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Review Paper

Binormal motions of inextensible curves in de-sitter space $\mathbb{S}^{2,1}$

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ABSTRACT

In the present work, we continue our study of motions of inextensible curves in de-sitter space $\mathbb{S}^{2,1}$, that we started in [1]. The binormal motions of timelike curves and spacelike curves with a timelike normal vector in $\mathbb{S}^{2,1}$ are described and studied. By the motions of these types of curves, new surfaces (we will call them Hasimoto surfaces) are constructed and plotted by using the hollow ball model by Mathematica 7.

1. Introduction

Many areas of physics and engineering used gaseous and liquid flows. Gaseous flows are very important in spacecraft, cars, and aircraft. Also, they used in the design of turbines and combustion engines. The study of liquid flow is very necessary for the applications of naval, such as the design of ships and many projects in civil engineering such as the design of the harbor and the protection of coastal. The curve flows are studied by many authors, Samah [1], studied and gave the general description the motions of spacelike curves with spacelike normal vector in 3-dimensional de-sitter space $\mathbb{S}^{2,1}$ and gave some explicit examples of motions of these curves in $\mathbb{S}^{2,1}$. Schief and Rogers [2], studied the binormal motion of curves with constant curvatures. Nassar et al. [3–6], constructed and studied new geometrical models of flows of curves and surfaces. Also, they constructed the Hasimoto surfaces in $\mathbb{R}^3$. T. Körpinar [7], used the Frenet frame of curves and constructed a new method for inextensible flows of timelike curves in Minkowski space-time $\mathbb{R}^{4,1}$.

In [8], we studied the motions of inextensible curves in spherical space $\mathbb{S}^3$. Rawya and Samah [9], studied the generated surfaces from the motions of inextensible curves in $\mathbb{R}^3$.

The outline of this paper is organized as follows: In Section 2, we give some geometric concepts in Minkowski space $\mathbb{R}^{3,1}$ and de-sitter space $\mathbb{S}^{2,1}$. Also, we study the geometry of timelike curves and spacelike curves with a timelike normal vector in $\mathbb{S}^{2,1}$. In Section 3, we study the binormal motions of timelike curves and spacelike curves with a timelike normal vector in $\mathbb{S}^{2,1}$. In Section 4, we construct Hasimoto surfaces that are generated by the binormal motions of timelike curves and spacelike curves with a timelike normal vector in $\mathbb{S}^{2,1}$. Finally, the last section is devoted to the conclusion.

2. Geometric preliminaries

The Minkowski space, or Lorentz space, is the space $\mathbb{R}^{3,1}$, which is defined as a four-dimensional $\mathbb{R}$-vector space consisting of vectors $\{X = (x_0, x_1, x_2, x_3) \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$, with the metric

$$g = dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2.$$

Definition 2.1 [10]**.** Let X, Y, Z be vectors in $\mathbb{R}^{3,1}$, where $X = (x_0, x_1, x_2, x_3)$, $Y = (y_0, y_1, y_2, y_3)$ and $Z = (z_0, z_1, z_2, z_3)$. The inner product is defined by

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_0y_0.$$

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The pseudo vector product of X, Y and Z is defined as

$$X \times Y \times Z = \det \begin{pmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{pmatrix},$$

where $e_0 = (1,0,0,0), e_1 = (0,1,0,0), e_2 = (0,0,1,0)$ and $e_3 = (0,0,0,1)$.

Definition 2.2 [10]**.** An arbitrary nonzero vector $v \in \mathbb{R}^{3,1}$ is **spacelike** if $\langle v, v\rangle > 0$, **timelike** if $\langle v, v\rangle < 0$ and **null** (lightlike) if $\langle v, v\rangle = 0$. The **signature** of a vector v is

$$sign(v) = \begin{cases} 1 & \text{v is spacelike,} \\ 0 & \text{v is lightlike,} \\ -1 & \text{v is timelike.} \end{cases}$$

The **norm** of the vector v is $\|v\| = \sqrt{|\langle v, v\rangle|}$.

Definition 2.3 [10]**.** The 3−dimensional de-sitter space $\mathbb{S}^{2,1}$ is defined by

$$\mathbb{S}^{2,1} = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^{3,1} \mid \sum_{j=1}^{3} x_j^2 - x_0^2 = 1\}.$$

The set of null vectors of $\mathbb{R}^{3,1}$ forms the light cone

$$L^3 = \{(x_0, x_1, x_2, x_3) \mid x_0^2 = x_1^2 + x_2^2 + x_3^2, \quad x_0 \neq 0\}.$$

Definition 2.4 [11]**.** For plotting surfaces in de-sitter space $\mathbb{S}^{2,1}$, we use the hollow ball model of $\mathbb{S}^{2,1}$ (it is a 3-dimensional ball in $\mathbb{R}^3$). For any point $(x_0, x_1, x_2, x_3) \in \mathbb{S}^{2,1} \longrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}$, define

$$y_k = e^{\arctan x_0} \frac{x_k}{\sqrt{1 + x_0^2}}, \qquad k = 1, 2, 3.$$

Then $e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}$. The identification $(x_0, x_1, x_2, x_3) \leftrightarrow ;(y_1, y_2, y_3)$ is then a bijection from $\mathbb{S}^{2,1}$ to the hollow ball

$$\mathcal{H} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}\}.$$

So $\mathbb{S}^{2,1}$ is identified with the hollow ball $\mathcal{H}$.

Definition 2.5. Consider the 3−dimensional de-sitter space $\mathbb{S}^{2,1}$ in $\mathbb{R}^{3,1}$. A regular parametrized curve $\hat{\gamma} = \hat{\gamma}(u)$, $\hat{\gamma} : I \longrightarrow \mathbb{S}^{2,1}$ is called **spacelike** if $\langle \dot{\hat{\gamma}}, \dot{\hat{\gamma}}\rangle > 0$, **timelike** if $\langle \dot{\hat{\gamma}}, \dot{\hat{\gamma}}\rangle < 0$, and **null** (light-like) if $\langle \dot{\hat{\gamma}}, \dot{\hat{\gamma}}\rangle = 0$, for all $\hat{u} \in I$, where $\hat{u}$ is the parameter of the curve $\hat{\gamma}$ and $\dot{\hat{\gamma}}(\hat{u})$ is the tangent vector to the curve $\hat{\gamma}$ and $\dot{} = \frac{d}{d\hat{u}}$ (see [10]).

Definition 2.6. Let $\hat{\gamma}(\hat{s}(\hat{u})) : I \to \mathbb{S}^{2,1}$ be a regular timelike curve or spacelike curve in de-sitter space $\mathbb{S}^{2,1}$. The arc-length of the curve $\hat{\gamma}$ with arbitrary parameter $\hat{u} \in I$ measured from $\hat{\gamma}(0)$, $0 \in I$ is defined by

$$\hat{s}(\hat{u}) = \int_0^{\hat{u}} \|\dot{\hat{\gamma}}(\hat{\sigma})\| d\hat{\sigma}.$$

Since $\hat{\gamma}$ is regular, then we define $\hat{g} > 0$ by $\frac{d\hat{s}}{d\hat{u}} = \|\dot{\hat{\gamma}}\| = \sqrt{\hat{g}}$.

Definition 2.7. If $\|\dot{\hat{\gamma}}\| = 1$ for all $\hat{u} \in I$, then $\hat{\gamma} = \hat{\gamma}(\hat{s})$ is said to be an arc-length parametrized or unit speed parametrized curve.

Consider that the curve $\hat{\gamma}$ is parametrized by the arc-length. Assume that $\langle \hat{\gamma}''(\hat{s}), \hat{\gamma}''(\hat{s})\rangle \neq 1$, where $' = \frac{d}{d\hat{s}}$. Let $\{\hat{\gamma}, \hat{T}, \hat{N}, \hat{B}\}$ be the Serret–Frenet frame of the curve $\hat{\gamma}$, where $\hat{\gamma}(\hat{s})$ is the position vector of the curve $\hat{\gamma}$ and $\hat{T}$, $\hat{N}$ and $\hat{B}$ are respectively, the unit tangent, the unit principal normal and the unit binormal vector field to the curve $\hat{\gamma}(\hat{s})$.

2.1. Geometric properties of the timelike curves in $\mathbb{S}^{2,1}$

Definition 2.8. Consider the timelike curve with the spacelike principal normal vector N and the spacelike binormal vector B. Then from [1], we have $\epsilon_1 = -1$ and $\epsilon_2 = 1$. Hence, the Frenet frame in $\mathbb{S}^{2,1}$ has the following properties:

- $\langle \hat{\gamma}, \hat{\gamma}\rangle = 1$, since the curve is in $\mathbb{S}^{2,1}$.
- $sign(\hat{T}) = -1$, $sign(\hat{T}' - \hat{\gamma}) = 1$, where $\hat{\gamma}' = \hat{T}$.
- $\hat{B}$ is chosen so that $\{\hat{\gamma}, \hat{T}, \hat{N}, \hat{B}\}$ is an oriented orthonormal basis of $\mathbb{R}^{3,1}$, so $\hat{B} = \hat{\gamma} \times \hat{T} \times \hat{N}$.

Definition 2.9. The unit normal vector to the timelike curve $\hat{\gamma}(\hat{s})$ is defined by

$$\hat{N} = \frac{\hat{T}' - \hat{\gamma}}{\|\hat{T}' - \hat{\gamma}\|}.$$

Lemma 2.10. *The inner product and the vector product have the following properties:*

- $\langle \hat{\gamma}, \hat{T}\rangle = \langle \hat{\gamma}, \hat{N}\rangle = \langle \hat{\gamma}, \hat{B}\rangle = \langle \hat{T}, \hat{N}\rangle = \langle \hat{T}, \hat{B}\rangle = \langle \hat{N}, \hat{B}\rangle = 0$, $\langle \hat{N}, \hat{N}\rangle = \langle \hat{B}, \hat{B}\rangle = 1$.
- $\hat{B} \times \hat{T} \times \hat{N} = -\hat{\gamma}$, $\hat{\gamma} \times \hat{B} \times \hat{N} = \hat{T}$, $\hat{\gamma} \times \hat{T} \times \hat{B} = -\hat{N}$.

Definition 2.11. The curvature and torsion of the timelike curves are defined by

- $\hat{k} = \langle \hat{T}' - \hat{\gamma}, \hat{N}\rangle$, i.e., $\hat{k} = \sqrt{\langle \hat{T}' - \hat{\gamma}, \hat{T}' - \hat{\gamma}\rangle} = \|\hat{T}' - \hat{\gamma}\|$.
- $\hat{\tau} = \langle \hat{N}', \hat{B}\rangle$, i.e., $\hat{\tau} = \frac{-1}{\hat{k}^2}\det(\hat{\gamma}, \hat{\gamma}', \hat{\gamma}'', \hat{\gamma}''')$.

Lemma 2.12. *The Serret–Frenet frame satisfies*

$$\hat{F}_{\hat{s}} = \hat{M} \cdot \hat{F}, \tag{2.1}$$

where $\hat{F} = \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{N} \\ \hat{B} \end{pmatrix}$ *and* $\hat{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \hat{k} & 0 \\ 0 & \hat{k} & 0 & \hat{\tau} \\ 0 & 0 & -\hat{\tau} & 0 \end{pmatrix}$.

2.2. Geometric properties of the spacelike curves with the timelike normal vector in $\mathbb{S}^{2,1}$

Definition 2.13. Consider the spacelike curve with the timelike normal vector N and the spacelike binormal vector B. Then from [1], we have $\epsilon_1 = 1$ and $\epsilon_2 = -1$. Hence, the Frenet frame in $\mathbb{S}^{2,1}$ has the following properties:

- $\langle \hat{\gamma}, \hat{\gamma}\rangle = 1$, since the curve is in $\mathbb{S}^{2,1}$.
- $sign(\hat{T}) = 1$, $sign(\hat{T}' + \hat{\gamma}) = -1$, where $\hat{\gamma}' = \hat{T}$.
- $\hat{B}$ is chosen so that $\{\hat{\gamma}, \hat{T}, \hat{N}, \hat{B}\}$ is an oriented orthonormal basis of $\mathbb{R}^{3,1}$, so $\hat{B} = \hat{\gamma} \times \hat{T} \times \hat{N}$.

Definition 2.14. The unit timelike normal vector to the spacelike curves $\hat{\gamma}(\hat{s})$ is defined by

$$\hat{N} = \frac{\hat{T}' + \hat{\gamma}}{\|\hat{T}' + \hat{\gamma}\|}.$$

Lemma 2.15. *The inner product and the vector product have the following properties:*

- $\langle \hat{\gamma}, \hat{T}\rangle = \langle \hat{\gamma}, \hat{N}\rangle = \langle \hat{\gamma}, \hat{B}\rangle = \langle \hat{T}, \hat{N}\rangle = \langle \hat{T}, \hat{B}\rangle = \langle \hat{N}, \hat{B}\rangle = 0$, $\langle \hat{N}, \hat{N}\rangle = -1$, $\langle \hat{B}, \hat{B}\rangle = 1$.
- $\hat{B} \times \hat{T} \times \hat{N} = -\hat{\gamma}$, $\hat{\gamma} \times \hat{B} \times \hat{N} = -\hat{T}$, $\hat{\gamma} \times \hat{T} \times \hat{B} = \hat{N}$.

Definition 2.16. The curvature and torsion of the spacelike curve with the timelike normal vector are defined by

- $\hat{k} = -\langle \hat{T}' + \hat{\gamma}, \hat{N}\rangle$, i.e., $\hat{k} = \sqrt{\langle \hat{T}' + \hat{\gamma}, \hat{T}' + \hat{\gamma}\rangle} = \|\hat{T}' + \hat{\gamma}\|$.
- $\hat{\tau} = \langle \hat{N}', \hat{B}\rangle$, i.e., $\hat{\tau} = \frac{-1}{\hat{k}^2}\det(\hat{\gamma}, \hat{\gamma}', \hat{\gamma}'', \hat{\gamma}''')$.

Lemma 2.17. *The Serret–Frenet frame satisfies*

$$\hat{F}_{\hat{s}} = \hat{M} \cdot \hat{F}, \tag{2.2}$$

$$\text{where } \hat{F} = \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{N} \\ \hat{B} \end{pmatrix} \text{ and } \hat{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \hat{k} & 0 \\ 0 & \hat{k} & 0 & \hat{\tau} \\ 0 & 0 & \hat{\tau} & 0 \end{pmatrix}.$$

3. Binormal motions of the timelike curves and the spacelike curves with the timelike normal vector in $\mathbb{S}^{2,1}$

Suppose that $\hat{\gamma}_0 : I \to \mathbb{S}^{2,1}$ is a regular timelike curve or spacelike curve with the timelike normal vector N in $\mathbb{S}^{2,1}$. Let $\hat{C}_t : \hat{\gamma}(\hat{s}, t)$ be a family of timelike curves or spacelike curves with the timelike normal vector where $\hat{\gamma}(\hat{s}, t) : I \times [0, \infty) \longrightarrow \mathbb{S}^{2,1}$, with initial curve $\hat{\gamma}_0 = \hat{\gamma}(\hat{s}, 0)$.

Let $\hat{\gamma}(\hat{s}, t)$ be the position vector of a point on the curve at time t and at the arc-length $\hat{s}$. The time parameter t is the parameter for the deformation $\hat{C}_t$ of the curve.

The arc-length of the timelike curve and the spacelike curve with the timelike normal vector is defined by

$$\hat{s}(\hat{u}, t) = \int_0^{\hat{u}} \sqrt{\hat{g}(\hat{\sigma}, t)} d\hat{\sigma},$$

where $\sqrt{\hat{g}} = \|\dot{\hat{\gamma}}(\hat{\sigma}, t)\|$. Then the element of the arc-length is $d\hat{s} = \sqrt{\hat{g}(\hat{u}, t)} d\hat{u}$, and the operator $\frac{\partial}{\partial \hat{s}}$ satisfies the following:

$$\frac{\partial}{\partial \hat{s}} = \frac{1}{\sqrt{\hat{g}}} \frac{\partial}{\partial \hat{u}} \quad , \quad \frac{\partial \hat{s}}{\partial \hat{u}} = \sqrt{\hat{g}}.$$

Definition 3.1. The curve $\hat{\gamma}(\hat{s}, t)$ and its flow $\frac{\partial \hat{\gamma}(\hat{s}, t)}{\partial t}$ in $\mathbb{S}^{2,1}$ are said to be inextensible if

$$\dot{\hat{s}} = \frac{\partial}{\partial t} \|\dot{\hat{\gamma}}(\hat{s}, t)\| = 0, \quad i.e, \hat{g}_t = 0.$$

Hence, the arclength of the curve $\hat{\gamma}(\hat{s}, t)$ is preserved.

The binormal motions of the timelike curves and the spacelike curves with the timelike normal vector can be expressed by the velocity vector field

$$\frac{\partial \hat{\gamma}}{\partial t} = \hat{V} \hat{B}, \tag{3.1}$$

where $\{\hat{\gamma}, \hat{T}, \hat{N}, \hat{B}\}$ is the orthonormal Frenet frame to the curve $\hat{C}_t$, and $\hat{V}$ is the velocity vector in the direction of the principal binormal vector $\hat{B}$ and it is a function of the curvature $\hat{k}(\hat{s}, t)$, torsion $\hat{\tau}(\hat{s}, t)$ of the curve, and the higher derivatives of $\hat{k}(\hat{s}, t)$ and $\hat{\tau}(\hat{s}, t)$.

3.1. Evolution equations of the timelike curves in $\mathbb{S}^{2,1}$

Theorem 3.2. *The time evolution of the Serret–Frenet frame for the timelike curve can be given through the matrix form:*

$$\hat{F}_t = \hat{Q} \cdot \hat{F}, \tag{3.2}$$

where

$$\hat{F} = \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{N} \\ \hat{B} \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} 0 & 0 & 0 & \hat{V} \\ 0 & 0 & -\hat{\tau}\hat{V} & \hat{V}_{\hat{s}} \\ 0 & -\hat{\tau}\hat{V} & 0 & \hat{\psi} \\ -\hat{V} & \hat{V}_{\hat{s}} & -\hat{\psi} & 0 \end{pmatrix} \quad and$$

$$\hat{\psi}(\hat{s}, t) = \frac{1}{\hat{k}} \Big(-(1 + \hat{\tau}^2)\hat{V} + \hat{V}_{\hat{s}\hat{s}} \Big).$$

Also, the time evolution equations of the curvature and torsion for the inextensible timelike curve $\hat{C}_t$ are given by:

$$\begin{aligned} \hat{k}_t &= -\hat{\tau}_{\hat{s}} \hat{V} - 2\hat{\tau} \hat{V}_{\hat{s}}, \\ \hat{\tau}_t &= -\hat{k} \hat{V}_{\hat{s}} + \frac{\partial \hat{\psi}}{\partial \hat{s}}. \end{aligned} \tag{3.3}$$

Proof. Take the $\hat{u}$ derivative of (3.1), then

$$\hat{\gamma}_{t\hat{u}} = \sqrt{\hat{g}} \Big(-\hat{\tau}\hat{V} \hat{N} + \hat{V}_{\hat{s}} \hat{B} \Big). \tag{3.4}$$

Since $\hat{\gamma}_{\hat{u}} = \sqrt{\hat{g}}\, \hat{\gamma}_{\hat{s}} = \sqrt{\hat{g}}\, \hat{T}$, then

$$\hat{\gamma}_{\hat{u}t} = \sqrt{\hat{g}}\, \hat{T}_t + \frac{\hat{g}_t}{2\sqrt{\hat{g}}} \hat{T}. \tag{3.5}$$

Since the derivatives with respect to $\hat{u}$ and t commute, then

$$\hat{\gamma}_{\hat{u}t} = \hat{\gamma}_{t\hat{u}}. \tag{3.6}$$

Substituting from (3.4) and (3.5) into (3.6), then

$$\begin{aligned} \frac{\partial \hat{g}}{\partial t} &= 0, \\ \frac{\partial \hat{T}}{\partial t} &= -\hat{\tau}\hat{V} \hat{N} + \hat{V}_{\hat{s}} \hat{B}. \end{aligned} \tag{3.7}$$

The time evolution equations for the timelike normal vector $\hat{N}$ and the curvature $\hat{k}$ of the curve $\hat{C}_t$ are computed as follows:

Take the $\hat{u}$ derivative of the second equation of (3.7), then

$$\hat{T}_{t\hat{u}} = \sqrt{\hat{g}} \Big(-\hat{k}\hat{\tau}\hat{V} \hat{T} - (\hat{V}\hat{\tau}_{\hat{s}} + 2\hat{V}_{\hat{s}}\hat{\tau}) \hat{N} + (\hat{V}_{\hat{s}\hat{s}} - \hat{\tau}^2\hat{V}) \hat{B} \Big). \tag{3.8}$$

Since

$$\hat{T}_{\hat{u}} = \sqrt{\hat{g}}\, \hat{T}_{\hat{s}} = \sqrt{\hat{g}}\, (\hat{\gamma} + \hat{k}\hat{N}). \tag{3.9}$$

Taking the t derivative of (3.9), then we have

$$\hat{T}_{\hat{u}t} = \sqrt{\hat{g}} \Big(\hat{V} \hat{B} + \hat{k} \hat{N}_t + \hat{k}_t \hat{N} \Big). \tag{3.10}$$

Since

$$\hat{T}_{t\hat{u}} = \hat{T}_{\hat{u}t} \tag{3.11}$$

Substitute from (3.8) and (3.10) into (3.11) and put $\frac{1}{\hat{k}}(\hat{V}_{\hat{s}\hat{s}} - (1 + \hat{\tau}^2)\hat{V}) = \hat{\psi}(\hat{s}, t)$, then

$$\begin{aligned} \hat{k}_t &= -\hat{V}\hat{\tau}_{\hat{s}} - 2\hat{V}_{\hat{s}}\hat{\tau}, \\ \hat{N}_t &= -\hat{\tau}\hat{V} \hat{T} + \hat{\psi} \hat{B}. \end{aligned} \tag{3.12}$$

The time evolution equation for the binormal vector $\hat{B}$ to the curve $\hat{C}_t$ is given as follows:
Since $\hat{B} = \hat{\gamma} \times \hat{T} \times \hat{N}$, so

$$\hat{B}_t = \hat{\gamma}_t \times \hat{T} \times \hat{N} + \hat{\gamma} \times \hat{T}_t \times \hat{N} + \hat{\gamma} \times \hat{T} \times \hat{N}_t. \tag{3.13}$$

Substitute from (3.1) and from the second equation of both (3.7) and (3.12) into (3.13), then

$$\hat{B}_t = -\hat{V} \hat{\gamma} + \hat{V}_{\hat{s}}\hat{T} - \hat{\psi} \hat{N}. \tag{3.14}$$

Take the $\hat{u}$ derivative of (3.14), then

$$\hat{B}_{t\hat{u}} = \sqrt{\hat{g}} \Big((-\hat{V} + \hat{V}_{\hat{s}\hat{s}} - \hat{k}\hat{V}_{\hat{s}}) \hat{T} + (\hat{k}\hat{V}_{\hat{s}} - \hat{\psi}_{\hat{s}}) \hat{N} - \hat{\psi}\hat{\tau} \hat{B} \Big). \tag{3.15}$$

Since

$$\hat{B}_{\hat{u}} = \sqrt{\hat{g}} \hat{B}_{\hat{s}} = \sqrt{\hat{g}}\, (-\hat{\tau}\hat{N}). \tag{3.16}$$

Taking the t derivative of (3.16), then we have

$$\hat{B}_{\hat{u}t} = -\sqrt{\hat{g}} (\hat{\tau}_t \hat{N} + \hat{\tau} \hat{N}_t). \tag{3.17}$$

Since $\hat{B}_{t\hat{u}} = \hat{B}_{\hat{u}t}$, then by substituting from (3.15) and from (3.17) into this equation, then we have the time evolution equation for the torsion $\hat{\tau}$:

$$\hat{\tau}_t = -\hat{k}\hat{V}_{\hat{s}} + \hat{\psi}_{\hat{s}}. \quad (3.18)$$

Hence, the theorem holds. □

3.2. Evolution equations of the spacelike curves with the timelike normal vector in $\mathbb{S}^{2,1}$

Theorem 3.3. *The time evolution of the Serret–Frenet frame for the spacelike curve with the timelike normal vector N can be given in matrix form:*

$$\hat{F}_t = \hat{Q} \cdot \hat{F}, \quad (3.19)$$

where

$$\hat{F} = \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{N} \\ \hat{B} \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} 0 & 0 & 0 & \hat{V} \\ 0 & 0 & \hat{\tau}\hat{V} & \hat{V}_{\hat{s}} \\ 0 & \hat{\tau}\hat{V} & 0 & \hat{\xi} \\ -\hat{V} & -\hat{V}_{\hat{s}} & \hat{\xi} & 0 \end{pmatrix} \quad \text{and}$$

$$\hat{\xi}(\hat{s}, t) = \frac{1}{\hat{k}}\Big((1+\hat{\tau}^2)\hat{V} + \hat{V}_{\hat{s}\hat{s}}\Big).$$

Also, the time evolution of the curvature and torsion for the inextensible spacelike curves with the timelike normal vector are given by:

$$\begin{aligned} \hat{k}_t &= \hat{\tau}_{\hat{s}}\hat{V} + 2\hat{\tau}\hat{V}_{\hat{s}}, \\ \hat{\tau}_t &= \hat{k}\hat{V}_{\hat{s}} + \hat{\xi}_{\hat{s}}. \end{aligned} \quad (3.20)$$

Proof. Take the $\hat{u}$ derivative of (3.1), then

$$\hat{\gamma}_{t\hat{u}} = \sqrt{\hat{g}}\Big(\hat{\tau}\hat{V}\,\hat{N} + \hat{V}_{\hat{s}}\,\hat{B}\Big). \quad (3.21)$$

Since $\hat{\gamma}_{\hat{u}} = \sqrt{\hat{g}}\,\hat{\gamma}_{\hat{s}} = \sqrt{\hat{g}}\,\hat{T}$, then

$$\hat{\gamma}_{\hat{u}t} = \sqrt{\hat{g}}\hat{T}_t + \frac{\hat{g}_t}{2\sqrt{\hat{g}}}\hat{T}. \quad (3.22)$$

Since the derivatives with respect to $\hat{u}$ and t commute, then

$$\hat{\gamma}_{\hat{u}t} = \hat{\gamma}_{t\hat{u}}. \quad (3.23)$$

Substituting from (3.21) and (3.22) into (3.23), then

$$\begin{aligned} \frac{\partial \hat{g}}{\partial t} &= 0, \\ \frac{\partial \hat{T}}{\partial t} &= \hat{\tau}\hat{V}\,\hat{N} + \hat{V}_{\hat{s}}\,\hat{B}. \end{aligned} \quad (3.24)$$

The time evolution equations for the timelike normal vector $\hat{N}$ and the curvature $\hat{k}$ are computed as follows:

Take the $\hat{u}$ derivative of the second equation of (3.24), then

$$\hat{T}_{t\hat{u}} = \sqrt{\hat{g}}\Big(\hat{k}\hat{\tau}\hat{V}\,\hat{T} + (\hat{V}\,\hat{\tau}_{\hat{s}} + 2\hat{V}_{\hat{s}}\hat{\tau})\,\hat{N} + (\hat{V}_{\hat{s}\hat{s}} + \hat{\tau}^2\hat{V})\,\hat{B}\Big). \quad (3.25)$$

Since

$$\hat{T}_{\hat{u}} = \sqrt{\hat{g}}\hat{T}_{\hat{s}} = \sqrt{\hat{g}}(-\hat{\gamma} + \hat{k}\hat{N}). \quad (3.26)$$

By taking the t derivative of (3.26), then we have

$$\hat{T}_{\hat{u}t} = \sqrt{\hat{g}}\Big(\hat{k}\,\hat{N}_t + \hat{k}_t\,\hat{N} - \hat{V}\,\hat{B}\Big). \quad (3.27)$$

Since

$$\hat{T}_{t\hat{u}} = \hat{T}_{\hat{u}t} \quad (3.28)$$

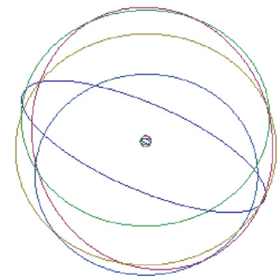

Fig. 1. Hollow ball model.

Substitute from (3.25) and (3.27) into (3.28) and put $\frac{1}{\hat{k}}\Big(\hat{V}_{\hat{s}\hat{s}} + (1 + \hat{\tau}^2)\hat{V}\Big) = \hat{\xi}(\hat{s}, t)$, then

$$\begin{aligned} \hat{k}_t &= \hat{V}\,\hat{\tau}_{\hat{s}} + 2\hat{V}_{\hat{s}}\hat{\tau}, \\ \hat{N}_t &= \hat{\tau}\hat{V}\,\hat{T} + \hat{\xi}\,\hat{B}. \end{aligned} \quad (3.29)$$

The time evolution equation for the unit binormal vector $\hat{B}$ to the curve $\hat{C}_t$ can be given by:

Since $\hat{B} = \hat{\gamma} \times \hat{T} \times \hat{N}$, so

$$\hat{B}_t = \hat{\gamma}_t \times \hat{T} \times \hat{N} + \hat{\gamma} \times \hat{T}_t \times \hat{N} + \hat{\gamma} \times \hat{T} \times \hat{N}_t. \quad (3.30)$$

Substitute from (3.1) and from the second equation of both (3.24) and (3.29) into (3.30), then

$$\frac{\partial \hat{B}}{\partial t} = -\hat{V}\,\hat{\gamma} - \hat{V}_{\hat{s}}\hat{T} + \hat{\xi}\,\hat{N}. \quad (3.31)$$

Take the $\hat{u}$ derivative of (3.31), then

$$\hat{B}_{t\hat{u}} = \sqrt{\hat{g}}\Big((-\hat{V} - \hat{V}_{\hat{s}\hat{s}} - \hat{k}\hat{V}_{\hat{s}})\,\hat{T} + (\hat{k}\hat{V}_{\hat{s}} + \hat{\xi}_{\hat{s}})\,\hat{N} - \hat{\xi}\,\hat{\tau}\,\hat{B}\Big). \quad (3.32)$$

Since

$$\hat{B}_{\hat{u}} = \sqrt{\hat{g}}\,\hat{B}_{\hat{s}} = \sqrt{\hat{g}}\,(\hat{\tau}\hat{N}). \quad (3.33)$$

By taking the t derivative of (3.33), then we have

$$\hat{B}_{\hat{u}t} = \sqrt{\hat{g}}(\hat{\tau}_t\,\hat{N} + \hat{\tau}\,\hat{N}_t). \quad (3.34)$$

Since $\hat{B}_{t\hat{u}} = \hat{B}_{\hat{u}t}$, then by substituting from (3.32) and (3.34) into this equation, then we have the time evolution equation for the torsion $\hat{\tau}$:

$$\hat{\tau}_t = \hat{k}\hat{V}_{\hat{s}} + \hat{\xi}_{\hat{s}}. \quad (3.35)$$

Hence, the theorem holds. □

4. Construction of Hasimoto surfaces in $\mathbb{S}^{2,1}$

In this section, we construct new kind of surfaces, we will call them Hasimoto surfaces as the case of binormal motion of curves in Euclidean space $\mathbb{R}^3$. These curves move with the binormal velocity equals to the curvature $k(\hat{s}, t)$ of the curve i.e., $\hat{V} = k(\hat{s}, t)$. Then the flow of curves (3.1) takes the form:

$$\frac{\partial \hat{\gamma}}{\partial t} = \hat{k}\,\hat{B}, \quad (4.1)$$

4.1. Model 1

For the binormal motions of inextensible timelike curves in $\mathbb{S}^{2,1}$, then (3.3) takes the form: (Fig. 1)

$$\hat{k}_t = -\hat{k}\hat{\tau}_{\hat{s}} - 2\hat{\tau}\,\hat{k}_{\hat{s}},$$

$$\hat{\tau}_t = -\hat{k}\hat{k}_{\hat{s}} + \frac{\partial}{\partial \hat{s}}\Big(-1 - \hat{\tau}^2 + \frac{\hat{k}_{\hat{s}\hat{s}}}{\hat{k}}\Big).$$

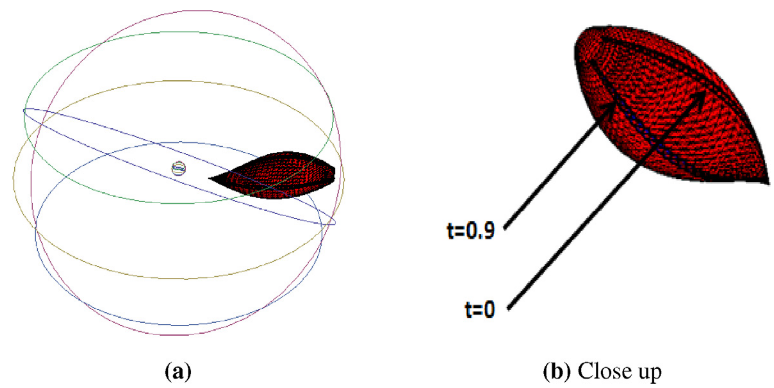

Fig. 2. The Hasimoto surface for $\hat{s} \in [0, 3]$, $t \in [0, 5]$, $c_1 = 0.6$, $c_2 = 0.0001$ and $c_3 = -0.00001$. The bold black curves on the surface represent the family of timelike curves $\hat{C}_t$ for $t = 0$ and $t = 0.9$.

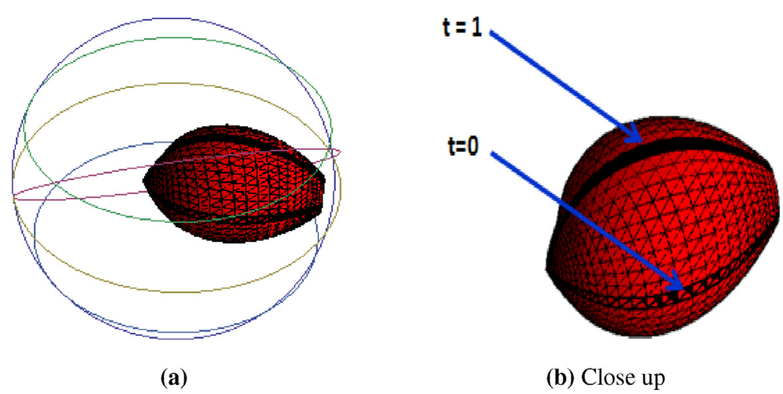

Fig. 3. The Hasimoto surface for $\hat{s} \in [0, 3]$, $t \in [0, 5]$, $c_1 = 0.9$, $c_2 = 0.001$ and $c_3 = 0.002$. The bold black curves on the surface represent the family of spacelike curves with the timelike normal vector $\hat{C}_t$ for $t = 0$ and $t = 1$.

One solution of this system is

$$\hat{k}(\hat{s}, t) = 2c_1 \tanh(c_1\hat{s} + c_2 t + c_3), \quad \hat{\tau}(\hat{s}, t) = \frac{c_2}{2c_1}. \tag{4.2}$$

where c_1, c_2 and c_3 are constants. For $\hat{V} = k$, substitute from (4.2) into (2.1), (3.2) and solve the systems (2.1) and (3.2) numerically. Then, we can determine the family of curves $\hat{C}_t = \hat{\gamma}(\hat{s}, t)$, hence, we can get the Hasimoto surface that is generated by these family of timelike curves (Fig. 2).

4.2. Model 2

For the binormal motions of inextensible spacelike curves with the timelike normal vector in $\mathbb{S}^{2,1}$, then (3.20) takes the form:

$$\hat{k}_t = \hat{k}\hat{\tau}_{\hat{s}} + 2\hat{\tau}\hat{k}_{\hat{s}},$$

$$\hat{\tau}_t = -\hat{k}\hat{k}_{\hat{s}} + \frac{\partial}{\partial \hat{s}}\left(1 + \hat{\tau}^2 + \frac{\hat{k}_{\hat{s}\hat{s}}}{\hat{k}}\right).$$

One solution of this system is

$$\hat{k}(\hat{s}, t) = 2c_1 \tanh(c_1\hat{s} + c_2 t + c_3), \quad \hat{\tau}(\hat{s}, t) = \frac{c_2}{2c_1}. \tag{4.3}$$

where c_1, c_2 and c_3 are constants. For $\hat{V} = k$, substitute from (4.3) into (2.2), (3.19) and solve the systems (2.2) and (3.19) numerically. Then, we can determine the family of curves $\hat{C}_t = \hat{\gamma}(\hat{s}, t)$, hence, we can get the Hasimoto surface that is generated by these family of spacelike curves with the timelike normal vector (Fig. 3).

5. Conclusion

In the present work, we focused our attention on:

- Study the binormal motions of inextensible timelike curves and inextensible spacelike curves with a timelike normal vector in de-sitter space $\mathbb{S}^{2,1}$.
- Determined the inextensible timelike curves and inextensible spacelike curves by solving their evolution equations.
- Constructed and plotted surfaces that are generated from the binormal motions of inextensible timelike curves and inextensible spacelike curves with a timelike normal vector in de-sitter space $\mathbb{S}^{2,1}$. These surfaces were called Hasimoto surfaces in $\mathbb{S}^{2,1}$.

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