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Comparison of Chebyshev and Legendre polynomials methods for solving two dimensional Volterra - Fredholm integral equations

Doaa Shokry Mohamed*, R.A. Taher

Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt, Egypt

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ABSTRACT

In this paper we apply Chebyshev and Legendre polynomials methods to obtain the approximate solution of mixed Volterra - Fredholm singular integral equations. Comparison between the two methods has been presented, we present some examples to determine the accuracy of the methods.

1. Introduction

Singular Volterra - Fredholm integral equations are used in many branches of science, like astronomy, quantum mechanics, optics and so on.

The approximate solution of singular integral equations for electrostatic problem are presented in [1].

The analytical solution of many singular integral equations is very difficult, many works have been focusing on the development of more advanced and efficients methods for two dimensional integral equations such as collocation method, Chebyshev polynomial method, Successive approximation method, Galerkin method , Variational iteration method, Adomian decompostion method and the Homotopy perturbation method and others see [2–11].

We investigate the numerical solution of singular integral equations by Chebyshev polynomial method, and Legendre polynomial method, some numerical examples are presented to illustrate the effectiveness of our methods.

Consider the following two dimensional Volterra - Fredholm integral equation:

$$u(x,t) = f(x,t) + \int_{-1}^{t}\int_{-1}^{1} (x-y)^{-\alpha}\ u(y,z)dydz,$$
$$(x,t) \in [-1,1]\times[-1,1], \tag{1.1}$$

where $\alpha \in]0,1[$.

The elements of the function $k(x,y) = (x-y)^{-\alpha}$ is the Abel kernal.

2. Chebyshev polynomials method

In this section we introduce some properties of Chebyshev polynomial of first kind, which will be used to find the numerical solution of two dimensional Volterra - Fredholm integral equations.

This method is based on approximating the unknown function $u(x,\ t)$ as:

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\ T_i(x)\ T_j(t), \qquad x,t \in [0,1], \tag{2.1}$$

where a_{ij} are unknowns parameters, to be determined.

$T_i(x)$ is Chebyshev polynomial of the first kind which is defined as:

$$T_i(x) = \cos i\theta, \quad x = \cos\theta,$$

* Corresponding author.
E-mail address: doaashokry@zu.edu.eg (D.S. Mohamed).

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and the following recurrence formulas:

$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad \forall n \geq 2.$

Substituting Eq. (2.1) into Eq. (1.1) we obtain:

$$\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\ T_i(x)\ T_j(t) - \int_{-1}^{t}\int_{-1}^{1} (x-y)^{-\alpha} \times \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\ T_i(y)\ T_j(z)dy\ dz = f(x,t). \tag{2.2}$$

For Gauss - Chebyshev - lobatto collocation points [12]

$$x_r = \cos\left(\frac{r\pi}{N}\right), \quad t_s = \cos\left(\frac{s\pi}{N}\right) \quad r,s = 0,1,\ldots,N. \tag{2.3}$$

Eq. (2.2) can be written as:

$$\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\left[T_i(x_r)\ T_j(t_s) - \int_{-1}^{t_s}\int_{-1}^{1} (x_r-y)^{-\alpha}\ T_i(y)\ T_j(z)\ dy\ dz\right] = f(x_r,t_s). \tag{2.4}$$

It is clear that the system of linear algebraic equations obtained contains $(N+1)^2$ equations with $(N+1)^2$ unknowns. Solving this system we obtain the value of the unknowns a_{ij} such that $i, j = 0,\ldots,N$.

3. Legendre polynomials method

In this section we introduce some properties of Legendre polynomials which will be applied to study the numerical solution of two dimensional singular integral equations.

In [13–16] this method is based on approximating the unknown function $u(x, t)$ as

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\ P_i(x)\ P_j(t), \tag{3.1}$$

c_{ij}, $i, j = 0,1,2,\ldots$ unknowns parameters, to be determined.where $P_i(x)$, $P_j(t)$ are Legendre polynomials which can be obtained as follows:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

with the initial conditions:

$P_0(x) = 1, \quad P_1(x) = x.$

Substituting Eq. (3.1) into Eq. (1.1) we obtain:

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\ P_i(x)\ P_j(t) - \int_{-1}^{t}\int_{-1}^{1} (x-y)^{-\alpha} \times \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\ P_i(y)\ P_j(z)dy\ dz = f(x,t). \tag{3.2}$$

We putting $x = x_r$, $t = t_s$ $r, s = 0,1,\ldots,N$.

We obtain

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\left[P_i(x_r)\ P_j(t_s) - \int_{-1}^{t_s}\int_{-1}^{1} (x_r-y)^{-\alpha}\ P_i(y)\ P_j(z)\ dy\ dz\right] = f(x_r,t_s). \tag{3.3}$$

The unknown coefficients c_{ij} are defined by selected several collocation points [17] from the interval $[-1, 1]$ as follows:

$$x_r = -1 + \frac{2r}{N}, \quad r = 0,1,2,\ldots,N,$$

$$t_s = -1 + \frac{2s}{N}, \quad s = 0,1,2,\ldots,N. \tag{3.4}$$

It is clear that the system of linear algebraic equations obtained contains $(N+1)^2$ equations with $(N+1)^2$ unknowns. Solving this system we obtain the value of the unknowns c_{ij} such that $i, j = 0,\ldots,N$.

4. Numerical examples

In this section some numerical examples of two dimensional Volterra - Fredholm integral equation are presented to illustrate the methods. All results are obtained by using Maple 17.

Example 1. Consider the following singular mixed Volterra - Fredholm integral equation:

$$u(x,t) = f(x,t) + \int_{-1}^{t}\int_{-1}^{1} (x-y)^{-\frac{1}{2}}\ u(y,z)dy\ dz, \tag{4.1}$$

where

$$f(x,t) = x^2 + t^2 - \frac{1}{3}(2\sqrt{x+1} - 2\sqrt{x-1})\ (t^3+1) + \sqrt{x+1}(t+1)\left(-\frac{16}{15}x^2 + \frac{8}{15}x - \frac{2}{5}\right) + \sqrt{x-1}(t+1)\left(\frac{16}{15}x^2 + \frac{8}{15}x + \frac{2}{5}\right),$$

with the exact solution is $u(x, t) = x^2 + t^2$.

1. Applying Chebyshev polynomial of the first kind for Eq. (4.1) when $N = 2$, and by using the collocation points (2.3) we obtain

   $x_0 = 1, \quad x_1 = 0, \quad x_2 = -1, \quad t_0 = 1, \quad t_1 = 0, \quad t_2 = -1.$

   when $N = 2$ we obtain:

   $$u(x,t) = \sum_{i=0}^{2}\sum_{j=0}^{2} a_{ij}T_i(x)T_j(t). \tag{4.2}$$

   Substituting Eq. (4.2) into Eq. (4.1) we have

   $$\sum_{i=0}^{2}\sum_{j=0}^{2} a_{ij}\left[T_i(x_r)T_j(t_s) - \int_{-1}^{t_s}\int_{-1}^{1}(x_r-y)^{-\frac{1}{2}}T_i(y)\ T_j(z)dydz\right] = x_r^2 + t_s^2 - \frac{1}{3}(2\sqrt{x_r+1} - 2\sqrt{x_r-1})\ (t_s^3+1) - \frac{16}{15}\sqrt{x_r+1}\ x_r^2\ (t_s+1) + \frac{8}{15}\sqrt{x_r+1}\ x_r\ (t_s+1) - \frac{2}{5}\sqrt{x_r+1}\ (t_s+1) + \frac{16}{15}\sqrt{x_r-1}\ x_r^2\ (t_s+1) + \frac{8}{15}\sqrt{x_r-1}\ x_r\ (t_s+1) + \frac{2}{5}\sqrt{x_r-1}\ (t_s+1). \tag{4.3}$$

   Applying the collocation points to Eq. (4.3) we obtain a system of linear algebraic equations contains nine equations with the same number of constants by solving this system we obtain the values of the constants as follows:

   $a_{00} = 1, \quad a_{01} = 0, \quad a_{02} = \frac{1}{2}, \quad a_{10} = 0, \quad a_{11} = 0, \quad a_{12} = 0,$

   $a_{20} = \frac{1}{2}, \quad a_{21} = 0, \quad a_{22} = 0.$

   Substituting from these values into Eq. (4.2) we obtain the approximate solution which is the exact solution.
2. Applying Legendre polynomial method for Eq. (4.1) when $N = 2$, and by using the collocation points (3.4) we obtain

   $x_0 = -1, \quad x_1 = 0, \quad x_2 = 1, \quad t_0 = -1, \quad t_1 = 0, \quad t_2 = 1.$

   when N = 2 we obtain:

   $$u(x,t) = \sum_{i=0}^{2}\sum_{j=0}^{2} c_{ij}P_i(x)P_j(t). \tag{4.4}$$

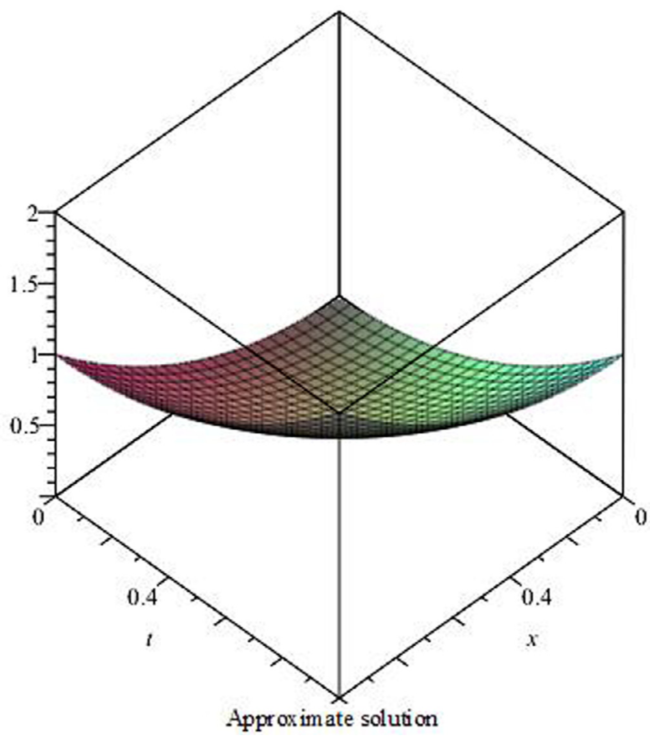

Fig. 1. Approximate solution of Example 1 by Chebyshev and Legendre polynomials methods.

Substituting Eq. (4.4) into Eq. (4.1) we have

$$\sum_{i=0}^{2}\sum_{j=0}^{2} c_{ij}\left[P_i(x_r)P_j(t_s) - \int_{-1}^{t_s}\int_{-1}^{1}(x_r-y)^{-\frac{1}{2}}P_i(y)\ P_j(z)dydz\right]$$
$$= x_r^2 + t_s^2 - \frac{1}{3}(2\sqrt{x_r+1} - 2\sqrt{x_r-1})\ (t_s^3+1)$$
$$- \frac{16}{15}\sqrt{x_r+1}\ x_r^2\ (t_s+1) + \frac{8}{15}\sqrt{x_r+1}\ x_r\ (t_s+1)$$
$$- \frac{2}{5}\sqrt{x_r+1}\ (t_s+1) + \frac{16}{15}\sqrt{x_r-1}\ x_r^2\ (t_s+1)$$
$$+ \frac{8}{15}\sqrt{x_r-1}\ x_r\ (t_s+1) + \frac{2}{5}\sqrt{x_r-1}\ (t_s+1). \tag{4.5}$$

Applying the collocation points to Eq. (4.5) we obtain a system of linear algebraic equations contains nine equations with the same number of constants by solving this system we obtain the values of the constants as follows:

$$c_{00} = \frac{2}{3},\quad c_{01} = 0,\quad c_{02} = \frac{2}{3},\quad c_{10} = 0,\quad c_{11} = 0,\quad c_{12} = 0,$$
$$c_{20} = \frac{2}{3},\quad c_{21} = 0,\quad c_{22} = 0.$$

Substituting $(c_{ij}), i, j = 0, \dots, 2$ into (4.4) we obtain the approximate solution which will be compared to the exact solution (Fig. 1).

Example 2. consider the following singular mixed Volterra - Fredholm integral equation:

$$u(x,t) = f(x,t) + \int_{-1}^{t}\int_{-1}^{1}\ (x-y)^{-\frac{1}{2}}\ u(y,z)dy\ dz, \tag{4.6}$$

where

$$f(x,t) = x\cos(t) + 1.121961313\ \sqrt{x-1}\ x$$
$$- 1.121961313\sqrt{x+1}x + 0.5669806566\sqrt{x-1}$$
$$+ 05669806566\sqrt{x+1} + 0.6666666667\sqrt{x+1}\ \sin(t)$$
$$- 1.333333333\sqrt{x+1}\sin(t)x$$
$$+ 0.6666666667\sqrt{x-1}\sin(t)$$
$$+ 1.333333333\sqrt{x-1}\sin(t)x$$

with the exact solution $u(x,t) = x\cos(t)$

1. Applying Chebyshev polynomial of the first kind for Eq. (4.6) when $N = 3$, we obtain the collocation points as:

$$x_0 = 1,\quad x_1 = \frac{1}{2},\quad x_2 = \frac{-1}{2},\quad x_3 = -1,$$
$$t_0 = 1,\quad t_1 = \frac{1}{2},\quad t_2 = \frac{-1}{2},\quad t_3 = -1,$$

we obtain the approximate solution as:

$$u(x,t) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}\ T_i(x)T_j(t). \tag{4.7}$$

Substituting Eq. (4.7) into Eq. (4.6) we have

$$\sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}\left[T_i(x_r)T_j(t_s) - \int_{-1}^{t_s}\int_{-1}^{1}(x_r-y)^{-\frac{1}{2}}T_i(y)T_j(z)\ dy\ dz\right]$$
$$= x_r\cos(t_s) + 1.121961313\sqrt{x_r-1}x_r$$
$$- 1.121961313\sqrt{x_r+1}x_r + 0.5669806566\sqrt{x_r-1}$$
$$+ 05669806566\sqrt{x_r+1} + 0.6666666667\sqrt{x_r+1}\sin(t_s)$$
$$- 1.333333333\sqrt{x_r+1}\sin(t_s)x_r$$
$$+ 0.6666666667\sqrt{x_r-1}\sin(t_s)$$
$$+ 1.333333333\sqrt{x_r-1}\sin(t_s)x_r \tag{4.8}$$

Applying the collocation points to Eq. (4.8) we obtain a system of 16 linear algebraic equations in 16 unknowns. The solution of this system we is given by:

$$a_{00} = -0.005115186960 - 0.003315087222I,$$
$$a_{01} = -0.01014046823 - 0.004913638641I,$$
$$a_{02} = -0.007493996125 + 0.001295682688I,$$
$$a_{03} = -0.002468714763 + 0.002894234195I,$$
$$a_{10} = 0.7674547701 - 0.004239671087I,$$
$$a_{11} = 0.003116499048 - 0.007908062023I,$$
$$a_{12} = -0.2260895630 - 0.006862161596I,$$
$$a_{13} = -0.002053597677 - 0.003193770877I,$$
$$a_{20} = 0.001744260914 + 0.0007584447555I,$$
$$a_{21} = 0.002826200785 + 0.0007396828277I,$$
$$a_{22} = 0.003268052279 - 0.0006362969686I,$$
$$a_{23} = 0.002186112580 - 0.0006175348690I,$$
$$a_{30} = 0.0002315661958 + 0.0002299389264I,$$
$$a_{31} = 0.0002307954973 - 0.00008013820224I,$$
$$a_{32} = 0.00006333007471 + 0.0001616749102I,$$
$$a_{33} = 0.00006410081618 + 0.0004717519959I, I^2 = -1$$

2. Applying Legendre polynomial method for Eq. (4.6) when $N = 3$, by using the collocation points (3.4) we obtain:

$$x_0 = -1,\quad x_1 = \frac{-1}{3},\quad x_2 = \frac{1}{3},\quad x_3 = 1$$
$$t_0 = -1,\quad t_1 = \frac{-1}{3},\quad t_2 = \frac{1}{3},\quad t_3 = 1,$$

when $N = 3$ we obtain:

Table 1
Numerical results of Example 2 by Chebyshev polynomials method and Legendre polynomials method when $N = 3$.

(x, t)	Abs.error of Chebyshev	Abs. error of Legendre
(0, 0)	$7.162154767 \times 10^{-3}$	6.98978×10^{-3}
(0, 0.2)	$9.493997174 \times 10^{-3}$	9.61688×10^{-3}
(0.2, 0.4)	3.932802×10^{-4}	3.671422×10^{-3}
(0.4, 0.4)	4.662206×10^{-4}	1.1677455×10^{-3}
(0.6, 0.2)	1.3244×10^{-4}	3.2323171×10^{-3}
(0.8, 0.6)	7.215766×10^{-4}	5.9599×10^{-3}
(1.0, 0.4)	4.098245×10^{-4}	1.3666×10^{-2}
(1.0, 0.8)	9.921778×10^{-4}	1.4414×10^{-2}

$$u(x,t) = \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij}\ P_i(x) P_j(t), \tag{4.9}$$

substituting Eq. (4.9) into Eq. (4.6) we have

$$\begin{aligned}
&\sum_{i=0}^{3}\sum_{j=0}^{3} c_{ij}\left[P_i(x_r)P_j(t_s) - \int_{-1}^{t_s}\int_{-1}^{1}(x-y)^{-\frac{1}{2}}P_i(y)P_j(z)dy\ dz\right] \\
&= x_r\cos(t_s) + 1.121961313\ \sqrt{x_r - 1}\ x_r \\
&\quad - 1.121961313\sqrt{x_r+1}x_r + 0.5669806566\sqrt{x_r - 1} \\
&\quad + 05669806566\ \sqrt{x_r + 1}0.6666666667\sqrt{x_r+1}\sin(t_s) \\
&\quad - 1.333333333\sqrt{x_r+1}\sin(t_s)x_r \\
&\quad + 0.6666666667\ \sqrt{x_r-1}\ \sin(t_s) \\
&\quad + 1.333333333\sqrt{x_r-1}\sin(t_s)\ x_r
\end{aligned} \tag{4.10}$$

Applying the collocation points to Eq. (4.10) we obtain a system of 16 linear algebraic equations in 16 unknowns. The solution of this system we is given by:

$$\begin{aligned}
c_{00} &= -0.002813133553 - 0.002684092484I, \\
c_{01} &= -0.01058670912 - 0.004782573130I, \\
c_{02} &= -0.01154928836 + 0.004425461190I, \\
c_{03} &= -0.003775712757 + 0.006523941876I, \\
c_{10} &= 0.8458663209 - 0.005229864755I, \\
c_{11} &= 0.002188219050 - 0.01159436111I, \\
c_{12} &= -0.3086873870 - 0.01066453726I, \\
c_{13} &= -0.005311590864 - 0.004300041110I, \\
c_{20} &= 0.005623096597 + 0.001801522285I, \\
c_{21} &= 0.009175015551 + 0.0009310276952I, \\
c_{22} &= 0.007168716114 - 0.003667880490I, \\
c_{23} &= 0.003616797387 - 0.002797385679I \\
c_{30} &= 0.001814340548 + 0.003961144437I, \\
c_{31} &= 0.003567723190 + 0.005291321020I, \\
c_{32} &= 0.002980867650 + 0.002155362282I, \\
c_{33} &= 0.001227485055 + 0.0008251856413I.
\end{aligned}$$

Then, by substituting these constants into Eq. (4.9) we obtain the approximate solution.

For the second example, the absolute error is calculated and shown in Table 1, Figs. 4 and 6.

Fig. 2 represented the exact solution of Example 2, Figs. 3 and 5 represented the approximate solution of Example 2 by using Chebyshev polynomials and Legendre polynomials methods and the absolute error is calculated and shown in Figs. 4 and 6.

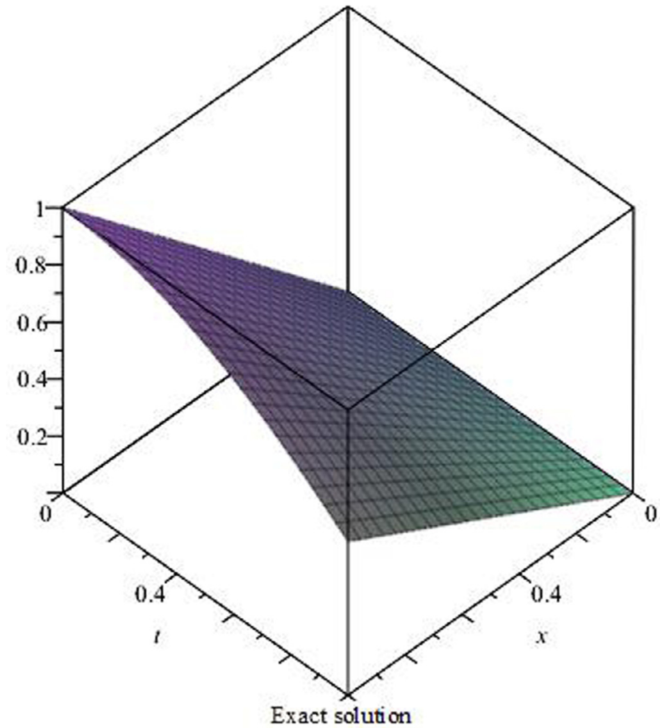

Fig. 2. Exact solution of Example 2 by Chebyshev and Legendre polynomials methods.

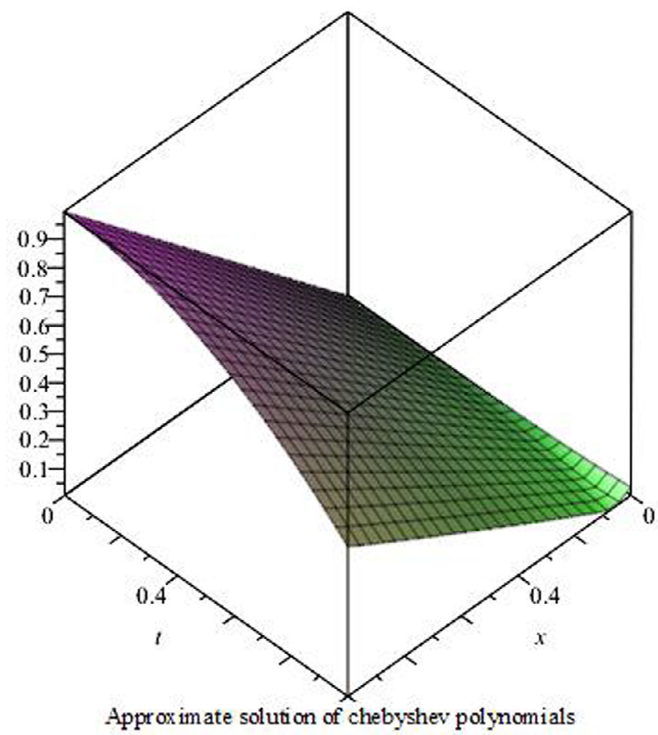

Fig. 3. Approximate solution of Example 2 by Chebyshev polynomials method.

Conclusion

This paper concerns the numerical solutions of two dimensional Volterra - Fredholm integral equations by using Chebyshev polynomial method and Legendre polynomial method, by comparing the results we find that Chebyshev polynomial method is better than

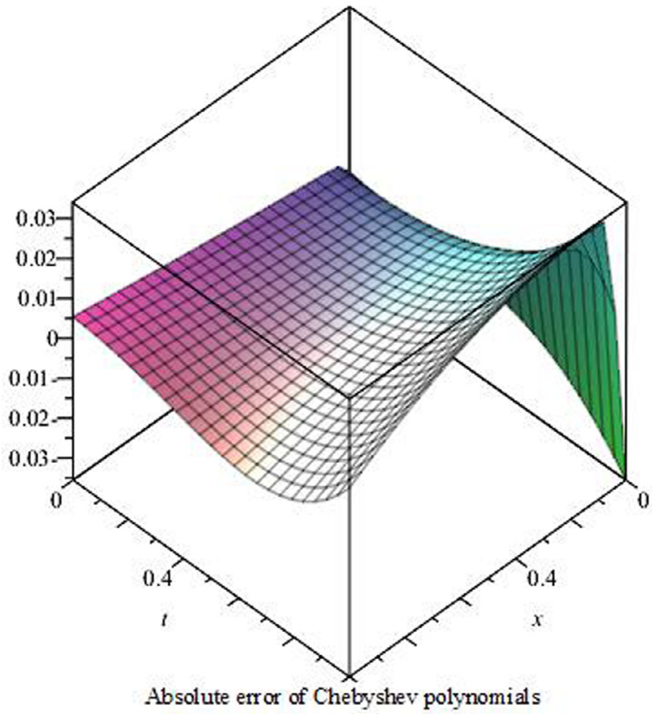

Fig. 4. Absolute error of Example 2 by Chebyshev polynomials method.

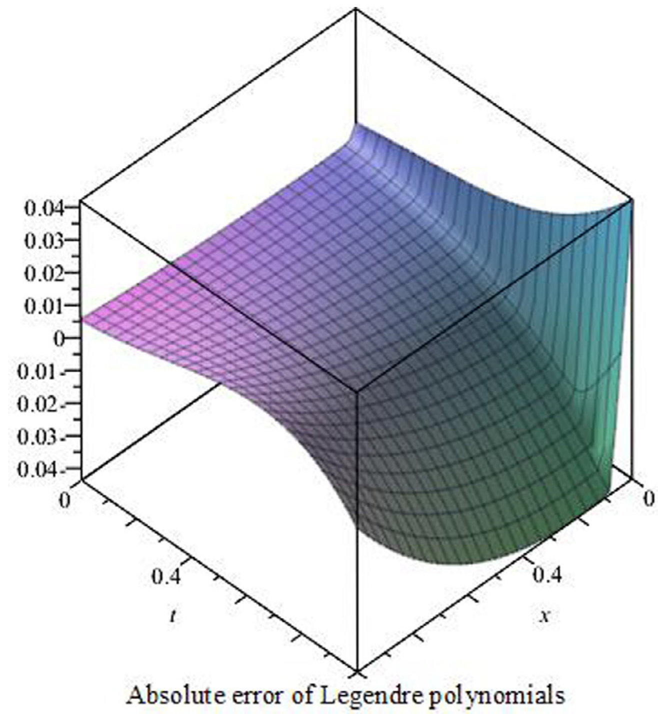

Fig. 6. Absolute error of Example 2 by Legendre polynomials method.

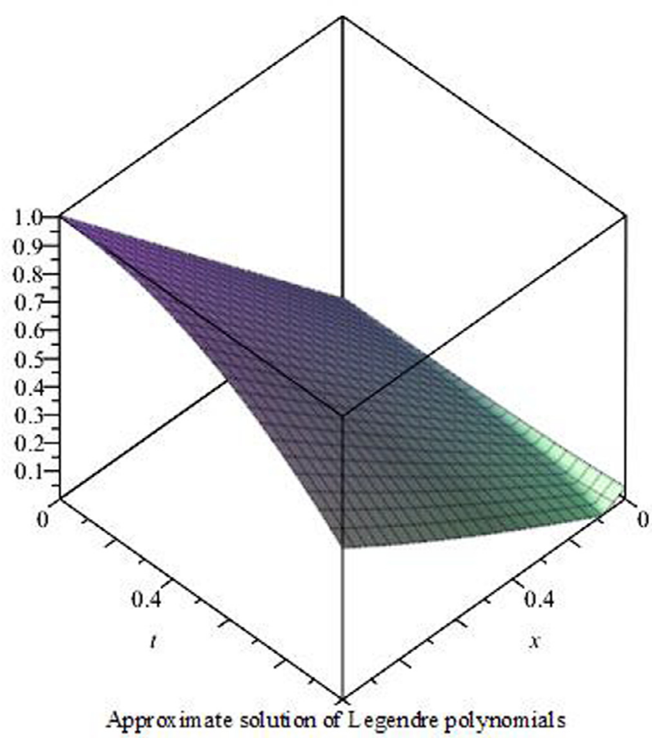

Fig. 5. Approximate solution of Example 2 by Legendre polynomials method.

Legendre polynomial method from Table 1 see the points (0.2,0.4), (0.4,0.4), (0.6,0.2) and others. The illustrative examples confirm the validity and efficiency of the methods. The author in [16] study the same problem by using Legendre Polynomials the Volterra - Fredholm Integral equations transform to a system of Fredholm integral equations of the second kind by applying Trapezoidal Method we show that our methods is faster than Legendre polynomial method with aid of Trapezoidal rule. In this paper the double integral are computed using maple programm also Volterra- Fredholm integral can be computed using numerical method such as quadrature method.

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