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Original article

Convolution conditions for subclasses of meromorphic functions of complex order associated with basic Bessel functions

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ABSTRACT

Making use of the operator $\mathcal{L}_{q,v}$ associated with functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^{k-1},$$

which are analytic in the punctured unit disc $\mathbb{U}^* := \mathbb{U}\setminus\{0\}$, we introduce two subclasses of meromorphic functions and investigate convolution properties and coefficient estimates for these subclasses.

1. Introduction

The q-theory has important role in various branches of mathematics and physics as for example, in the areas of special functions, ordinary fractional calculus, optimal control problems, q-difference, q-integral equations, q-transform analysis and in quantum physics (see for instance [1–8]).

Let Σ denote the class of meromorphic functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^{k-1}, \tag{1.1}$$

which are analytic in the punctured unit disc $\mathbb{U}^* := \mathbb{U}\setminus\{0\}$, where $\mathbb{U} := \{z\in\mathbb{C} : |z|<1\}$. If $g\in\Sigma$ is given by

$$g(z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_k z^{k-1},$$

then the *Hadamard (or convolution) product* of f and g is defined by

$$(f*g)(z) := \frac{1}{z}+\sum_{k=1}^{\infty}a_k b_k z^{k-1}.$$

We recall some definitions which will be used in our paper.

Definition 1.1. For two functions f and g analytic in $\mathbb{U}$, we say that the function f is *subordinate* to g, written $f(z)\prec g(z)$, if there exists a *Schwarz function* w, that is w is analytic in $\mathbb{U}$, with $w(0)=0$ and $|w(z)|<1$ for all $z\in\mathbb{U}$, such that $f(z)=g(w(z))$, $z\in\mathbb{U}$.

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [9]):

$$f(z)\prec g(z) \Leftrightarrow f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}).$$

Gasper and Rahman [5] defined the *q-derivative* $(0<q<1)$ of a function f of the form (1.1) by

$$D_q f(z) := \frac{f(qz)-f(z)}{(q-1)z},\ z\in\mathbb{U}^*. \tag{1.2}$$

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From (1.2) we deduce that for a function f of the form (1.1) the q-derivative $D_q f$ is given by

$$D_q f(z) = -\frac{1}{qz^2} + \sum_{k=1}^{\infty}[k-1]_q\, a_k z^{k-2},\ z \in \mathbb{U}^*,$$

where

$$[k-1]_q := \frac{1-q^{k-1}}{1-q} = 1 + q + \cdots + q^{k-2}.$$

As $q \to 1^-$, then $[k-1]_q \to k-1$, hence we have

$$\lim_{q\to 1^-} D_q f(z) = f'(z),\ z \in \mathbb{U}^*.$$

Definition 1.2. For $0 < q < 1$, $0 \le \lambda < 1$, $-1 \le B < A \le 1$ and $b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, let $\Sigma\mathcal{S}^*_{q,\lambda}[b; A, B]$ be the subclass of Σ consisting of functions f of the form (1.1) and satisfying

$$1 - \frac{1}{b}\left[\frac{zD_q f(z)}{\left(1-\frac{\lambda}{q}\right) f(z) - \lambda z D_q f(z)} + \frac{1}{q}\right] \prec \frac{1+Az}{1+Bz}. \tag{1.3}$$

Also, let $\Sigma\mathcal{K}_{q,\lambda}[b; A, B]$ be the subclass of Σ consisting of functions f of the form (1.1) and satisfying

$$1 - \frac{1}{b}\left[\frac{zD_q(-qzD_q f(z))}{\left(1-\frac{\lambda}{q}\right)(-qzD_q f(z)) - \lambda z D_q(-qzD_q f(z))} + \frac{1}{q}\right] \prec \frac{1+Az}{1+Bz}. \tag{1.4}$$

It is easy to verify from (1.3) and (1.4) that

$$f \in \Sigma\mathcal{K}_{q,\lambda}[b; A, B] \Leftrightarrow -qzD_q f \in \Sigma\mathcal{S}^*_{q,\lambda}[b; A, B]. \tag{1.5}$$

Remarks 1.1. We remark the following special cases:

(i) $\Sigma\mathcal{S}^*_{q,0}[b; A, B] =: \Sigma\mathcal{S}^*_q[b; A, B]$ and $\Sigma\mathcal{K}_{q,0}[b; A, B] =: \Sigma\mathcal{K}_q[b; A, B]$ (see Mostafa et al. [10]);

(ii) $\lim_{q\to 1^-} \Sigma\mathcal{S}^*_{q,\lambda}[b; A, B] =: \Sigma\mathcal{S}^*_\lambda[b; A, B]$ and $\lim_{q\to 1^-} \Sigma\mathcal{K}_{q,\lambda}[b; A, B] =: \Sigma\mathcal{K}_\lambda[b; A, B]$ (see Aouf et al. [11]);

(iii) $\lim_{q\to 1^-} \Sigma\mathcal{S}^*_{q,0}[b; A, B] =: \Sigma\mathcal{S}^*[b; A, B]$ and $\lim_{q\to 1^-} \Sigma\mathcal{K}_{q,0}[b; A, B] =: \Sigma\mathcal{K}[b; A, B]$ (see Bulboacă et al. [12]);

(iv) $\lim_{q\to 1^-} \Sigma\mathcal{S}^*_{q,0}[b; 1, -1] =: \Sigma\mathcal{S}(b)$ and $\lim_{q\to 1^-} \Sigma\mathcal{K}_{q,0}[b; 1, -1] =: \Sigma\mathcal{K}(b)$ (see Aouf [13]);

(v) $\lim_{q\to 1^-} \Sigma\mathcal{S}^*_{q,0}\left[(1-\alpha)e^{-i\mu}\cos\mu; 1, -1\right] =: \Sigma\mathcal{S}^\mu(\alpha)$ and $\lim_{q\to 1^-} \Sigma\mathcal{K}_{q,0}[(1-\alpha)e^{-i\mu}\cos\mu; 1, -1] =: \Sigma\mathcal{K}^\mu(\alpha)$ $\left(\mu \in \mathbb{R},\ |\mu| < \frac{\pi}{2},\ 0 \le \alpha < 1\right)$ (see Ravichandran et al. [14] with $p = 1$).

For any complex number α, the *q-shifted factorials* are defined by

$$(\alpha; q)_0 := 1; \quad (\alpha; q)_n := \prod_{k=0}^{n-1}(1-\alpha q^k),\ n \in \mathbb{N} := \{1, 2, \dots\}. \tag{1.6}$$

If $|q| < 1$, the definition (1.6) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j).$$

In terms of the *q-analogue of the gamma function*

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)},\ n > 0,$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty (1-q)^{1-x}}{(q^x; q)_\infty},\ 0 < q < 1.$$

We note that

$$\lim_{q\to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n,$$

where

$$(\alpha)_n = \begin{cases} 1, & \text{if} \quad n = 0, \\ \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1), & \text{if} \quad n \in \mathbb{N}. \end{cases}$$

Consider the *q-analogue of Bessel function* defined by (see Jackson [15])

$$\mathcal{J}^{(1)}_\nu(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(q; q)_k \left(q^{\nu+1}; q\right)_k}\left(\frac{z}{2}\right)^{2k+\nu},\ 0 < q < 1,$$

and let us define

$$\mathcal{L}_\nu(z; q) := \frac{2^\nu (q; q)_\infty}{(q^{\nu+1}; q)_\infty (1-q)^\nu z^{\nu/2+1}} \mathcal{J}^{(1)}_\nu\left(z^{1/2}(1-q); q\right) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k (1-q)^{2k}}{4^k (q; q)_k (q^{\nu+1}; q)_k} z^{k-1},\ z \in \mathbb{U}^*.$$

Using the Hadamard product, we define the linear operator $\mathcal{L}_{q,\nu} : \Sigma \to \Sigma$ by

$$(\mathcal{L}_{q,\nu} f)(z) := \mathcal{L}_\nu(z; q) * f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k (1-q)^{2k}}{4^k (q; q)_k (q^{\nu+1}; q)_k} a_k z^{k-1},\ z \in \mathbb{U}^*, \tag{1.7}$$

where $f \in \Sigma$ has the form (1.1).

The operator $\mathcal{L}_{q,\nu}$ was introduced and studied by Mostafa et al [10]. Also, as $q \to 1^-$, the linear operator $\mathcal{L}_{q,\nu}$ reduces to the operator $\mathcal{L}_\nu$ introduced and studied by Aouf et al. [11].

Definition 1.3. For $0 < q < 1$, $0 \le \lambda < 1$, $-1 \le B < A \le 1$, $b \in \mathbb{C}^*$ and ν is an unrestricted (real or complex) number, let

$$\Sigma\mathcal{S}^*_{q,\lambda,\nu}[b; A, B] := \{f \in \Sigma : \mathcal{L}_{q,\nu} f \in \Sigma\mathcal{S}^*_{q,\lambda}[b; A, B]\},$$

and

$$\Sigma\mathcal{K}_{q,\lambda,\nu}[b; A, B] := \{f \in \Sigma : \mathcal{L}_{q,\nu} f \in \Sigma\mathcal{K}_{q,\lambda}[b; A, B]\}.$$

It is easy to show that

$$f \in \Sigma\mathcal{K}_{q,\lambda,\nu}[b; A, B] \Leftrightarrow -qzD_q f \in \Sigma\mathcal{S}^*_{q,\lambda,\nu}[b; A, B].$$

We note that: (i) $\Sigma\mathcal{S}^*_{q,0,\nu}[b; A, B] =: \Sigma\mathcal{S}^*_{q,\nu}[b; A, B]$ and $\Sigma\mathcal{K}_{q,0,\nu}[b; A, B] =: \Sigma\mathcal{K}_{q,\nu}[b; A, B]$ (see Mostafa et al. [10]);

(ii) $\lim_{q\to 1^-} \Sigma\mathcal{S}^*_{q,\lambda,\nu}[b; A, B] =: \Sigma\mathcal{S}^*_{\lambda,\nu}[b; A, B]$ and $\lim_{q\to 1^-} \Sigma\mathcal{K}_{q,\lambda,\nu}[b; A, B] =: \Sigma\mathcal{K}_{\lambda,\nu}[b; A, B]$ (see Aouf et al. [11]).

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $0 < q < 1$, $0 \le \lambda < 1$, $-1 \le B < A \le 1$, $b \in \mathbb{C}^*$ and ν is an unrestricted (real or complex) number.

Theorem 2.1. *If $f \in \Sigma$, then $f \in \Sigma\mathcal{S}^*_{q,\lambda}[b; A, B]$ if and only if*

$$z\left[f(z) * \frac{1 + \left[\left(1-\frac{\lambda}{q}\right)M(\theta) - \left(\frac{\lambda}{q}+q\right)\right]z}{z(1-z)(1-qz)}\right] \neq 0, \quad \text{for all } z \in \mathbb{U} \text{ and } \theta \in [0, 2\pi), \tag{2.1}$$

where

$$M(\theta) = M^{b;A,B}_q(\theta) := \frac{e^{-i\theta} + B}{(A-B)qb}. \tag{2.2}$$

Proof. It is easy to verify that for any $f \in \Sigma$ the next relations hold:

$$f(z) * \frac{1}{z(1-z)} = f(z), \tag{2.3}$$

and

$$f(z) * \left[\frac{1}{z(1-z)(1-qz)} - \frac{q+1}{(1-z)(1-qz)}\right] = -qzD_q f(z). \quad (2.4)$$

First, if $f \in \Sigma\mathcal{S}^*_{q,\lambda}[b;A,B]$, in order to prove that (2.1) holds we will write (1.3) by using the definition of the subordination, that is

$$-\frac{qzD_q f(z)}{(1-\frac{\lambda}{q})f(z) - \lambda zD_q f(z)} = \frac{1+[B+(A-B)qb]w(z)}{1+Bw(z)},$$

where w is a Schwarz function, hence

$$z\left\{-qzD_q f(z)(1+Be^{i\theta}) - [1+[B+(A-B)qb]e^{i\theta}] \cdot \left[\left(1-\frac{\lambda}{q}\right)f(z) - \lambda zD_q f(z)\right]\right\} \neq 0, \quad (2.5)$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$. Using (2.3) and (2.4), the relation (2.5) may be written as

$$z\left\{(1+Be^{i\theta})\left(f(z) * \left[\frac{1}{z(1-z)(1-qz)} - \frac{q+1}{(1-z)(1-qz)}\right]\right) - [1+[B+(A-B)qb]e^{i\theta}] \cdot \left[\left(1-\frac{\lambda}{q}\right)\left(f(z) * \frac{1}{z(1-z)}\right) - \frac{\lambda}{q}\left(f(z) * \left[\frac{1}{z(1-z)(1-qz)} - \frac{q+1}{(1-z)(1-qz)}\right]\right)\right]\right\} \neq 0,$$

which is equivalent to

$$z\left[f(z) * \frac{1 + \frac{(1+Be^{i\theta})(1-\frac{\lambda}{q})z}{(A-B)qbe^{i\theta}} - \frac{(\frac{\lambda}{q}+q)(A-B)qbe^{i\theta}z}{(A-B)qbe^{i\theta}}}{z(1-z)(1-qz)}[-(A-B)qbe^{i\theta}]\right] \neq 0,$$

or

$$z\left[f(z) * \frac{1 + [\frac{(1-\frac{\lambda}{q})(e^{-i\theta}+B)}{(A-B)qb} - (\frac{\lambda}{q}+q)]z}{z(1-z)(1-qz)}\right] \neq 0, \; z \in \mathbb{U}, \; \theta \in [0, 2\pi),$$

and thus the first part of Theorem 2.1 was proved.

Reversely, suppose that $f \in \Sigma$ satisfy the condition (2.1). Like it was previously shown, the assumption (2.1) is equivalent to (2.5), hence

$$-\frac{qzD_q f(z)}{(1-\frac{\lambda}{q})f(z) - \lambda zD_q f(z)} \neq \frac{1+[B+(A-B)qb]e^{i\theta}}{1+Be^{i\theta}},$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$. (2.6)

Denoting

$$\varphi(z) = -\frac{qzD_q f(z)}{(1-\frac{\lambda}{q})f(z) - \lambda zD_q f(z)} \quad \text{and}$$

$$\psi(z) = \frac{1+[B+(A-B)qb]z}{1+Bz},$$

the relation (2.6) means that $\varphi(\mathbb{U}) \cap \psi(\partial\mathbb{U}) = \emptyset$. Thus, the simply connected domain is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. Therefore, using the fact that $\varphi(0) = \psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$, which implies that $f \in \Sigma\mathcal{S}^*_{q,\lambda}[b;A,B]$. Thus, the proof of Theorem 2.1 is completed. □

Remarks 2.1.

(i) Taking $q \to 1^-$ and $\lambda = 0$ in Theorem 2.1 we obtain the result of Bulboacă et al. [12, Theorem 1];

(ii) Putting $q \to 1^-$, $b = 1$, $\lambda = 0$ and $e^{i\theta} = x$ in the above theorem we obtain the result due to Ponnusamy [16, Theorem 2.1];

(iii) Considering $q \to 1^-$, $\lambda = 0$, $b = (1-\alpha)e^{-i\mu}\cos\mu$ ($\mu \in \mathbb{R}, |\mu| < \frac{\pi}{2}, 0 \le \alpha < 1$), $A = 1$, $B = -1$ and $e^{i\theta} = x$ in Theorem 2.1 we obtain the result of Ravichandran et al. (see [14, Theorem 1.2] with $p = 1$);

(iv) If we take $q \to 1^-$, the above theorem reduces to the result from [5, Theorem 4].

Theorem 2.2. *If $f \in \Sigma$, then $f \in \Sigma\mathcal{K}_{q,\lambda}[b;A,B]$ if and only if*

$$z\left[f(z) * \frac{1-(1+q+q^2)z - (q+1)[(1-\frac{\lambda}{q})M(\theta) - (\frac{\lambda}{q}+q)]qz^2}{z(1-z)(1-qz)(1-q^2z)}\right] \neq 0, \quad (2.7)$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$, where $M(\theta)$ is given by (2.2).

Proof. From (1.5) it follows that $f \in \Sigma\mathcal{K}_{q,\lambda}[b;A,B]$ if and only if $\Phi_q(z) := -qzD_q f(z) \in \Sigma\mathcal{S}^*_{q,\lambda}[b;A,B]$. Then, according to Theorem 2.1, the function Φ_q belongs to $\Sigma\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$z[\Phi_q(z) * g(z)] \neq 0, \text{ for all } z \in \mathbb{U} \text{ and } \theta \in [0, 2\pi), \quad (2.8)$$

where

$$g(z) = \frac{1+[(1-\frac{\lambda}{q})M(\theta) - (\frac{\lambda}{q}+q)]z}{z(1-z)(1-qz)}.$$

A simple computation shows that

$$D_q g(z) = \frac{g(qz)-g(z)}{(q-1)z} = \frac{-1+(1+q+q^2)z + qz^2(q+1)[(1-\frac{\lambda}{q})M(\theta) - (\frac{\lambda}{q}+q)]}{z(qz)(1-z)(1-qz)(1-q^2z)},$$

and therefore

$$-qzD_q g(z) = \frac{1-(1+q+q^2)z - (q+1)[(1-\frac{\lambda}{q})M(\theta) - (\frac{\lambda}{q}+q)]qz^2}{z(1-z)(1-qz)(1-q^2z)}.$$

Using the above relation and the identity

$$[-qzD_q f(z)] * g(z) = f(z) * [-qzD_q g(z)],$$

it is easy to check that (2.8) is equivalent to (2.7). □

Remarks 2.2.

(i) Putting $q \to 1^-$ and $\lambda = 0$ in Theorem 2.2 we obtain the result of Bulboacă et al. [12, Theorem 2];

(ii) Taking $q \to 1^-$, $b = 1$, $\lambda = 0$ and $e^{i\theta} = x$ in the above theorem we obtain the result due to Ponnusamy [16, Theorem 2.2];

(iii) If we take $q \to 1^-$, this theorem reduces to the result from [11, Theorem 6].

Theorem 2.3. *If $f \in \Sigma$ has the form (1.1), then $f \in \Sigma\mathcal{S}^*_{q,\lambda,\nu}[b;A,B]$ if and only if*

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k(1-q)^{2k}}{4^k (q;q)_k (q^{\nu+1};q)_k} \cdot \frac{(1-\frac{\lambda}{q}[k]_q)(A-B)qb + (1-\frac{\lambda}{q})[k]_q(e^{-i\theta}+B)}{(A-B)qb} a_k z^k \neq 0, \quad (2.9)$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$.

Proof. If $f \in \Sigma$, then from Theorem 2.1 we have $f \in \Sigma\mathcal{S}^*_{q,\lambda,\nu}[b;A,B]$ if and only if

$$z\left[(\mathcal{L}_{q,\nu} f)(z) * \frac{1+[(1-\frac{\lambda}{q})M(\theta) - (\frac{\lambda}{q}+q)]z}{z(1-z)(1-qz)}\right] \neq 0, \quad (2.10)$$

for all $z \in \mathbb{U}$ and all $\theta \in [0, 2\pi)$, where $M(\theta)$ is given by (2.2). Since

$$\frac{1}{z(1-z)(1-qz)} = \frac{1}{z} + (1+q) + \left(1+q+q^2\right)z + (1+q+q^2+q^3)z^2 + \dots,\ z \in \mathbb{U}^*, \tag{2.11}$$

it follows that

$$\frac{1+[(1-\frac{\lambda}{q})M(\theta) - (\frac{\lambda}{q}+q)]z}{z(1-z)(1-qz)} = \frac{1}{z} + \sum_{k=1}^{\infty}\left\{1+\left[\left(1-\frac{\lambda}{q}\right)M(\theta) - \frac{\lambda}{q}\right][k]_q\right\}z^{k-1},$$

and we may easily check that (2.10) is equivalent to (2.9). □

Theorem 2.4. *If $f \in \Sigma$ has the form (1.1), then $f \in \Sigma\mathcal{K}_{q,\lambda,\nu}[b; A, B]$ if and only if*

$$1 - q\sum_{k=1}^{\infty}\frac{(-1)^k(1-q)^{2k}[k-1]_q}{4^k\,(q;q)_k(q^{\nu+1};q)_k} \cdot \frac{(1-\frac{\lambda}{q}[k]_q)(A-B)qb + (1-\frac{\lambda}{q})[k]_q(e^{-i\theta}+B)}{(A-B)qb}\,a_k z^k \neq 0, \tag{2.12}$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$.

Proof. If $f \in \Sigma$, then from Theorem 2.2 we have $f \in \Sigma\mathcal{K}_{q,\lambda,\nu}[b; A, B]$ if and only if

$$z\left[(\mathcal{L}_{q,\nu}f)(z) * \frac{1-(1+q+q^2)z-(q+1)[(1-\frac{\lambda}{q})M(\theta)-(\frac{\lambda}{q}+q)]qz^2}{z(1-z)(1-qz)(1-q^2z)}\right] \neq 0, \tag{2.13}$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$, where $M(\theta)$ is given by (2.2). Using the power series expansion (2.11) we get

$$\frac{1}{z(1-z)(1-qz)} \cdot \frac{1}{1-q^2z} = \frac{1}{z} + (1+q+q^2) + (1+q+2q^2+q^3+q^4)z + (1+q+2q^2+2q^3+2q^4+q^5+q^6)z^2 + \dots, z \in \mathbb{U}^*,$$

hence

$$\frac{1-(1+q+q^2)z-(q+1)[(1-\frac{\lambda}{q})M(\theta)-(\frac{\lambda}{q}+q)]qz^2}{z(1-z)(1-qz)(1-q^2z)} = \frac{1}{z} - q\sum_{k=1}^{\infty}[k-1]_q\left\{1+\left[\left(1-\frac{\lambda}{q}\right)M(\theta)-\frac{\lambda}{q}\right][k]_q\right\}z^{k-1},\ z \in \mathbb{U}^*.$$

Now, we may easily check that (2.13) is equivalent to (2.12), which proves our result. □

Unless otherwise mentioned, we assume throughout the remainder of this section that ν is a real number, with $\nu > -1$.

Theorem 2.5. *If $f \in \Sigma$ has the form (1.1) and satisfies the inequality*

$$\sum_{k=1}^{\infty}\frac{(1-q)^{2k}[(A-B)q|b(1-\frac{\lambda}{q}[k]_q)| + |1-\frac{\lambda}{q}|(1+|B|)[k]_q]}{4^k\,(q;q)_k\,(q^{\nu+1};q)_k}|a_k| < (A-B)q|b|, \tag{2.14}$$

then $f \in \Sigma\mathcal{S}^*_{q,\lambda,\nu}[b; A, B]$.

Proof. If $f \in \Sigma$ has the form (1.1), assuming that (2.14) holds we obtain

$$\left|1+\sum_{k=1}^{\infty}\frac{(-1)^k(1-q)^{2k}}{4^k\,(q;q)_k\,(q^{\nu+1};q)_k} \cdot \frac{(1-\frac{\lambda}{q}[k]_q)(A-B)qb + (1-\frac{\lambda}{q})[k]_q(e^{-i\theta}+B)}{(A-B)qb}\,a_k z^k\right|$$

$$\geq 1 - \left|\sum_{k=1}^{\infty}\frac{(-1)^k(1-q)^{2k}}{4^k\,(q;q)_k\,(q^{\nu+1};q)_k} \cdot \frac{(1-\frac{\lambda}{q}[k]_q)(A-B)qb + (1-\frac{\lambda}{q})[k]_q(e^{-i\theta}+B)}{(A-B)qb}\,a_k z^k\right|$$

$$\geq 1 - \sum_{k=1}^{\infty}\frac{(1-q)^{2k}}{4^k\,(q;q)_k\,(q^{\nu+1};q)_k} \cdot \frac{(A-B)q|b(1-\frac{\lambda}{q}[k]_q)| + |1-\frac{\lambda}{q}|(1+|B|)[k]_q}{(A-B)q|b|}|a_k| > 0,$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$. Thus, the relation (2.9) holds and our conclusion follows from Theorem 2.3. □

Using similar arguments to those in the proof of Theorem 2.5, we obtain the following theorem:

Theorem 2.6. *If $f \in \Sigma$ has the form (1.1) and satisfies the inequality*

$$\sum_{k=1}^{\infty}\frac{(1-q)^{2k}[k-1]_q[(A-B)q|b(1-\frac{\lambda}{q}[k]_q)| + |1-\frac{\lambda}{q}|(1+|B|)[k]_q]}{4^k\,(q;q)_k\,(q^{\nu+1};q)_k}|a_k| < (A-B)q|b|,$$

then $f \in \Sigma\mathcal{K}_{q,\lambda,\nu}[b; A, B]$.

Remarks 2.3.

(i) Putting $q \to 1^-$ in Theorem 2.3 we obtain the result of Aouf et al. [11, Theorem 8];
(ii) Taking $q \to 1^-$ in Theorem 2.4 we obtain the result due to Aouf et al. [11, Theorem 9];
(iii) If we take $q \to 1^-$, Theorem 2.5 reduces to the result of [11, Theorem 10];
(iv) Considering $q \to 1^-$ in the last theorem we obtain the result from [11, Theorem 11];
(v) Taking $\lambda = 0$ in our results, we obtain the results due to Mostafa et al. [10].

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