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The development of a differential game related to terrorism: Min-Max differential game

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A B S T R A C T

In this work, we study a differential game related to terrorism: Min-Max differential game taking into account the governmental activities such as the education quality, increasing the chances of labor, social justice, religious awareness and security arrangements. A Min-Max differential game between government and terrorist organizations is considered in this study. To obtain the optimal strategy of solving this problem, we study the analytic form of a Min-Max differential game and the governmental activities. Furthermore, a saddle point of a Min-Max differential game is studied.

1. Introduction

Terrorism is the use of violence to cause political, religiouse and ideological change, or control the wealth of nations. Terrorism problems have become very large and dangerous all over the world. Governments have taken important procedures, such as the education quality, increasing the chances of labor, social justice, religious awareness and security arrangements so as to fight terrorism. Some mathematical subjects are applied to get methods for combating terrorism, particularly 'Operations Research'.

Counter-terrorism measures range from security arrangements and the governmental activities to freezing assets of a terrorist organization or even invading their territories and assassinating them.

However, any of the a fore-mentioned actions has to be thoroughly investigated for consecutive reaction. In this paper, a differential game approach is used for studying the reciprocal strategies of governments on one hand and those terrorist of organizations on the second hand.

The power of organizations is measured by the terrorist attacks, Caulkins et al. [1] proposed that the combating terrorism relies on the community opinion and Caulkins et al. [2] introduced the comparison between the efficiency of water and fire strategies. The decreasing rate of terrorists is affected by their own actions and the anti-terrorist actions of the government through the education quality, increasing the chances of labor, social justice, religious awareness and security arrangements. The government benefits from the loss of the terrorist resources and their activities but incurs costs for combating terrorism and disutilities cause by terrorist organizations. These organizations try to maximize their power, both by enlarging their size as well as terrorist attacks.

Therefore,this study is an attempt to help governments fight terrorism better effectively. A min-max differential game plays the main role to combat terrorism. Hsie et al. [3,4] introduce the first approach to fuzzy differential game problem: guarding a territory and guarding a movable territory. Youness et al. [5] discuss a parametric-Nash-collative differential game. A study on a fuzzy differential game, a study on large scale continuous differential game, min-max zero-sum fuzzy continuous differential game and min-max zero-sum continuous differential game with fuzzy control are presented in [6–9]. Nova et al. [10] introduce a differential game related to terrorism namely 'Nash and Stackelberg strategies', Roy et al. [11] present a terrorism deterrence in a two country framework: strategic interactions between R&D, defense and preemption. Ahmed et al. [12] introduce a complex adaptive system to study the terrorism phenomena. Megahed [13] presents a Min-Max dif-

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ferential game approach for getting the optimal strategy to fight terrorist organizations.

2. Problem formulation

Consider the following differential game with the state variable, $x(t)$ which describes the resource of an International Terror Organization (ITO). It may also include weapons, financial capital, network of supporters, etc. and another state variable $M(t)$ which describe the governmental activities, the education quality, increasing the chances of labor, social justice, religious awareness and security arrangements, $t \in [0, \infty)$ is the time. Two players are the government with non-negative strategy $u(t)$ and the other side ITO with non-negative strategy $\nu(t)$ as the opponent. The stock of resources of ITO grows according to the growth of a linear function $g(x)$, i.e. $g(x) = rx, r > 0$, and the governmental activities grows according to a linear function $A(M) = \mu M$, where $\mu > 0$ is the growth rate of the governmental activities. Carrying out attacks make a reduction the growth of the resource stock as it affects negatively the number of terrorists (e.g. due to suicide bombing or terrorists being caught or killed) as well as weapons and financial means, it may even include a reduction of the network of supporters. The reduction of the growth of the resource stock, however does not only depend on the intensity of attacks $nu(t)$ but it is also influenced by the counter-terror measures $u(t)$ This influence of the control variables of the two players on the growth of the resource may be denoted as "harvest function" $h(u, \nu)$. As a consequence the dynamics of the resource stock $x(t)$can be written as

$$\dot{x} = rx(t) - h(u(t), \nu(t)), \qquad x(0) = x_0 > 0 \tag{1}$$

$$\dot{M} = \mu M + au - b\nu \qquad M(0) = M_0 > 0 \tag{2}$$

where x_0 denotes the initial stock of terrorist's resources, M_0 is the initial government's activities and a, b are positive constants. Moreover,we assume that along trajectories the non-negativity constraints

$$x(t) \geq 0, M(t) \geq 0, t \geq 0 \tag{3}$$

Since a higher intensity of counter-terror measure and attacks leads to a reduction of growth, we assume that the partial derivatives are greater than zero :$h_u(u, \nu) > 0$, $h_\nu(u, \nu) > 0$.The counter-terror measures exhibit marginally decreasing efficiency $h_{uu} < 0$. Moreover a higher rate of attacks induces disproportional higher losses of resources i.e. $h_{\nu\nu} > 0$. Finally the instruments reinforce each other, i.e. $h_{u\nu} > 0$ which makes economically sense. This positive interaction means that the marginal efficiency of counter-terror activities are increasing with the intensity of terrorist's attacks as active visible terrorists can be more easily controlled than hidden ones. Additionally, we assume that the Inada conditions in the economic literature are fulfilled

$$\lim_{u \to 0} h_u(u, \nu) = \infty, \quad \lim_{u \to \infty} h_u(u, \nu) = 0 \tag{4}$$

$$\lim_{\nu \to 0} h_u(u, \nu) = 0, \ \lim_{u \to \infty} h_u(u, \nu) = \infty \tag{5}$$

This guarantees that the optimal strategies are nonnegative, $u(t) \geq 0$, and $\nu(t) \geq 0$, $t > 0$.

Player 1(Government) draw utility from its activities, $M(t)$ and the loss of the terrorist's resources but disutility from the size of ITO, terrorists activities and their own costs of counter-terror measures. For simplicity all these terms are assumed to be linear. Thus, the objective of the government

$$\max_{u(t)} \left\{ J_1 = \int_0^\infty e^{-\rho_1 t} \times [\omega h(u(t), \nu(t)) + qM(t) - cx(t) - k\nu(t) - \alpha u(t)]dt \right\} \tag{6}$$

where ω, c, k, q and $\alpha > 0$.

The second player (ITO) derives utility from the resource stock $x(t)$ and the terrorist actions at intensity $\nu(t)$ and disutility from Government's activities. This leads to the following maximization problem

$$\max_{\nu(t)} \left\{ J_2 = \int_0^\infty e^{-\rho_2 t}[\sigma x(t) + \beta \nu(t) - \omega M(t)]dt \right\} \tag{7}$$

where $\sigma,\ \beta,\ \omega \, and \, \eta > 0$.

The decreasing rates ρ_i, $i = 1, 2$ are assumed to be greater than the growth and activity rates r, μ respectively i.e.,

$$\rho_i > r, \quad \rho_i > \mu \quad \text{for} \quad i = 1, 2 \tag{8}$$

In this paper, we will calculate min-max equilibrium. The solution procedure relies on Pontryagin's maximum [9].

3. Min-Max equilibrium

A min-max game is called antagonistic game for two persons(two players), In this paper, player 1 is the government and player 2 is the International Terror Organization (ITO). There are two cases for studying of this problem.

3.1. The game of the government view

In this case, the government is going to find the strategic variable $u(t)$ to maximize its payoff,namely the maximizing player, but the ITO tries to find the strategic variable $\nu(t)$ to minimize this payoff, namely the minimizing player. The game takes the following form

$$\left.\begin{array}{ll}
\min_{\nu(t)} \ \max_{u(t)} J_1 & \\
= \int_0^\infty e^{-\rho_1 t}[wh(u(t), \nu(t)) + qM(t) & \\
- cx(t) - k\nu(t) - \alpha u(t)]dt & \\
\dot{x} = rx(t) - h(u(t), \nu(t)), & x(0) = x_0 > 0 \ , x(t) \geq 0 \text{ for all } t \\
\dot{M} = \mu M(t) + au - b\nu(t), & M(0) = M_0 > 0 \ , M(t) \geq 0 \text{ for all } t
\end{array}\right\} \tag{9}$$

Note: It's denoted that

$$I_1(x(t), u(t), \nu(t)) = \omega h(u(t), \nu(t)) + qM(t) - cx(t) - k\nu(t) - \alpha u(t)$$

and

$$f(x, u, \nu) = rx(t) - h(u(t), \nu(t))$$

Definition 3.1. The point (u^*, ν^*) is said to be the saddle point of the min-max continuous differential game problem (9) if

$$J_1(u^*, \nu) \leq J_1(u^*, \nu^*) \leq J_1(u, \nu^*) \tag{10}$$

3.2. The necessary conditions of an open saddle point solution

Theorem 3.1. *Let $I_1(x(t), u(t), \nu(t))$ and $f(x, u, \nu)$ are continuous differentiable functions. If (u^*, ν^*) is saddle point with the state trajectories $x^*(t)$ and $M^*(t)$ for the problem (Government). Then there exists a costate vectors $\lambda_1(t)$, $P_1(t)$ and the Hamiltonian function H_1 defined by*

$$H_1(x(t), u(t), \nu(t), \lambda_1(t), P_1(t)) = I_1(x(t), u(t), \nu(t)) + \lambda_1(t) f(x, u, \nu) + P_1(t)(\mu M + au - b\nu) \tag{11}$$

such that the following conditions are satisfied

$$\left.\begin{array}{ll} \frac{\partial H_1}{\partial u} = 0, & \frac{\partial H_1}{\partial v} = 0 \\ \frac{\partial^2 H_1}{\partial u^2}\frac{\partial^2 H_1}{\partial v^2} - \left(\frac{\partial^2 H_1}{\partial u \partial v}\right)^2 \le 0, & \frac{\partial^2 H_1}{\partial u^2} \le 0, \quad \frac{\partial^2 H_1}{\partial v^2} \ge 0 \\ \dot{\lambda_1} = \rho_1\lambda_1 - \frac{\partial H_1}{\partial x} & \\ \dot{P_1} = \rho_1 P_1 - \frac{\partial H_1}{\partial M} & \\ \min_{v(t)} H_1(x(t), u^*(t), v(t), \lambda_1(t), P_1(t)) & \\ \quad = H_1(x(t), u^*(t), v^*(t), \lambda_1(t), P_1(t)) & \\ \quad = \max_{u(t)} H_1(x(t), u(t), v^*(t), \lambda_1(t), P_1(t)) & \end{array}\right\} \quad (12)$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.1 in [9]. □

Since the optimal strategy of the government and ITO have to maximize and minimize the Hamiltonian function H_1. Then

$$\left.\begin{array}{l} \frac{\partial H_1}{\partial u} = (\omega - \lambda_1)h_u - \alpha + P_1 a = 0 \Longrightarrow h_u = \frac{\alpha - P_1 a}{\omega - \lambda_1} \\ \frac{\partial H_1}{\partial v} = (\omega - \lambda_1)h_v - k - P_1 b = 0 \Longrightarrow h_v = \frac{k + P_1 b}{\omega - \lambda_1} \end{array}\right\} \quad (13)$$

the adjoint variables satisfy the differential equations

$$\dot{\lambda_1} = \rho_1\lambda_1 - \frac{\partial H_1}{\partial x} = \lambda_1(\rho_1 - r) + c \quad (14)$$

$$\dot{P_1} = \rho_1 P_1 - \frac{\partial H_1}{\partial M} = (\rho_1 - \mu)P_1 - q \quad (15)$$

and the limiting transversality conditions

$$\lim_{t\to\infty} e^{-\rho_1 t} x(t)\lambda_1(t) = 0 \quad (16)$$

$$\lim_{t\to\infty} e^{-\rho_1 t} M(t)P_1(t) = 0 \quad (17)$$

then the solution of the adjoint equation is

$$\lambda_1(t) = \left(\lambda_0 + \frac{c}{(\rho_1 - r)}\right)e^{(\rho_1 - r)t} - \frac{c}{\rho_1 - r} \quad (18)$$

$$P_1(t) = \left(P_0 + \frac{q}{\rho_1 - \mu}\right)e^{(\rho_1 - \mu)t} + \frac{q}{\rho_1 - \mu} \quad (19)$$

where $\lambda_1(0) = \lambda_0$ and $P_1(0) = P_0$, since $\rho_1 > r$ and $\rho_1 > \mu$. then $\lambda_1(t) \to \infty$ and $M(t) \to \infty$ as $t \to \infty$ which is violating the transversality conditions except when choosing the constant steady state values

$$\lambda_1 = \lambda_0 = -\frac{c}{\rho_1 - r}$$
$$P_1 = P_0 = \frac{q}{\rho_1 - \mu}$$

The Hamiltonian H_1 is concave with respect to the strategy u and convex with respect to the strategy v and therefore, we find the maximization of H_1 with respect to u and the minimization of H_1 with respect to v.

Consider the harvest function $h(u, v) = u^\tau v^\delta$, with $0 < \tau\ 1 < \delta$

Remark 1. Since $H_{1uu} = (w - \lambda)^2\tau(\tau - 1)u^{\tau-2}v^\delta < 0$ and $H_{1vv} = (w - \lambda)^2\delta(\delta - 1)u^\tau v^{\delta-2} > 0$, then H_1 is concave with respect to u and convex with respect to v

Proposition 3.1. *The optimal strategies of the game 9 are given by*

$$\left.\begin{array}{l} u = \left[\left(\frac{\alpha - P_1 a}{\tau(\omega - \lambda_1)}\right)^{\delta-1}\left(\frac{k + bP_1}{\delta(\omega - \lambda_1)}\right)^{-\delta}\right]^{\frac{1}{1-\tau-\delta}} \\ v = \left[\left(\frac{k + bP_1}{\delta(\omega - \lambda_1)}\right)^{\tau-1}\left(\frac{\alpha - P_1 a}{\tau(\omega - \lambda_1)}\right)^{-\tau}\right]^{\frac{1}{1-\tau-\delta}} \end{array}\right\} \quad (20)$$

with the harvest function

$$h(u, v) = \left(\frac{\alpha - P_1 a}{\tau(\omega - \lambda_1)}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{k + bP_1}{\delta(\omega - \lambda_1)}\right)^{\frac{-\delta}{1-\tau-\delta}} \quad (21)$$

Proof. From the necessary conditions we have

$$h_u = \tau u^{\tau-1}v^\delta = \frac{\alpha - P_1 a}{(\omega - \lambda_1)}, \text{ then } u = \left(\frac{\alpha - P_1 a}{\tau(\omega - \lambda_1)}\right)^{\frac{1}{\tau-1}} v^{\frac{-\delta}{\tau-1}}$$

$$\text{and } h_v = \delta u^\tau v^{\delta-1} = \frac{k + bP_1}{\omega - \lambda_1}, \text{ then } v = \left(\frac{k + bP_1}{\delta(\omega - \lambda_1)}\right)^{\frac{1}{\delta-1}} u^{\frac{-\tau}{\delta-1}}$$

and thus

$$\left.\begin{array}{l} u = \left[\left(\frac{\alpha - P_1 a}{\tau(\omega - \lambda_1)}\right)^{\delta-1}\left(\frac{k + bP_1}{\delta(\omega - \lambda_1)}\right)^{-\delta}\right]^{\frac{1}{1-\tau-\delta}} \\ v = \left[\left(\frac{k + bP_1}{\delta(\omega - \lambda_1)}\right)^{\tau-1}\left(\frac{\alpha - P_1 a}{\tau(\omega - \lambda_1)}\right)^{-\tau}\right]^{\frac{1}{1-\tau-\delta}} \\ h(u, v) = \left(\frac{\alpha - P_1 a}{\tau(\omega - \lambda_1)}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{k + bP_1}{\delta(\omega - \lambda_1)}\right)^{\frac{-\delta}{1-\tau-\delta}} \end{array}\right\} \quad (22)$$

and

$$\left.\begin{array}{l} \begin{vmatrix} H_{1uu} & H_{1uv} \\ H_{1uv} & H_{1vv} \end{vmatrix} = (w - \lambda)^2 \begin{vmatrix} \tau(\tau - 1)u^{\tau-2}v^\delta & \tau\delta u^{\tau-1}v^{\delta-1} \\ \tau\delta u^{\tau-1}v^{\delta-1} & \delta(\delta - 1)u^\tau v^{\delta-2} \end{vmatrix} \\ \quad = (w - \lambda)^2\tau\delta(1 - \tau - \delta)u^{2(\tau-1)}v^{2(\delta-1)} < 0 \end{array}\right\} \quad (23)$$

i.e., (u, v) is saddle point of the problem (9). □

Lemma 3.1. *The objective of the government (player 1) for the constant strategies u, v is*

$$J_1 = \frac{h}{\rho_1}\left(w + \frac{c}{\rho_1 - r}\right) - \frac{\alpha u}{\rho_1} - \frac{kv}{\rho_1} - \frac{cx_0}{\rho_1 - r} - \frac{M_0\mu q(\rho_1 - \mu) + (au - bv)(\rho_1 q - \rho_1 + \mu)}{\mu\rho_1(\mu - \rho_1)} \quad (24)$$

Proof. The solution of the ordinary differential equations

$$\dot{x} = r\,x(t) - h(u(t), v(t))$$
$$\dot{M} = \mu M + au - bv$$

are

$$x(t)e^{-rt} = \frac{1}{r}e^{-rt}h(u, v) + c_1(\text{constant})$$
$$M(t)e^{-\mu t} = -\frac{au - bv}{\mu}e^{-\mu t} + c_2(constant)$$

for $t \to 0, c_1 = x_0 - \frac{1}{r}h(u, v)$ and $c_2 = M_0 + \frac{au - bv}{\mu}$, then

$$\left.\begin{aligned} x(t) &= \left(x_0 - \frac{1}{r}h(u,v)\right)e^{rt} + \frac{h}{r} \\ M(t) &= \left(M_0 + \frac{au - bv}{\mu}\right)e^{\mu t} - \frac{au - bv}{\mu} \end{aligned}\right\} \tag{25}$$

and thus

$$J_1 = \frac{h}{\rho_1}\left(w + \frac{c}{\rho_1 - r}\right) - \frac{\alpha u}{\rho_1} - \frac{kv}{\rho_1} - \frac{cx_0}{\rho_1 - r} - \frac{M_0\mu q(\rho_1 - \mu) + (au - bv)(\rho_1 q - \rho_1 + \mu)}{\mu\rho_1(\mu - \rho_1)} \tag{26}$$

where u, v and $h(u, v)$ are defined in (20). □

3.3. The game of ITO view

In this case, the ITO is going to find the strategic variable $v(t)$ to maximize his payoff and it's called the maximizing player but the government tries to find the strategic variable $u(t)$ to minimize its payoff and it's called the minimizing player, the game takes the following form

$$\left.\begin{aligned} &\min_{u(t)} \max_{v(t)} \{ J_2 = \textstyle\int_0^\infty e^{-\rho_2 t}[\sigma x(t) + \beta v(t) - \omega M(t)]dt\} \\ &\dot{x} = r\,x(t) - h(u(t), v(t)), \qquad x(0) = x_0 > 0\ , x(t) \geq 0 \text{ for all } t \\ &\dot{M} = \mu M + au - bv, \qquad M(0) = M_0 > 0\ , M(t) \geq 0 \text{ for all } t \end{aligned}\right\} \tag{27}$$

Note: it's denoted that

$$I_2(x(t), u(t), v(t)) = \sigma x(t) + \beta v(t) - \omega M(t)$$

Definition 3.2. The point (u^*, v^*) is said to be saddle point of the min-max continuous differential game problem (27) if

$$J_2(u^*, v) \leq J_2(u^*, v^*) \leq J_2(u, v^*) \tag{28}$$

Theorem 3.2. *Let $I_2(x(t), u(t), v(t))$and $f(x, u, v)$ are continuous differentiable functions. If (u^*, v^*) is saddle point with the state trajectory $x^*(t)$ for the problem (ITO). Then there exists a costate vector $\lambda_2(t)$ and the Hamiltonian function H_2 defined by*

$$H_2(x(t), u(t), v(t), \lambda(t)) = I_2(x(t), u(t), v(t)) + \lambda_2(t) f(x, u, v) + P_2(t)(\mu M + au - bv) \tag{29}$$

such that the following conditions are satisfied

$$\left.\begin{aligned} &\frac{\partial H_2}{\partial u} = 0, \qquad \frac{\partial H_2}{\partial v} = 0 \\ &\frac{\partial^2 H_2}{\partial u^2}\frac{\partial^2 H_2}{\partial v^2} - \left(\frac{\partial^2 H_2}{\partial u \partial v}\right)^2 \leq 0, \quad \frac{\partial^2 H_2}{\partial u^2} \geq 0, \quad \frac{\partial^2 H_2}{\partial v^2} \leq 0 \\ &\dot{\lambda_2} = \rho_2\lambda_2 - \frac{\partial H_2}{\partial x} \\ &\max_{v(t)} H_2(x(t), u^*(t), v(t), \lambda(t)) \\ &\quad = H_2(x(t), u^*(t), v^*(t), \lambda(t)) \\ &\quad = \min_{u(t)} H_2(x(t), u(t), v^*(t), \lambda(t)) \end{aligned}\right\} \tag{30}$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.1 in [9]. □

Since the optimal strategy of the ITO and the government have to maximize and minimize the Hamiltonian function H_2 respectively. Then

$$\left.\begin{aligned} \frac{\partial H_2}{\partial u} &= -\lambda_2 h_u + aP_2 = 0 \Longrightarrow h_u = \frac{aP_2}{\lambda_2} \\ \frac{\partial H_2}{\partial v} &= \beta - \lambda_2 h_v - bP_2 = 0 \Longrightarrow h_v = \frac{1}{\lambda_2}[\beta - bP_2] \end{aligned}\right\} \tag{31}$$

the adjoint variables satisfy the differential equations

$$\left.\begin{aligned} \dot{\lambda_2} &= \rho_2\lambda_2 - \frac{\partial H_2}{\partial x} = \lambda_2(\rho_2 - r) - \sigma \\ \dot{P_2} &= \rho_2 P_2 - \frac{\partial H_2}{\partial M} = P_2(\rho_2 - \mu) + \omega \end{aligned}\right\} \tag{32}$$

and the limiting transversality conditions

$$\lim_{t\to\infty} e^{-\rho_2 t} x(t)\lambda_2(t) = 0$$

$$\lim_{t\to\infty} e^{-\rho_2 t} M(t) P_2(t) = 0$$

then the solution of the adjoint equations are

$$\left.\begin{aligned} \lambda_2(t) &= (\lambda_{20} - \frac{\sigma}{(\rho_2 - r)})e^{(\rho_2 - r)t} + \frac{\sigma}{\rho_2 - r} \\ P_2(t) &= (P_{20} + \frac{\omega}{\rho_2 - \mu})e^{(\rho_2 - \mu)t} - \frac{\omega}{\rho_2 - \mu} \end{aligned}\right\} \tag{33}$$

where $\lambda_2(0) = \lambda_{20}$ and $P_2(0) = P_{20}$.

The Hamiltonian H_2 is concave with respect to the strategy v and convex with respect to the strategy u and therefore, we find the maximization of H_2 with respect to v and the minimization of H_2 with respect to u.

Consider the harvest function $h(u, v) = u^\tau v^\delta$, with $0 < \delta < 1 < \tau$.

Remark 2. Since $H_{2uu} = (w - \lambda)^2\tau(\tau - 1)u^{\tau-2}v^\delta > 0$ and $H_{2vv} = (w - \lambda)^2\delta(\delta - 1)u^\tau v^{\delta-2} < 0$, then H_2 is convex with respect to u and concave with respect to v

Proposition 3.2. *The optimal strategies of the problem (27) are given by*

$$\left.\begin{aligned} u &= \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{\delta-1}{1-\tau-\delta}}\left(\frac{\beta - P_2 b}{\delta\lambda_2}\right)^{\frac{-\delta}{1-\tau-\delta}} \\ v &= \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{\beta - P_2 b}{\delta\lambda_2}\right)^{\frac{1-\tau}{1-\tau-\delta}} \\ h(u, v) &= \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{\beta - P_2 b}{\delta\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}} \end{aligned}\right\} \tag{34}$$

Proof. The proof is similar to the previous proposition.

From (34) and the following condition, we find that (u, v) is saddle point of the problem (27)

$$\left.\begin{aligned} \begin{vmatrix} H_{2uu} & H_{2uv} \\ H_{2vu} & H_{2vv} \end{vmatrix} &= (w - \lambda)^2 \begin{vmatrix} \tau(\tau - 1)u^{\tau-2}v^\delta & \tau\delta u^{\tau-1}v^{\delta-1} \\ \tau\delta u^{\tau-1}v^{\delta-1} & \delta(\delta - 1)u^\tau v^{\delta-2} \end{vmatrix} \\ &= (w - \lambda)^2\tau\delta(1 - \tau - \delta)u^{2(\tau-1)}v^{2(\delta-1)} < 0 \end{aligned}\right\} \tag{35}$$

and (u, v) is saddle point of the problem (27).

Similarly from the above lemma we have

$$\left.\begin{aligned} x(t) &= \left(x_0 - \frac{1}{r}h(u,\nu)\right)e^{rt} + \frac{h}{r} \\ M(t) &= \left(M_0 + \frac{au-b\nu}{\mu}\right)e^{\mu t} - \frac{au-b\nu}{\mu} \\ J_2 &= \frac{\sigma}{\rho_2 - r}\left(x_0 - \frac{1}{r}h(u,\nu)\right) + \frac{h(u,\nu)}{r\rho_2} + \frac{\beta\nu}{\rho_2} \\ &\quad + \frac{\omega}{\mu-\rho_2}\left(M_0 + \frac{au-b\nu}{\mu}\right) - \frac{au-b\nu}{\mu\rho_2} \end{aligned}\right\} \tag{36}$$

where

$$\left.\begin{aligned} u &= \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{\delta-1}{1-\tau-\delta}}\left(\frac{\beta-P_2b}{\delta\lambda_2}\right)^{\frac{-\delta}{1-\tau-\delta}} \\ \nu &= \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{\beta-P_2b}{\delta\lambda_2}\right)^{\frac{1-\tau}{1-\tau-\delta}} \\ h(u,\nu) &= \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{\beta-P_2b}{\delta\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}} \end{aligned}\right\} \tag{37}$$

□

4. Comparison

A comparison between the game of the government view and the game of the ITO view is presented in the following table.

Table 1
The table provides a comparison between the game of the government view and the game of the ITO of view.

The game of the government view	The game of the ITO of view
$u = \left[\left(\frac{\alpha-P_1a}{\tau(w-\lambda_1)}\right)^{\delta-1}\left(\frac{k+bP_1}{\delta(w-\lambda_1)}\right)^{-\delta}\right]^{\frac{1}{1-\tau-\delta}}$	$u = \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{\delta-1}{1-\tau-\delta}}\left(\frac{\beta-P_2b}{\delta\lambda_2}\right)^{\frac{-\delta}{1-\tau-\delta}}$
$\nu = \left[\left(\frac{k+bP_1}{\delta(w-\lambda_1)}\right)^{\tau-1}\left(\frac{\alpha-P_1a}{\tau(w-\lambda_1)}\right)^{-\tau}\right]^{\frac{1}{1-\tau-\delta}}$	$\nu = \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{\beta-P_2b}{\delta\lambda_2}\right)^{\frac{1-\tau}{1-\tau-\delta}}$
$h(u,\nu) = \left(\frac{\alpha-P_1a}{\tau(w-\lambda_1)}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{k+bP_1}{\delta(w-\lambda_1)}\right)^{\frac{-\delta}{1-\tau-\delta}}$	$h(u,\nu) = \left(\frac{aP_2}{\tau\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}}\left(\frac{\beta-P_2b}{\delta\lambda_2}\right)^{\frac{-\tau}{1-\tau-\delta}}$
$x(t) = (x_0 - \frac{1}{r}h(u,\nu))e^{rt} + \frac{h}{r}$	$x(t) = (x_0 - \frac{1}{r}h(u,\nu))e^{rt} + \frac{1}{r}h(u,\nu)$
$M(t) = (M_0 + \frac{au-b\nu}{\mu})e^{\mu t} - \frac{au-b\nu}{\mu}$	$M(t) = (M_0 + \frac{au-b\nu}{\mu})e^{\mu t} - \frac{au-b\nu}{\mu}$
$J_1 = \frac{h}{\rho_1}(w - \frac{c}{r-\rho_1}) - \frac{k\nu}{\rho_1} - \frac{\alpha u}{\rho_1} + \frac{cx_0}{r-\rho_1} - \frac{M_0\mu\rho_1 + (au-b\nu)(2\rho_1-\mu)}{\mu\rho_1(\mu-\rho_1)}$	$J_2 = \frac{\sigma}{\rho_2-r}(x_0 - \frac{1}{r}h(u,\nu)) + \frac{h(u,\nu)}{r\rho_2} + \frac{\beta\nu}{\rho_2} + \frac{\omega}{\mu-\rho_2}(M_0 + \frac{au-b\nu}{\mu}) - \frac{au-b\nu}{\mu\rho_2}$

5. Conclusions

In this study, the governmental performance is essential in fighting terrorism. If the government solves the problems with unemployment, the social justice, religious awareness, the education quality and security measurements, so the combating terrorism will be better powerful and effective. However, if the government ignores these problems, the combating terrorism becomes very hard and countries will be fertile ground for the growth of those terrorist organizations.

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