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On initial coefficient inequalities for certain new subclasses of bi-univalent functions

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ABSTRACT

In the present paper, we introduce two interesting subclasses of the class of bi-univalent functions defined on the open unit disk $\mathbb{U}$ and obtain improved estimates on the initial coefficients $|a_2|$, $|a_3|$ and $|a_4|$ for the functions in these subclasses.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{A}$ denote the class of functions in $\mathcal{H}$ given by:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions *f*(*z*) in $\mathbb{U}$. Clearly, due to the Koebe one quarter theorem [1], every function $f \in \mathcal{S}$ has an inverse f^{-1} such that $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq 1/4)$. In fact, some simple calculations gives:

$$\begin{aligned} g(w) &= f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \end{aligned} \tag{1.2}$$

Let $\Sigma = \{f \in \mathcal{S} : \textit{both } f \textit{ and } f^{-1} \textit{ are univalent in } \mathbb{U}\}$ denote the class of bi-univalent functions in $\mathbb{U}$. For more information and examples on the class Σ, see the work of Srivastava et al. [2] (see also [3,4]). Recently many researchers (viz [5–26]) obtained initial coefficient estimates for the functions in various subclasses of bi-univalent functions. In 1972, the following univalence criterion was proved by Ozaki and Nunokawa [27].

Lemma 1.1. *If for* $f(z) \in \mathcal{A}$

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

then f(z) is univalent in $\mathbb{U}$ *and hence* $f(z) \in \mathcal{S}$.

Also, let $\mathcal{T}(\mu)$ denote the class of functions $f(z) \in \mathcal{A}$ such that:

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu \quad (z \in \mathbb{U}),$$

where μ is real number with $0 < \mu \leq 1$ and $\mathcal{T}(1) = \mathcal{T}$. Clearly, $\mathcal{T}(\mu) \subset \mathcal{T} \subset \mathcal{S}$.

Further (see Kuroki et al. [28]), for $f(z) \in \mathcal{T}(\mu)$ see that:

$$\Re\left(\frac{z^2 f'(z)}{(f(z))^2} \right) > 1 - \mu \quad (z \in \mathbb{U}).$$

In 1967, Lewin [29] investigated the class Σ and proved that $|a_2| < 1.51$; in 1969, Netanyahu [30] showed that $max_{f\in\Sigma} |a_2| = 4/3$ and in 1979, Brannan and Clunie [31] conjectured that $|a_2| \leq \sqrt{2}$. But still the problem of coefficient estimate for $|a_n|$, $(n = 3, 4, \cdots)$ is an open problem.

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The object of the present paper is to obtain the improved estimates on the coefficients $|a_2|$, $|a_3|$ and $|a_4|$ for two new subclasses of the bi-univalent function class Σ.

We need the following lemma (see [32]) to prove our main results.

Lemma 1.2. *If $\phi(z) \in \mathcal{P}$, the class of functions analytic in $\mathbb{U}$ with positive real part, given by:*

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (z \in \mathbb{U});$$

then $|c_n| \leq 2$ for each $n \in \mathbb{N}$.

2. Main results

Definition 2.1. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{T}_\Sigma(\mu)$ if the following conditions are satisfied:

$$\Re\left(\frac{z^2 f'(z)}{(f(z))^2}\right) > 1 - \mu \quad (z \in \mathbb{U}; 0 < \mu \leq 1)$$

and

$$\Re\left(\frac{w^2 g'(w)}{(g(w))^2}\right) > 1 - \mu \quad (w \in \mathbb{U}; 0 < \mu \leq 1),$$

where the function g is an extension of f^{-1} to $\mathbb{U}$ defined by (1.2).

Definition 2.2. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{T}_\Sigma^\alpha$ if the following conditions are satisfied:

$$\left|arg\left(\frac{z^2 f'(z)}{(f(z))^2}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}; 0 < \alpha \leq 1)$$

and

$$\left|arg\left(\frac{w^2 g'(w)}{(g(w))^2}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}; 0 < \alpha \leq 1),$$

where the function g is an extension of f^{-1} to $\mathbb{U}$ defined by (1.2).

Theorem 2.3. *Let the function $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_\Sigma(\mu)$, $(0 < \mu \leq 1)$. Then,*

$$|a_2| \leq 1,$$

$$|a_3| \leq 2\mu,$$

$$|a_4| \leq 3\mu.$$

Proof. Using Definition 2.1 we can write:

$$\frac{z^2 f'(z)}{(f(z))^2} = (1 - \mu) + \mu\, s(z) \tag{2.1}$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = (1 - \mu) + \mu\, t(w), \tag{2.2}$$

where $s(z), t(w) \in \mathcal{P}$ such that:

$$s(z) = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \ (z \in \mathbb{U}); \tag{2.3}$$

$$t(w) = 1 + t_1 w + t_2 w^2 + t_3 w^3 + \cdots, \ (w \in \mathbb{U}). \tag{2.4}$$

Hence we have:

$$(1 - \mu) + \mu\, s(z) = 1 + \mu s_1 z + \mu s_2 z^2 + \mu s_3 z^3 + \cdots,$$

$$(1 - \mu) + \mu\, t(w) = 1 + \mu t_1 w + \mu t_2 w^2 + \mu t_3 w^3 + \cdots.$$

Using (1.1) and (1.2), we obtain:

$$\frac{z^2 f'(z)}{(f(z))^2} = 1 + \left(a_3 - a_2^2\right) z^2 + 2\left(a_2^3 + a_4 - 2a_2 a_3\right) z^3 + \cdots,$$

$$\frac{w^2 g'(w)}{(g(w))^2} = 1 - \left(a_3 - a_2^2\right) w^2 - 2\left(2a_2^3 + a_4 - 3a_2 a_3\right) w^3 + \cdots.$$

Now, equating the coefficients in (2.1) and (2.2) we get $s_1 = t_1 = 0$ and also:

$$\left(a_3 - a_2^2\right) = \mu s_2, \tag{2.5}$$

$$2\left(a_2^3 + a_4 - 2a_2 a_3\right) = \mu s_3, \tag{2.6}$$

$$-\left(a_3 - a_2^2\right) = \mu t_2, \tag{2.7}$$

$$-2\left(2a_2^3 + a_4 - 3a_2 a_3\right) = \mu t_3. \tag{2.8}$$

Eq. (2.5) in light of Lemma 1.2 gives:

$$\left|a_3 - a_2^2\right| = |\mu s_2| = \mu |s_2| \leq 2\mu. \tag{2.9}$$

By adding (2.6) in (2.8), we obtain:

$$2a_2\left(a_3 - a_2^2\right) = \mu(s_3 + t_3), \tag{2.10}$$

which, by using Lemma 1.2 gives:

$$\left|a_2\left(a_3 - a_2^2\right)\right| = |a_2|\left|a_3 - a_2^2\right| \leq 2\mu. \tag{2.11}$$

See that, (2.9) and (2.11) together yields:

$$|a_2| \leq 1. \tag{2.12}$$

By subtracting (2.8) from (2.6), we get:

$$2\left(3a_2^3 + 2a_4 - 5a_2 a_3\right) = \mu(s_3 - t_3). \tag{2.13}$$

Eliminating a_2^3 by using (2.10) and (2.13), we get:

$$4(a_4 - a_2 a_3) = \mu(s_3 - t_3) + 3\mu(s_3 + t_3) = \mu(4s_3 + 2t_3) \tag{2.14}$$

which, by using Lemma 1.2 gives:

$$|a_4 - a_2 a_3| \leq 3\mu. \tag{2.15}$$

Also, eliminating $a_2 a_3$ by using (2.10) and (2.13), we get:

$$4\left(a_4 - a_2^3\right) = \mu(s_3 - t_3) + 5\mu(s_3 + t_3) = \mu(6s_3 + 4t_3) \tag{2.16}$$

which, by using Lemma 1.2 gives:

$$\left|a_4 - a_2^3\right| \leq 5\mu. \tag{2.17}$$

Now, using the inequality:

$$||z_1| - |z_2|| \leq |z_1 - z_2| \tag{2.18}$$

in Eq. (2.9), we can write:

$$|a_3| - \left|a_2^2\right| \leq \left|a_3 - a_2^2\right| \leq 2\mu,$$

from which, it is obvious that:

$$|a_3| \leq 2\mu.$$

Similarly, by using the inequality (2.18) in (2.15) and (2.17), we get:

$$|a_4| \leq 3\mu.$$

This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let the function $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_\Sigma^\alpha$, $(0 < \alpha \leq 1)$. Then,*

$$|a_2| \leq 1,$$

$$|a_3| \leq 2\alpha,$$

$$|a_4| \leq 3\alpha.$$

Proof. Definition 2.2 implies that there exist functions $s(z)$ and $t(w)$ given by (2.3) and (2.4) respectively, such that:

$$\frac{z^2 f'(z)}{(f(z))^2} = [s(z)]^\alpha \tag{2.19}$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = [t(w)]^{\alpha}. \tag{2.20}$$

Clearly, we have:

$$[s(z)]^{\alpha} = 1 + \alpha s_1 z + \left[\alpha s_2 + \frac{\alpha(\alpha-1)}{2} s_1^2\right] z^2 + \left[\alpha s_3 + \alpha(\alpha-1) s_1 s_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} s_1^3\right] z^3 + \cdots,$$

$$[t(w)]^{\alpha} = 1 + \alpha t_1 w + \left[\alpha t_2 + \frac{\alpha(\alpha-1)}{2} t_1^2\right] w^2 + \left[\alpha t_3 + \alpha(\alpha-1) t_1 t_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} t_1^3\right] w^3 + \cdots.$$

Also, we have:

$$\frac{z^2 f'(z)}{(f(z))^2} = 1 + \left(a_3 - a_2^2\right) z^2 + 2\left(a_2^3 + a_4 - 2a_2 a_3\right) z^3 + \cdots,$$

$$\frac{w^2 g'(w)}{(g(w))^2} = 1 - \left(a_3 - a_2^2\right) w^2 - 2\left(2a_2^3 + a_4 - 3a_2 a_3\right) w^3 + \cdots.$$

Observe that, by equating the coefficients in (2.19) and (2.20) we get $s_1 = t_1 = 0$ and hence the further equalities becomes:

$$\left(a_3 - a_2^2\right) = \alpha s_2,$$

$$2\left(a_2^3 + a_4 - 2a_2 a_3\right) = \alpha s_3,$$

$$-\left(a_3 - a_2^2\right) = \alpha t_2,$$

$$-2\left(2a_2^3 + a_4 - 3a_2 a_3\right) = \alpha t_3.$$

Now, proceeding similarly as in the proof of Theorem 2.3, we can complete the further proof. □

3. Conclusions

- The estimate $|a_2| \leq 1$ is independent of μ and α for the subclasses $\mathcal{T}_{\Sigma}(\mu)$ and $\mathcal{T}_{\Sigma}^{\alpha}$ respectively.
- By observing the estimates on $|a_3|$ and $|a_4|$, it is interesting to see here that, can we generalize it to $|a_n| \leq (n-1)\mu$, $(n \geq 3)$ for the subclass $\mathcal{T}_{\Sigma}(\mu)$ and $|a_n| \leq (n-1)\alpha$, $(n \geq 3)$ for the subclass $\mathcal{T}_{\Sigma}^{\alpha}$?
- It is interesting that the inequalities obtained for the subclasses $\mathcal{T}_{\Sigma}(\mu)$ and $\mathcal{T}_{\Sigma}^{\alpha}$ are similar.

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References

[1] P.L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften, Springer, New York, 1983.
[2] H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192.
[3] D.A. Brannan, J.G. Clunie, W.E. Kirwan, Coefficient estimates for a class of starlike functions, Canad. J. Math. 22 (1970) 476–485.
[4] T.S. Taha, Topics in Univalent Function Theory, University of London, London, 1981 Ph.d. Thesis.
[5] D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math. 31 (2) (1986) 70–77.
[6] B.A. Frasin, M.K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. 24 (2011) 1569–1573.
[7] R.M. Ali, S.K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. 25 (2012) 344–351.
[8] Q.-H. Xu, Y.-C. Gui, H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. 25 (2012) 990–994.
[9] X.F. Li, A.P. Wang, Two new subclasses of bi-univalent functions, Int. Math. Forum 7 (2012) 1495–1504.
[10] S.P. Goyal, P. Goswami, Estimates for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, J. Egypt. Math. Soc. 20 (2012) 179–182.
[11] S. Porwal, M. Darus, On a new subclass of bi-univalent functions, J. Egypt. Math. Soc. 21 (3) (2013) 190–193.
[12] M. Çağlar, H. Orhan, N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, Filomat 27 (7) (2013) 1165–1171.
[13] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, Novi Sad J. Math. 43 (2) (2013) 59–65.
[14] S. Bulut, Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator, J. Funct. Spaces Appl. (2013) 1–7. Article ID 181932.
[15] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Classical Anal. 2 (1) (2013) 49–60.
[16] H.M. Srivastava, G. Murugusundaramoorthy, N. Magesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, Global J. Math. Anal. 1 (2) (2013) 67–73.
[17] H. Tang, G.-T. Deng, N. Magesh, S.-H. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, J. Inequal. Appl. 2013 (317) (2013).
[18] S.S. Kumar, V. Kumar, V. Ravichandran, Estimates for the initial coefficients of bi-univalent functions, Tamsui Oxf. J. Inf. Math. Sci. 29 (4) (2013) 487–504.
[19] R.M. El-Ashwah, Subclasses of bi-univalent functions defined by convolution, J. Egypt. Math. Soc. 22 (2014) 348–351.
[20] H.M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egypt. Math. Soc. 23 (2) (2015) 242–246.
[21] H. Orhan, N. Magesh, V.K. Balaji, Initial coefficient bounds for a general class of bi-univalent functions, Filomat 29 (6) (2015) 1259–1267.
[22] A.B. Patil, U.H. Naik, Initial coefficient bounds for a general subclass of bi-univalent functions defined by Al-Oboudi differential operator, J. Anal. 23 (2015) 111–120.
[23] S. Altinkaya, S. Yalcin, Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points, J. Funct. Spaces (2015) 1–5. Article ID 145242
[24] S. Joshi, S. Joshi, H. Pawar, On some subclasses of bi-univalent functions associated with pseudo-starlike functions, J. Egypt. Math. Soc. 24 (2016) 522–525.
[25] A.B. Patil, U.H. Naik, On initial coefficient estimates of subclass of analytic bi-univalent functions defined by quasi-subordination, Bull. Cal. Math. Soc. 108 (4) (2016) 259–268.
[26] A.Y. Lashin, On certain subclasses of analytic and bi-univalent functions, J. Egypt. Math. Soc. 24 (2016) 220–225.
[27] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, Proc. Amer. Math. Soc. 33 (1972) 392–394.
[28] K. Kuroki, T. Hayami, N. Uyanik, S. Owa, Some properties for a certain class concerned with univalent functions, Comput. Math. Appl. 63 (2012) 1425–1432.
[29] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967) 63–68.
[30] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Ration. Mech. Anal. 32 (1969) 100–112.
[31] D.A. Brannan, J.G. Clunie (Eds.), Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Institute Held at the University of Durham, Durham; July 1-20, 1979, Academic Press, New York and London, 1980.
[32] C. Pommerenke, Univalent Functions, Vandenhoeck and Rupercht, Göttingen, 1975.