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$L^p - L^r$ estimates for $\bar{\partial}$ on q-convex intersections in a Stein manifold

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ABSTRACT

In this work, we obtain L^p-regularity with gain for the $\bar{\partial}$-equation on q-convex intersection in a Stein manifold.

1. Introduction

The estimates of growth and regularity of the $\bar{\partial}$-equation plays a very prominent role in the theory of functions of several complex variables. In particular, L^p estimates of the solutions to the $\bar{\partial}$-equation has a long history in this field going back to the classical results of Kerzman [1] and Øvrelid [2]. $L^p - L^r$ estimates (or L^p-estimates with gain) for $\bar{\partial}$ were first obtained by Krantz [3], who proved that for every $\bar{\partial}$-closed (0, 1)-form f with L^p coefficients, $1 \le p < \infty$, on a strictly pseudoconvex domain Ω with $\mathcal{C}^5$ boundary in $\mathbb{C}^n$, there exists a function u in $L^r(\Omega)$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2(n+1)}$ such that $\bar{\partial}u = f$ in Ω and $\|u\|_{L^r(\Omega)} \lesssim \|f\|_{L^p_{0,1}(\Omega)}$, see [3] for the precise formulations.

This result strongly improves the $L^p - L^p$ result obtained by Øvrelid [2], because it gives a gain $r > p$ and this is the key for the "raising steps method" to work. This kind of results has been recently extended by Amar [4] to (r, s)-forms.

Minini [5] obtained $L^p - L^r$ estimates for solutions of the $\bar{\partial}$-equation on a finite transverse intersection of strictly pseudoconvex bounded domains in $\mathbb{C}^n$.

In the case of bounded convex domains of finite type m, by using support functions, Diederich et al.[6] proved optimal $\frac{1}{m}$-Hölder estimate and Fischer [7] obtained optimal L^p estimates for $\bar{\partial}$.

On generalizing the domains introduced in [5] to q-convex setting, $q \ge 1$, Lan Ma and Vassiliadou [8] introduced the so called q-convex intersection below and then they obtained $L^p - L^r$ estimates for $\bar{\partial}$ on such domains.

Definition 1.1. A bounded domain D in a complex manifold X of complex dimension n is called a $\mathcal{C}^3$ q-convex intersection $(q \ge 1)$ in the sense of Grauert if there exist a bounded neighborhood U of $\bar{D}$ and a finite number of real-valued $\mathcal{C}^3$ functions $\rho_1(z), \dots, \rho_N(z)$, where $n \ge N+2$, defined on U such that

$$D = \{z \in U \,|\, \rho_1(z) < 0, \dots, \rho_N(z) < 0\}$$

and the following conditions are fulfilled:

(1) For $1 \le i_1 < i_2 < \cdots < i_\ell \le N$ the 1-forms $d\rho_{i_1}, \dots, d\rho_{i_\ell}$ are $\mathbb{R}-$linearly independent on the set $\bigcap_{j=1}^{\ell} \{\rho_{i_j}(z) \le 0\}$.

(2) For $1 \le i_1 < i_2 < \cdots < i_\ell \le N$ and every $z \in \bigcap_{j=1}^{\ell} \{\rho_{i_j}(z) \le 0\}$, if we set $I = (i_1, \dots, i_\ell)$, there exists a linear subspace T_z^I of X of complex dimension at least $n-q+1$ such that for $i \in I$ the Levi forms L_{ρ_i} restricted on T_z^I are positive definite.

Theorem 1.2 ([8]). *Let Ω be a $\mathcal{C}^3$ q-convex intersection in $\mathbb{C}^n$ with $1 \le q \le n$. Let f be a $\bar{\partial}$-closed form in $L^p_{0,s}(\Omega)$, where $1 \le p \le \infty$ and $s \ge q$. Then there exist a $\nu \in \mathbb{N}^+$, depending on the maximal number of nonempty intersections of $\{\rho_i = 0\}_{i=1}^{\ell+1}$, a form u in $L^r_{0,s-1}(\Omega)$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{\lambda} - 1$, and a positive constant C such that $\bar{\partial}u = f$ and*

$$\|u\|_{L^r_{0,s-1}(\Omega)} \le C\|f\|_{L^p_{0,s}(\Omega)},$$

where $1 \le \lambda < \frac{2\nu+2n}{2n-1+2\nu}$. More precisely, we have

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(1) *For any* $1 < p < 2(n+\nu)$, *there exists a positive constant* $C_p(\Omega)$ *(depends on* $\max\{\|\rho_i\|_{\mathcal{C}^3}\}_{i=1}^{\ell+1}$, Ω *and* p*) such that*

$$\|u\|_{L^r_{0,s-1}(\Omega)} \le C_p(\Omega)\|f\|_{L^p_{0,s}(\Omega)} \quad \text{with} \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{2n+2\nu}.$$

(2) *For* $p \ge 2(n+\nu)$, *there is a constant* $C_p(\Omega) > 0$ *(depends on* $\max\{\|\rho_i\|_{\mathcal{C}^3}\}_{i=1}^{\ell+1}$, Ω *and* p*) such that*

$$\|u\|_{L^\infty_{0,s-1}(\Omega)} \le A_p(\Omega)\|f\|_{L^p_{0,s}(\Omega)}.$$

The main goal of this paper is to extend their results above to Stein manifolds. Namely, we aim to prove the following $L^p - L^r$ existence theorem.

Theorem 1.3. *Let* Ω *be a* $\mathcal{C}^3$ *q-convex intersection* $(q \ge 1)$ *in a Stein manifold X of complex dimension n with* $n \ge 2$ *and f a* $\bar\partial$*-closed* (0, *s*)*-form on* Ω *with* $s \ge q$. *Then we have the following assertions.*

(a) *If* $1 \le p \le 2$ *and f belongs to* $L^p_{0,s}(\Omega)$, *then the equation* $\bar\partial u = f$ *has a solution u in* $L^r_{0,s-1}(\Omega)$ *with* $\frac{1}{r} = \frac{1}{p} - \gamma$, *where* $\gamma = \min(\frac{1}{2(n+\nu)}, \frac{1}{p} - \frac{1}{2})$, *moreover, there is a constant* $C_p(\Omega) > 0$ *such that*

$$\|u\|_{L^r_{0,s-1}(\Omega)} \le C_p(\Omega)\|f\|_{L^p_{0,s}(\Omega)}.$$

(b) *If* $2 < p < 2(n+\nu)$ *and f belongs to* $L^{p,c}_{0,s}(\Omega)$ *with* $\bar\partial\omega = 0$ *for* $q \le s < n$ *and f belongs to* $L^p_{0,s}(\Omega)$ *with* $f \perp \mathcal{H}^{p'}_n(\Omega)$ *for* $s = n$, *then the equation* $\bar\partial u = f$ *has a solution u in* $L^r_{0,s-1}(\Omega)$ *with* $\frac{1}{r} = \frac{1}{p} - \frac{1}{2(n+\nu)}$, *moreover, there is a positive constant* $A_p(\Omega)$ *such that*

$$\|u\|_{L^r_{0,s-1}(\Omega)} \le A_p(\Omega)\|f\|_{L^p_{0,s}(\Omega)}.$$

(c) *If* $p \ge 2(n+\nu)$ *and f belongs to* $L^p_{0,s}(\Omega)$, *then the equation* $\bar\partial u = f$ *has a solution u in* $L^\infty_{0,s-1}(\Omega)$, *moreover, there is a constant* $E_p(\Omega) > 0$ *such that*

$$\|u\|_{L^\infty_{0,s-1}(\Omega)} \le E_p(\Omega)\|f\|_{L^p_{0,s}(\Omega)},$$

The constants $C_p(\Omega)$, $A_p(\Omega)$, *and* $E_p(\Omega)$ *depend on* $\max\{\|\rho_i\|_{\mathcal{C}^3}\}_{i=1}^{\ell+1}$, Ω *and p.*

The proof relies heavily on the L^2-Hilbert space techniques of Hörmander [9] and on applying the raising steps method introduced in Amar [10], we first recall this method to make our paper reasonably self-contained. Let X be a smooth manifold admitting a partition of unity and a decreasing scale $\{B_p\}_{p\ge1}$, $r \ge p \Rightarrow B_r(\Omega) \subset B_p(\Omega)$ of Banach spaces of functions or forms defined on relatively compact open set Ω in X such that $\Omega' \subset \Omega$ implies $B_p(\Omega) \subset B_p(\Omega')$. These Banach spaces must be "strong" modules over $\mathcal{D}$, the space of $\mathcal{C}^\infty$ functions with compact support, i.e., let Ω, U be two open sets and $\Omega' = \Omega \cap U$; if $f \in B_p(\Omega')$ and $\chi \in \mathcal{D}(U)$, then $\chi f \in B_p(\Omega)$ which is stronger than $\chi f \in B_p(\Omega')$, with $\|\chi f\|_{B_p(\Omega)} \le C(\chi)\|f\|_{B_p(\Omega')}$. This means that the smooth extension of f by 0 in $\Omega\backslash\Omega'$ is also in $B_p(\Omega)$. For instance, $B_p(\Omega) = L^p(\Omega)$ the Lebesgue spaces, or $B_p(\Omega) = W^{p,t}(\Omega)$ the Sobolev spaces are such spaces.

The problem is to solve the linear equation $Du = f$, where D is a linear operator and $f \in B_p(\Omega)$ with eventually the constraint $\Delta f = 0$, where Δ is also a linear operator such that $\Delta D = D\Delta = 0$. In case there is no constraint we take $\Delta \equiv 0$. We put the following hypotheses on D for any domain $\Omega \subset X$:

(i) $\forall \chi \in \mathcal{D}(\Omega)$, $D\chi \in \mathcal{D}(\Omega)$;
(ii) $\forall \chi \in \mathcal{D}(\Omega)$, $\forall \alpha \in B_p(\Omega)$, $D(\chi\alpha) = D\chi \cdot \alpha + \chi D\alpha$.
It can be easily seen that a linear differential operator D verifies these assumptions.
Let Ω be a relatively compact domain in X, we put the following assumptions on X and Ω. There is a $p_0 > 1$ and a $\delta > 0$ such that
(iii) There is an open covering $\{U_j\}_{j=1,\dots,N}$ of $\overline{\Omega}$ such that, $\forall p \le p_0$, setting $\frac{1}{r} = \frac{1}{p} - \delta$, for given $f \in B_p(\Omega)$ with $\Delta f = 0$, we can solve $Du_j = f$ locally in $\Omega_j := U_j \cap \Omega$ with $B_p(\Omega) - B_r(\Omega_j)$ estimates, that is, there exist u_j in $B_s(\Omega_j)$ and a constant $C_0 > 0$ such that
$Du_j = f$ in Ω_j and $\|u_j\|_{B_r(\Omega_j)} \le C_0\|f\|_{B_p(\Omega)}$.
(iv) We can solve $D\beta = f$, $\Delta f = 0$, globally in Ω with $B_{p_0} - B_{p_0}$ estimates, i.e., there exist β in $B_{p_0}(\Omega)$ and a constant $E > 0$ such that
$D\beta = f$ in Ω and $\|\beta\|_{B_{p_0}(\Omega)} \le E\|f\|_{B_{p_0}(\Omega)}$.

We therefore have the following key theorem.

Theorem 1.4 (Raising steps theorem, [10]). *Under the above assumptions. If f is in* $B_p(\Omega)$ *with* $\Delta f = 0$, $p \le p_0$, *there exist u in* $B_r(\Omega)$ *with* $\gamma := \min(\delta, \frac{1}{p} - \frac{1}{p_0})$ *and* $\frac{1}{r} = \frac{1}{p} - \gamma$, *and a constant* $C > 0$ *such that* $Du = f$ *in* Ω *and*

$$\|u\|_{B_r(\Omega)} \le C\|f\|_{B_p(\Omega)}.$$

2. Proof of Theorem 1.3

We apply the raising steps method above to the case when Ω is a $\mathcal{C}^3$ q-convex intersection in a Stein manifold X of complex dimension $n \ge 2$, $D = \Delta = \bar\partial$, and $B_p(\Omega) = L^p_{0,s}(\Omega)$ is the space of $(0, s)$-forms with coefficients in $L^p(\Omega)$. This will be done in several steps. Clearly (i) and (ii) are verified. Then we begin by using the L^2-theory for $\bar\partial$.

2.1. Use of L^2-estimates and Serre duality for $\bar\partial$

Let Ω be given as above. Choose finitely many holomorphic coordinates system $h_j : U_j \to \mathbb{C}^n$ on X and choose also open subsets $V_j \Subset U_j$ such that $\overline{\Omega} \subseteq \cup_j V_j$ and $\Omega_j = V_j \cap \Omega$ is a local q-convex intersection in X and $\Omega'_j = h_j(\Omega_j)$ is a $\mathcal{C}^3$ q-convex intersection in $\mathbb{C}^n$ for each j. Let f be a $\bar\partial$-closed form in $L^p_{0,s}(\Omega)$. Theorem 1.2 is now applied to each $\Omega'_j \subset \mathbb{C}^n$ which yields a solution u'_j in $L^r_{0,s-1}(\Omega'_j)$ to the equation $\bar\partial u'_j = h_j f$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2(n+\nu)}$. So we have here $\delta = \frac{1}{2(n+\nu)}$. The resulting solution u'_j is then pulled-back to Ω_j using the holomorphic map h_j, we then have a solution $u_j \in L^r_{0,s-1}(\Omega_j)$ to the equation $\bar\partial u_j = f$ in Ω_j with control of the norm. So assumption (iii) is fulfilled.

The assumption (iv) follows from the following L^2-setting. Since the q-convexity is stable with respect to $\mathcal{C}^3$ small perturbations, by arguing as in the proof of Lemma 2.1 in [11], we can exhaust Ω from inside by a sequence of $\mathcal{C}^3$ strictly q-convex domains $\{\Omega_k\}$ such that

$$\Omega_k \Subset \Omega_{k+1} \Subset \Omega \quad \text{and} \quad \Omega = \cup_k \Omega_k.$$

It follows from [9, Theorem 3.4.1] that the operator $\bar\partial : L^2_{0,s-1}(\Omega_k) \to L^2_{0,s}(\Omega_k)$ has closed range for each k and all $s \ge q$. For all $f \in L^2_{0,s}(\Omega) \cap \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$, $s \ge q$, we then have the estimate

$$\int_{\Omega_k} |f|^2 dv \le C_s(\Omega_k)\left(\|\bar\partial f\|^2_{L^2(\Omega_k)} + \|\bar\partial^* f\|^2_{L^2(\Omega_k)}\right),$$

where dV is the volume element on X and $C_s(\Omega_k)$ is a positive constant depends on the diameter of Ω_k and s.

Using this estimate and standard arguments, we deduce that the $\bar\partial$ operator has L^2 closed range on Ω for all $(0, s)$-forms with $s \ge q$. Proposition 1.1.2 in [9] and the fact that $\mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*) = \{0\}$ imply that the estimate

$$\|f\|^2_{L^2(\Omega)} \le K(\Omega)\left(\|\bar\partial f\|^2_{L^2(\Omega)} + \|\bar\partial^* f\|^2_{L^2(\Omega)}\right)$$

holds for all $f \in L^2_{0,s}(\Omega) \cap \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ with $s \geq q$.

This last estimate together with Theorem 1.1.4 in [9] enable us to have the following L^2-existence theorem.

Theorem 2.1. *Let Ω be a $\mathcal{C}^3$ q-convex intersection ($q \geq 1$) in a Stein manifold X of complex dimension n with $n \geq 2$. Then for every $\bar\partial$-closed form f in $L^2_{0,s}(\Omega)$, $s \geq q$, there exist a form u in $L^2_{0,s-1}(\Omega)$ solving the equation $\bar\partial u = f$ in Ω and a constant $C > 0$ (depending on Ω and s) satisfying the following L^2-estimate*

$$\|u\|_{L^2_{0,s-1}(\Omega)} \leq C\|f\|_{L^2_{0,s}(\Omega)}.$$

On applying Theorem 1.4, we therefore get the assertion (a) in Theorem 1.3.

We turn now to the case when $p > 2$, we proceed by duality and ask that f has compact support, that is, f belongs to $L^{p,c}_{(0,s)}(\Omega)$ when $s < n$. Denote by $\mathcal{H}^p_m(\Omega)$ the space of all $\bar\partial$-closed $(m, 0)$-forms in Ω with coefficients in $L^p(\Omega)$.

To prove the assertion (b) in Theorem 1.3, we use the same duality approach which we have already used in [12] inspired by the Serre duality [13]. As usual let r' the conjugate exponent to r.

Lemma 2.2. *Let Ω be given as in the statement of Theorem 1.3 and f be as in the assertion (b) of that theorem. Consider the form $\mathcal{L} = \mathcal{L}_f$ defined on $\bar\partial$-closed $(n, n-s+1)$-form α with $L^{p'}(\Omega)$-components by*

$$\mathcal{L}_f(\alpha) := (-1)^{s-1}\langle f, \varphi\rangle,$$

where $\varphi \in L^{r'}_{n,n-s}(\Omega)$ is such that $\bar\partial\varphi = \alpha$ in Ω. Then $\mathcal{L}$ is well defined and linear.

Proof. First notice that if $\frac{1}{p} = \frac{1}{r} - \frac{1}{2(n+\nu)}$ then $\frac{1}{r'} = \frac{1}{p'} - \frac{1}{2(n+\nu)}$, we assume also that $r < 2(n+\nu)$ to insure that p being finite. Such a current φ with $\bar\partial\varphi = \alpha$ exists since p' the conjugate exponent of p satisfies $p' \leq 2$, hence we can apply Theorem 1.3 (a) that we have just proved, with the remark that $p' \leq r'$.

Suppose first that $s < n$, to prove that $\mathcal{L}$ is well defined we have to show

$$\forall \varphi, \psi \in L^{r'}_{n,n-s}(\Omega),\ \bar\partial\varphi = \bar\partial\psi \Rightarrow \langle f, \varphi\rangle = \langle f, \psi\rangle.$$

This is meaningful because $f \in L^{p,c}_{0,s}(\Omega)$, $p > 1$, $\mathrm{Supp} f \Subset \Omega$. Since $\bar\partial\varphi = \bar\partial\psi$, then the difference $\varphi - \psi$ is $\bar\partial$-closed form in $L^{r'}_{n,n-s}(\Omega) \subset L^{p'}_{n,n-s}(\Omega)$ so the $\bar\partial$-equation is solvable in $L^{r'}(\Omega)$, because $s' \leq 2$ by Theorem 1.3 (a), and hence there exists a form γ in $L^{r'}_{n,n-s-1}(\Omega)$ such that $\bar\partial\gamma = \varphi - \psi$. Therefore

$$\langle f, \varphi - \psi\rangle = \left\langle f, \bar\partial\gamma\right\rangle = (-1)^{s-1}\left\langle\bar\partial f, \gamma\right\rangle = 0.$$

As f being compactly supported in Ω there is no boundary term. Then $\mathcal{L}_f$ is well defined in this case.

For $s = n$, we have $\bar\partial f = 0$ (because in this case f is of bidegree $(0, n)$). Again, let φ, $\psi \in L^{r'}_{n,0}(\Omega)$ with $\bar\partial\varphi = \bar\partial\psi$, hence $\varphi - \psi$ is a $\bar\partial$-closed $(n, 0)$-form, i.e., $\varphi - \psi \in \mathcal{H}^{r'}_n(\Omega)$. Since, by hypothesis, $f \perp \mathcal{H}^{r'}_n(\Omega)$, we have $\langle\varphi - \psi, f\rangle = 0$. Then $\mathcal{L}_f$ is also well defined in this case.

We show next that $\mathcal{L}$ is a linear form, let α_1 and α_2 be in $L^{p'}_{n,n-s+1}(\Omega)$ such that $\bar\partial\alpha_1 = \bar\partial\alpha_2 = 0$ and put $\alpha = \alpha_1 + \lambda\alpha_2$, with $\lambda \in \mathbb{C}$, then $\bar\partial\alpha = 0$ and so there are φ, φ_1, and φ_2 in $L^{r'}_{n,n-s}(\Omega)$ such that $\alpha = \bar\partial\varphi$, $\alpha_1 = \bar\partial\varphi_1$, and $\alpha_2 = \bar\partial\varphi_2$,. So, because $\bar\partial(\varphi - \varphi_1 - \lambda\varphi_2) = 0$, if $s < n$, there is a form ψ in $L^{r'}_{n,n-s-1}(\Omega)$ such that $\varphi = \varphi_1 + \varphi_2 + \bar\partial\psi$.

Therefore

$$\begin{aligned}\mathcal{L}_f(\alpha) = (-1)^{s-1}\langle f, \varphi\rangle &= (-1)^{s-1}\left\langle f, \varphi_1 + \lambda\varphi_2 + \bar\partial\psi\right\rangle \\ &= \mathcal{L}(\alpha_1) + \lambda\mathcal{L}(\alpha_2) + (-1)^{s-1}\left\langle f, \bar\partial\psi\right\rangle \\ &= \mathcal{L}_f(\alpha_1) + \lambda\mathcal{L}_f(\alpha_2).\end{aligned}$$

This is because $\left\langle f, \bar\partial\psi\right\rangle = \left\langle\bar\partial f, \psi\right\rangle = 0$, since $\mathrm{Supp} f \Subset \Omega$ implies there is no boundary term.

If $s = n$, because $\bar\partial(\varphi - \varphi_1 - \varphi_2) = 0$, we have $h = (\varphi - \varphi_1 - \lambda\varphi_2) \in \mathcal{H}^{r'}_n(\Omega)$ and the hypothesis $f \perp \mathcal{H}^{r'}_n(\Omega)$ gives $\langle f, h\rangle = 0$. Thus

$$\begin{aligned}\mathcal{L}_f(\alpha) &= (-1)^{s-1}\langle f, \varphi\rangle = (-1)^{s-1}\langle f, \varphi_1 + \lambda\varphi_2 + h\rangle \\ &= \mathcal{L}_f(\alpha_1) + \lambda\mathcal{L}_f(\alpha_2) + (-1)^{s-1}\langle f, h\rangle \\ &= \mathcal{L}_f(\alpha_1) + \lambda\mathcal{L}_f(\alpha_2).\end{aligned}$$

By repeating the same arguments for $\alpha = \lambda\alpha_1$, we get the linearity of $\mathcal{L}$. □

Lemma 2.3. *Under the same hypotheses as above, there is a $(0, s-1)$-form u such that*

$$\forall\ \alpha \in L^{p'}_{n,n-s+1}(\Omega),\ \langle u, \alpha\rangle = \mathcal{L}_f(\alpha) = (-1)^{s-1}\langle f, \varphi\rangle,$$

and

$$\sup_{\alpha \in L^{p'}(\Omega),\ \|\alpha\|_{L^{p'}(\Omega)} \leq 1} |\langle u, \alpha\rangle| \leq C\|f\|_{L^r(\Omega)}. \tag{2.1}$$

Proof. It follows from Lemma 2.2 that $\mathcal{L}$ is a linear form on $\bar\partial$-closed $(n, n-s+1)$-forms α with $L^{p'}(\Omega)$-coefficients, then there exists an $(n, n-s)$-form φ with $L^{r'}(\Omega)$-coefficients solving the equation $\bar\partial\varphi = \alpha$ and satisfying the estimate

$$\|\varphi\|_{L^{r'}(\Omega)} \leq K\|\alpha\|_{L^{p'}(\Omega)}. \tag{2.2}$$

By definition of $\mathcal{L}$ and Hölder inequality, we get

$$|\mathcal{L}(\alpha)| = |\langle f,\ \varphi\rangle| \leq \|f\|_{L^r(\Omega)}\|\varphi\|_{L^{r'}(\Omega)} \leq C\|f\|_{L^r(\Omega)}\|\alpha\|_{L^{s'}(\Omega)}.$$

The last inequalities follows from (2.2). So we see that the norm of $\mathcal{L}$ is bounded on the subspace of $\bar\partial$ closed forms in $L^{p'}(\Omega)$ by $C\|f\|_{L^r(\Omega)}$.

By the Hahn–Banach Theorem, we can extend $\mathcal{L}$ with the same norm to all $(n, n-s+1)$-forms in $L^{p'}(\Omega)$. As in Serre duality theorem (see [13], p. 20) this is one of the main ingredient in the proof. This means, in the sense of currents, that there is a $(0, s-1)$-form u represents the extended form $\mathcal{L}$, i.e., u satisfies the required properties. □

2.2. End Proof of Theorem 1.3

Let us now apply Lemma 2.3 to forms $\varphi \in \mathcal{D}_{n,n-s}(\Omega)$, we then get $\alpha = \bar\partial\varphi \in \mathcal{D}_{n,n-s+1} \subset L^{p'}(\Omega)$ and

$$\left\langle u, \bar\partial\varphi\right\rangle = \mathcal{L}(\bar\partial\varphi) = (-1)^{s-1}\langle f, \varphi\rangle \Rightarrow \left\langle\bar\partial u, \varphi\right\rangle = \langle f, \varphi\rangle$$

Since φ has compact support in Ω, then $\bar\partial u = f$ in the sense of distributions. In addition, the estimate (2.1) implies by duality arguments that

$$\|u\|_{L^p_{0,s-1}(\Omega)} \leq C\|f\|_{L^r_{0,s}(\Omega)}.$$

This proves the assertion (b) of Theorem 1.3.

For the assertion (c), arguing as above with $p' = 1$ we then find

$$\sup_{\alpha \in L^1(\Omega),\ \|\alpha\|_{L^1(\Omega)} \leq 1} |\langle u, \alpha\rangle| \leq C\|\omega\|_{L^r(\Omega)},$$

which implies by duality that $u \in L^\infty_{0,s-1}(\Omega)$ with control of the norm. The proof of Theorem 1.3 is now complete. □

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