the commutator bracket $DerA$ becomes a graded Lie algebra. Using the grading convention $A^n = A_{-n}$, we may regard a derivation of degree k as increasing the lower degree by k. There is a differential $\delta : (DerA)_k \to (DerA)_{k-1}$ defined by $\delta\theta = [d, \theta]$.

Moreover $L = Der\ A$ is a differential graded A-module with the action given by $(a\theta)(x) = a\theta(x)$. That is, L is a graded A-module which satisfies the relation

$$[d, a\theta] = (da)\theta + (-1)^{|a|}a[d, \theta].$$

With the above grading convention, if $\theta \in L_k$ and $a \in A^i$, then $a\theta \in L_{k-i}$. Consider $\theta, \theta' \in L$ and $a \in A$. Then

$$[\theta, a\theta'] = \theta(a)\theta' + (-1)^{|a||\theta|}a[\theta, \theta']. \quad (6)$$

We note there is an induced A-module structure on $s^{-1}L$ defined by $a(s^{-1}\theta) = (-1)^{|a|}s^{-1}(a\theta)$. On $s^{-1}L$, we define a bracket of degree 1 by

$$\{\alpha, \beta\} = -(-1)^{|\alpha|}s^{-1}[s\alpha, s\beta]. \quad (7)$$

The differential δ' on $s^{-1}L$ can be expressed by

$$\delta'(\alpha) = -s^{-1}[d, s\alpha] = \{\tilde{d}, \alpha\},$$

where $\tilde{d} = s^{-1}d \in (s^{-1}L)_{-2}$.

Consider the commutative differential graded algebra $\wedge_A(s^{-1}L) = T_A(s^{-1}L)/I$ where I is the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|}y \otimes x$, $x, y \in T_A(s^{-1}L)$. We extend the bracket to $\wedge_A s^{-1}L = A \oplus s^{-1}L \oplus \wedge_A^2 s^{-1}L \oplus \cdots$ by $\{a, b\} = 0$ for $a, b \in \wedge_A^0 s^{-1}L = A$, and for $\alpha \in s^{-1}L$, and using Eq. (6) we define $\{\alpha, a\} = (s\alpha)(a)$. Hence

$$\{\alpha, a\beta\} = \{\alpha, a\}\beta + (-1)^{|a|(|\alpha|+1)}a\{\alpha, \beta\}, a \in A, \text{ and } \alpha, \beta \in s^{-1}L. \quad (8)$$

It is then defined inductively on $\wedge_A^{\geq 2}s^{-1}L$ by forcing the Leibniz rule

$$\{\alpha, \beta\gamma\} = \{\alpha, \beta\}\gamma + (-1)^{(|\alpha|+1)|\beta|}\beta\{\alpha, \gamma\}. \quad (9)$$

The differential δ' on $s^{-1}L$ extends to d_0 on $\wedge_A(s^{-1}L)$ in a similar manner, that is, $d_0\alpha = \{\tilde{d}, \alpha\}$. The above bracket (called Nijenhuis–Schouten bracket) turns $(\wedge_A(s^{-1}L), d_0)$ into a differential Gerstenhaber algebra. We have the following result [8].

Theorem 2. *Let $A = \wedge V$ be a Sullivan algebra. There is a canonical map $\phi : (\Lambda_A s^{-1}L, d_0) \to (C^*(A, A), D_0 + D_1)$ which induces an isomorphism of Gerstenhaber algebras in homology.*

Assume that $A = \wedge V$ is a Sullivan algebra where V is finite dimensional and let $Z = s^{-1}V^{\#}$. The isomorphism $\wedge V \otimes Z \cong s^{-1}Der \wedge V$ defines a bracket on $\wedge V \otimes Z$. That is,

$$\{a, b\} = 0, \quad \{\gamma, a\} = (s\gamma)(a) \text{ for } a, b \in A, \gamma \in Z.$$

The bracket is then extended inductively on $\wedge V \otimes \wedge^{\geq 2} Z$ by forcing the Leibniz rule. In particular, if $a \in \wedge V$ and $\gamma_1, \gamma_2 \in Z$, using Eq. (4) or (6) and taking into account that $\{\gamma_i, \gamma_j\} = 0$ (see Lemma 4), one can deduce the formulae

$$\{a\gamma_1, \gamma_2\} = (-1)^{|\gamma_1|(|\gamma_2|+1)}\{a, \gamma_2\}\gamma_1 \quad (10)$$

and

$$\{\gamma_1, a\gamma_2\} = \{\gamma_1, a\}\gamma_2. \quad (11)$$

In the same way the differential δ' on $\wedge V \otimes Z$ extends into a derivation of algebras on $\wedge V \otimes \wedge Z$, which is also denoted by d_0. Moreover $\wedge_A s^{-1}(DerA)$ and $\wedge V \otimes \wedge Z$ are isomorphic as differential Gerstenhaber algebras [9].

2. The BV structure

Definition 3. A graded commutative algebra B is called a Batalin–Vilkovisky algebra if there is a linear map $\Delta : B_n \to B_{n+1}$ such that $\Delta^2 = 0$ and B becomes a Gerstenhaber algebra with the bracket

$$\{x, y\} = (-1)^{|x|}(\Delta(xy) - \Delta(x)y - (-1)^{|x|}x\Delta(y)). \tag{12}$$

We say that the BV structure extends the Gerstenhaber structure on B.

If X is simply connected and coefficients are taken in a field $\Bbbk$ of characteristic 0, and $(\wedge V, d)$ is the minimal Sullivan model of X, then one has an isomorphism of Gerstenhaber algebras [10]

$$HH^*(C^*X; C^*X) \cong HH^*(\wedge V; \wedge V).$$

Therefore one can compute $\mathbb{H}_*(LX) \cong HH^*(\wedge V; \wedge V)$ in terms of $(\wedge V, d)$. We show first the following result.

Lemma 4. *Let $(\wedge V, d)$ be a Sullivan algebra where V is spanned by $\{x_1, \dots, x_k\}$.*

1. *$Der \wedge V$ is generated as a $\wedge V$-module by $\theta_1, \dots, \theta_k$, where θ_i is the unique derivation of $\wedge V$ defined by $\theta_i(x_j) = \delta_{ij}$,*
2. *Furthermore if $(\wedge V, d)$ is minimal, then*
$$[d, \theta_i] = \sum a_{ij}\theta_j, \quad \text{where} \quad \theta_j(a_{ij}) = 0,$$
3. $[\theta_i, \theta_j] = 0$.

Proof.

1. If θ is a derivation of $\wedge V$ such that $\theta(x_i) = a_i$, then $\theta = \sum_i a_i\theta_i$.
2. We assume that $(\wedge(x_1, \dots, x_k), d)$ satisfies $dx_i \in \wedge(x_1, \dots, x_{i-1})$. From (1) above, there are $a_{ij} \in A$ such that $[d, \theta_i] = \sum_j a_{ij}\theta_j$. Therefore,
$$a_{ij} = [d, \theta_i](x_j) = d\theta_i(x_j) + (-1)^{|\theta_i|}\theta_i(dx_j)$$
$$= 0 + (-1)^{|\theta_i|}\theta_i(dx_j) = \pm\frac{\partial(dx_j)}{\partial x_i}.$$
If $j \le i$, then dx_j does not contain x_i as a factor for degree reasons, hence $a_{ij} = 0$.
If $j > i$, then $a_{ij} = \frac{\partial(dx_j)}{\partial x_i}$ cannot contain x_j as a factor. In both cases we have $\theta_j(a_{ij}) = 0$.
3. We observe that
$$[\theta_i, \theta_j](x_l) = \theta_i(\theta_j(x_l)) \pm \theta_j(\theta_i(x_l))$$
$$= \theta_i(\delta_{jl}) \pm \theta_j(\delta_{il}) = 0.$$

□

In the sequel we define a BV structure on $\wedge V \otimes \wedge Z$ that extends the Gerstenhaber bracket. Let $a \in \wedge V$ and $\gamma_1, \dots, \gamma_n \in Z$, not necessarily distinct. For $a\gamma_1 \cdots \gamma_i \cdots \gamma_n \in \wedge V \otimes \wedge Z$, define

$$\Delta : \wedge V \otimes \wedge^n Z \to \wedge V \otimes \wedge^{n-1} Z$$

by

$$\Delta(a\gamma_1 \cdots \gamma_i \cdots \gamma_n) = (-1)^{|a|}\sum_i (-1)^{\epsilon_i}\{a, \gamma_i\}\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_n, \tag{13}$$

where $\epsilon_i = |\gamma_i|(|\gamma_1| + \dots + |\gamma_{i-1}|)$ and Δ is zero when restricted to $\wedge V$. Note that Δ increases the lower degree by 1.

Lemma 5. $\Delta^2 = 0$.

Proof. Consider $a \in \wedge V$ and $\gamma_1, \dots, \gamma_n \in Z$

$$\Delta(\Delta(a\gamma_1 \cdots \gamma_n))$$
$$= (-1)^{|a|}\Delta\left(\sum_i (-1)^{\epsilon_i}\{a, \gamma_i\}\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_n\right)$$
$$= -\sum_{j<i}(-1)^{\epsilon_i+\epsilon_j+|\gamma_i|}\{\{a, \gamma_i\}, \gamma_j\}\gamma_1 \cdots \hat{\gamma}_i \cdots \hat{\gamma}_j \cdots \gamma_n$$
$$- \sum_{j>i}(-1)^{\epsilon_i+\epsilon_j+|\gamma_i||\gamma_j|+|\gamma_i|}\{\{a, \gamma_i\}, \gamma_j\}\gamma_1 \cdots \hat{\gamma}_i \cdots \hat{\gamma}_j \cdots \gamma_n.$$

By Lemma 4, $\{\gamma_i, \gamma_j\} = 0$. Hence

$$0 = \{a, \{\gamma_i, \gamma_j\} = \{\{a, \gamma_i\}, \gamma_j\} - (-1)^{(|a|+1)(|\gamma_i|+1)}\{\{a, \gamma_j\}, \gamma_i\}.$$

Therefore each summand in the sum

$$-\sum_{j<i}(-1)^{\epsilon_i+\epsilon_j+|\gamma_i|}\{\{a, \gamma_i\}, \gamma_j\}\gamma_1 \cdots \hat{\gamma}_i \cdots \hat{\gamma}_j \cdots \gamma_n$$

will cancel with a corresponding term in

$$-\sum_{j>i}(-1)^{\epsilon_i+\epsilon_j+|\gamma_i||\gamma_j|+|\gamma_i|}\{\{a, \gamma_i\}, \gamma_j\}\gamma_1 \cdots \hat{\gamma}_i \cdots \hat{\gamma}_j \cdots \gamma_n.$$

We deduce that $\Delta^2 = 0$. □

Theorem 6. *$(\wedge V \otimes \wedge Z, \Delta)$ is a BV-algebra.*

Proof. It remains to verify that

$$\{\alpha, \beta\} = (-1)^{|\alpha|}(\Delta(\alpha\beta) - \Delta(\alpha)\beta - (-1)^{|\alpha|}\alpha\Delta(\beta)). \tag{14}$$

Let $\alpha = a\gamma_1\gamma_2 \cdots \gamma_p$, $\beta = b\gamma_{p+1}\gamma_{p+2} \cdots \gamma_{p+q}$ and $\xi(p) = |\gamma_1| + \dots + |\gamma_p|$.

$$\Delta(\alpha\beta) = (-1)^{|b|\xi(p)}\Delta(ab\gamma_1\gamma_2 \cdots \gamma_p\gamma_{p+1}\gamma_{p+2} \cdots \gamma_{p+q})$$
$$= (-1)^{|b|\xi(p)+|ab|}\sum_{i=1}^{p+q}(-1)^{\epsilon(i)}\{ab, \gamma_i\}\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_{p+q}$$
$$= (-1)^{|b|\xi(p)+|ab|}\sum_{i=1}^{p+q}(-1)^{\epsilon(i)}\ (a\{b, \gamma_i\}\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_{p+q}$$
$$+ (-1)^{|a||b|}b\{a, \gamma_i\}\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_{p+q}).$$

On the other hand

$$\Delta(\alpha)\beta = (-1)^{|a|}\sum_{i=1}^{p}(-1)^{\epsilon(i)}\{a, \gamma_i\}\gamma_1\hat{\gamma}_i \cdots \gamma_p(b\gamma_{p+1} \cdots \gamma_{p+q})$$
$$= (-1)^{|a|+|b|(\xi(p)+1)}\left(\sum_{i=1}^{p}(-1)^{\epsilon(i)}b\{a, \gamma_i\}\gamma_1\hat{\gamma}_i \cdots \gamma_p\gamma_{p+1} \cdots \gamma_{p+q}\right).$$

$$\alpha\Delta(\beta) = (a\gamma_1 \cdots \gamma_p)\left((-1)^{|b|}\sum_{i=p+1}^{p+q}(-1)^{\epsilon(i)}\{b, \gamma_i\}\gamma_{p+1} \cdots \hat{\gamma}_i \cdots \gamma_{p+q}\right)$$
$$= \sum_{i=p+1}^{p+q}(-1)^{|b|+(|b|+1)\xi(p)+\epsilon(i)} \times a\{b, \gamma_i\}\gamma_1 \cdots \gamma_p\gamma_{p+1} \cdots \hat{\gamma}_i \cdots \gamma_{p+q}.$$

$$\{\alpha, \beta\} = \{a\gamma_1 \cdots \gamma_p, b\gamma_{p+1} \cdots \gamma_{p+q}\}$$
$$= \{a\gamma_1 \cdots \gamma_p, b\}\gamma_{p+1} \cdots \gamma_{p+q} + (-1)^{|b|(|a|+\xi(p)+1)}b\{a\gamma_1 \cdots \gamma_p, \gamma_{p+1} \cdots \gamma_{p+q}\}$$
$$= \sum_{i=1}^{p}(-1)^{|b|+(|b|+1)\xi(p)+\epsilon(i)} \times a\{b, \gamma_i\}\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_p\gamma_{p+1} \cdots \gamma_{p+q}$$
$$+ \sum_{i=p+1}^{p+q}(-1)^{|b||a|+(|b|+1)\xi(p)+\epsilon(i)} \times b\{a, \gamma_i\}\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_p\gamma_{p+1} \cdots \gamma_{p+q}.$$

Hence Eq. (14) is satisfied. □

Lemma 7. *The map* $\Delta : (\wedge V \otimes \wedge Z, d_0)_i \to (\wedge V \otimes \wedge Z, d_0)_{i+1}$ *commutes with differentials.*

Proof. Recall that $d_0\gamma = \{\tilde{d}, \gamma\}$, where $\tilde{d} = s^{-1}d$. Moreover, it comes from the definition that $\Delta(a) = \Delta(\gamma) = 0$, for $a \in A$ and $\gamma \in Z$. From Lemma 4(2), we also have the relationship $\Delta(\{\tilde{d}, \gamma\}) = 0$.

We show by induction on n that

$$(d_0\Delta + \Delta d_0)(x) = 0 \quad \text{for} \quad x \in \wedge V \otimes \wedge^n Z.$$

Let $a\gamma \in \wedge V \otimes Z$.

$$\begin{aligned} d_0(\Delta(a\gamma)) &= (-1)^{|a|} d_0(\{a, \gamma\}) = (-1)^{|a|}\{\tilde{d}, \{a, \gamma\}\} \\ &= (-1)^{|a|}\left(\{\{\tilde{d}, a\}, \gamma\} - (-1)^{|a|}\{a, \{\tilde{d}, \gamma\}\right) \\ &= (-1)^{|a|}\{\{\tilde{d}, a\}, \gamma\} - \{a, \{\tilde{d}, \gamma\}\}. \end{aligned} \tag{15}$$

On the other hand

$$\Delta(d_0(a\gamma)) = \Delta(\{\tilde{d}, a\}\gamma + (-1)^{|a|} a\{\tilde{d}, \gamma\}).$$

Using Eq. (12), one gets

$$\begin{aligned} \Delta(d_0(a\gamma)) &= -(-1)^{|a|}\{\{\tilde{d}, a\}, \gamma\} + (-1)^{|a|}\Delta(a\{\tilde{d}, \gamma\}) \\ &= -(-1)^{|a|}\{\{\tilde{d}, a\}, \gamma\} + \{a, \{\tilde{d}, \gamma\}\} \\ &\quad + (-1)^{|a|}\Delta(a)\gamma + a\Delta(\{\tilde{d}, \gamma\}). \end{aligned}$$

As $\Delta(a) = 0$ and $\Delta(\{\tilde{d}, \gamma\}) = 0$, we deduce that

$$\Delta(d_0(a\gamma)) = -(-1)^{|a|}\{\{\tilde{d}, a\}, \gamma\} + \{a, \{\tilde{d}, \gamma\}\}. \tag{16}$$

Combining Eqs. (15) and (16) yields

$$(\Delta d_0 + d_0\Delta)(a\gamma) = 0.$$

Assume that $(\Delta d_0 + d_0\Delta)(x) = 0$ for $x \in \wedge V \otimes \wedge^n Z$, we need to show that $(\Delta d_0 + d_0\Delta)(x\gamma) = 0$, for $\gamma \in Z$.

$$\begin{aligned} (d_0\Delta)(x\gamma) &= d_0((-1)^{|x|}\{x, \gamma\} + \Delta(x)\gamma) \\ &= (-1)^{|x|}\{\tilde{d}, \{x, \gamma\}\} + d_0(\Delta(x))\gamma - (-1)^{|x|}\Delta(x)d_0(\gamma). \end{aligned}$$

Moreover

$$\begin{aligned} \Delta(d_0(x\gamma)) &= \Delta(d_0(x)\gamma) + (-1)^{|x|}\Delta(xd_0(\gamma)) \\ &= -(-1)^{|x|}\{d_0(x), \gamma\} + \Delta(d_0(x))\gamma - (-1)^{|x|}d_0(x)\Delta\gamma \\ &\quad + \{x, \{\tilde{d}, \gamma\}\} + (-1)^{|x|}\Delta(x)d_0(\gamma) + x\Delta(d_0\gamma) \end{aligned}$$

As $\Delta(\gamma) = \Delta(d_0(\gamma)) = 0$ and $\Delta d_0 x = -d_0\Delta(x)$ by induction hypothesis, therefore

$$\begin{aligned} \Delta(d_0(x\gamma)) &= -(-1)^{|x|}\{\{\tilde{d}, x\}, \gamma\} + \{x, \{\tilde{d}, \gamma\}\} - d_0(\Delta(x))\gamma \\ &\quad + (-1)^{|x|}\Delta(x)d_0(\gamma). \end{aligned}$$

Using Jacobi identity, one deduces the identity $(\Delta d_0 + d_0\Delta)(x\gamma) = 0$. □

We deduce the following result.

Theorem 8. *The induced map*

$$H_*(\Delta) : H_*(\wedge V \otimes \wedge Z, d_0) \to H_{*+1}(\wedge V \otimes \wedge Z, d_0)$$

defines a BV structure that extends the Gerstenhaber bracket on $H_*(\wedge V \otimes \wedge Z, d_0)$.

Remark 9. The action of S^1 on LX induces a BV structure on $\mathbb{H}_*(LX)$ that extends the Gerstenhaber bracket. Moreover

$$\mathbb{H}_*(LX) \cong H_*(\wedge V \otimes \wedge Z, d_0) \cong HH^*(\wedge V; \wedge V).$$

There is a BV structure on $HH^*(\wedge V; \wedge V)$ that induces an isomorphism $\mathbb{H}_*(LX) \cong HH^*(\wedge V; \wedge V)$ of BV-algebras [7]. We do not know if there is an isomorphism of BV-algebras $\mathbb{H}_*(LX) \cong H_*(\wedge V \otimes \wedge Z, d_0)$.

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