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Original Article

On some generalizations of certain nonlinear retarded integral inequalities for Volterra–Fredholm integral equations and their applications in delay differential equations

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ABSTRACT

By some new analysis techniques, we generalize the results presented by Pachpatte in [Integral and Finite Difference Inequalities and Applications, volume 205, Elsevier, 2006] and and by S. D. Kendre in [Some nonlinear integral inequalities for Volterra–Fredholm integral equations, Adv. Inequal. App, 2014:Article 21, 2014.] to nonlinear retarded inequalities, and investigate also some new forms. Some applications are also presented in order to illustrate the usefulness of some of our results.

1. Introduction

We know that Gronwall-Bellman inequality [1,2], play a considerable role in the study of qualitative properties of the solutions of some certain differential equations see, (e.g., [3–7]). Many others results on its generalizations may be seen in [8–14].

In [1] Gronwall introduced the following inequality which is one of the best widely and known in the study of many qualitative and quantitative properties of solutions of differential equations.

Theorem 1.1. [1] *Let u be a continuous function defined on the interval $J=[\alpha,\alpha+h]$ and*

$$0\le u(t)\le\int_\alpha^t[bu(s)+a]ds,\qquad\forall t\in J,\tag{1.1}$$

where a, b are nonnegative constants. Then

$$0\le u(t)\le ah\exp[bh],\qquad\forall t\in J.\tag{1.2}$$

In [2] Bellman proved the following inequality:

Theorem 1.2. [2] *Let u and f be continuous and nonnegative functions defined on $J=[0,h]$, and let c be a nonnegative constant. Then the inequality*

$$u(t)\le c+\int_0^t f(s)u(s)ds,\qquad\forall t\in J,\tag{1.3}$$

implies that

$$u(t)\le c\exp\left(\int_0^t f(s)ds\right),\qquad\forall t\in J.\tag{1.4}$$

Lipovan [3], studied the retarded Gronwall-type inequality in the following theorem.

Theorem 1.3. [3] *Let $u,\ h\in\mathcal{C}([t_0,T_0],\mathbb{R}_+)$, $\alpha\in\mathcal{C}([t_0,T_0],[t_0,T_0])$, with $\alpha(t)\le t$ on $[t_0,T_0]$, and k be constant. Then the inequality*

$$u(t)\le k+\int_{\alpha(t_0)}^{\alpha(t)}h(s)u(s)ds,\quad t_0<t<T_0,$$

implies that

$$u(t)\le k\ \exp\left(\int_{\alpha(t_0)}^{\alpha(t)}h(s)ds\right),\quad t_0<t<T_0.$$

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In this paper, several retarded nonlinear integral inequalities are proved, however, the inequalities presented in [15] and [16] is not directly applicable in the study of certain nonlinear retarded differential and integral equations. In some problems, it is desirable to establish some new inequalities of the above type where non-retarded case t in [15] and [16] is replaced by retarded case $\alpha(t)$ and the linear case $u(t)$ in integral functions in [15] is replaced by the nonlinear case $u^p(t)$, and also investigate a slight generalization of the celebrated Gronwall–Bellman type inequalities which can be used more effectively in the study the qualitative behavior of the solutions of certain classes of nonlinear retarded differential and integral equations. Some applications of some of our results are also introduced to illustrate the benefits of this work.

We start by giving the following inequality which used in the proofs of the basic theorems of our results:

Lemma 1.4. *[17] If $x \geq 0$, $p \geq 1$, then*

$$x^{\frac{1}{p}} \leq m_1 x + m_2, \tag{1.5}$$

where $m_1 = \frac{1}{p}K^{\frac{1-p}{p}}$ and $m_2 = \frac{p-1}{p}K^{\frac{1}{p}}$, $K > 0$.

Now we are ready to state and proof our results:

2. Main results

Throughout this article, $\mathbb{R}$ denoted the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $J_1 = [a, b]$ and $'$ denotes the first derivative. $C(J_1, \mathbb{R}_+)$ denotes the set of all nonnegative real-valued continuous functions from J_1 into $\mathbb{R}_+$ and $C^1(J_1, \mathbb{R}_+)$ denotes the set of all nonnegative real-valued continuously differentiable functions from J_1 into $\mathbb{R}_+$.

Theorem 2.1. *Let $\omega, g, f \in C(J_1, \mathbb{R}_+)$, $c(t)$, $\alpha(t) \in C^1(J_1, \mathbb{R}_+)$, with $\alpha(t) \leq t$, $\alpha(a) = a$ and $p \geq 1$ be a constant. And*

$$\omega^p(t) \leq c(t) + \int_a^{\alpha(t)} g(s)\omega(s)ds + \int_a^b f(s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.1}$$

If

$$E_1 = \int_a^b f(s)\exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]ds < 1,$$

then

$$\begin{aligned}\omega^p(t) \leq &\Big\{c(a) + \int_a^b f(s)\Big(\exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]\int_a^s\Big[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}\\ &+ m_2 g(r)\Big]\exp\Big[-\int_a^r m_1 g(\tau)d\tau\Big]dr\Big)ds\Big\}\Big\{1 - E_1\Big\}^{-1}\\ &\times \exp\Big[\int_a^{\alpha(t)} m_1 g(s)ds\Big] + \exp\Big[\int_a^{\alpha(t)} m_1 g(s)ds\Big]\\ &\times \int_a^{\alpha(t)}\Big[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2 g(s)\Big]\exp\Big[-\int_a^s m_1 g(r)dr\Big]ds,\end{aligned} \tag{2.2}$$

for all $t \in J_1$. Where m_1, m_2 are defined as in Lemma 1.4.

Proof. Define a function $z_1(t)$ by:

$$z_1(t) = c(t) + \int_a^{\alpha(t)} g(s)\omega(s)ds + \int_a^b f(s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.3}$$

We observe, $z_1(t) \geq 0$ nondecreasing on J_1 with

$$z_1(a) = c(a) + \int_a^b f(s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.4}$$

Then from (2.1) and (2.3) and by using the monotonicity of $z_1(t)$, we get

$$\omega(t) \leq z_1^{\frac{1}{p}}(t), \omega(\alpha(t)) \leq z_1^{\frac{1}{p}}(\alpha(t)) \leq z_1^{\frac{1}{p}}(t), \qquad \forall t \in J_1. \tag{2.5}$$

By differentiating (2.3) and using (2.5), we obtain

$$\begin{aligned}z_1'(t) &= c'(t) + \Big[g(\alpha(t))\omega(\alpha(t))\Big]\alpha'(t)\\ &\leq c'(t) + \Big[g(\alpha(t))z_1^{\frac{1}{p}}(t)\Big]\alpha'(t), \qquad \forall t \in J_1,\end{aligned} \tag{2.6}$$

from (2.6) and using Lemma (1.4), we get

$$z_1'(t) \leq c'(t) + \Big[m_1 g(\alpha(t))z_1(t) + m_2 g(\alpha(t))\Big]\alpha'(t), \forall t \in J_1. \tag{2.7}$$

The inequality (2.7) can be written as the following

$$z_1'(t) - m_1 g(\alpha(t))\alpha'(t)z_1(t) \leq c'(t) + m_2 g(\alpha(t))\alpha'(t), \qquad \forall t \in J_1. \tag{2.8}$$

The inequality (2.8) gives us the following estimation

$$\begin{aligned}z_1(t) \leq\ & z_1(a)\exp\Big[\int_a^{\alpha(t)} m_1 g(s)ds\Big] + \exp\Big[\int_a^{\alpha(t)} m_1 g(s)ds\Big]\\ &\times \int_a^{\alpha(t)} \frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2 g(s)\Big]\exp\Big[-\int_a^s m_1 g(r)dr\Big]ds,\end{aligned} \tag{2.9}$$

for all $t \in J_1$. Using $\omega^p(t) \leq z_1(t)$ in (2.9), reduces

$$\begin{aligned}\omega^p(t) \leq\ & z_1(a)\exp\Big[\int_a^{\alpha(t)} m_1 g(s)ds\Big] + \exp\Big[\int_a^{\alpha(t)} m_1 g(s)ds\Big]\\ &\times \int_a^{\alpha(t)}\Big[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2 g(s)\Big]\exp\Big[-\int_a^s m_1 g(r)dr\Big]ds,\end{aligned} \tag{2.10}$$

for all $t \in J_1$. From (2.4) and (2.10), we get

$$\begin{aligned}z_1(a) \leq\ & c(a) + \int_a^b f(s)\Big(z_1(a)\exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]\\ &+ \exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]\\ &\times \int_a^s\Big[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2 g(r)\Big]\exp\Big[-\int_a^r m_1 g(\tau)d\tau\Big]dr\Big)ds\\ \leq\ & c(a) + z_1(a)\int_a^b f(s)\exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]ds\\ &+ \int_a^b f(s)\Big(\exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]\int_a^s\Big[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2 g(r)\Big]\\ &\times \exp\Big[-\int_a^r m_1 g(\tau)d\tau\Big]dr\Big)ds\\ \leq\ & \Big\{c(a) + \int_a^b f(s)\Big(\exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]\\ &\times \int_a^s\Big[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2 g(r)\Big]\\ &\times \exp\Big[-\int_a^r m_1 g(\tau)d\tau\Big]dr\Big)ds\Big\}\\ &\times \Big\{1 - \int_a^b f(s)\exp\Big[\int_a^s m_1 g(\tau)d\tau\Big]ds\Big\}^{-1},\end{aligned} \tag{2.11}$$

for all $t \in J_1$. We obtain the required inequality (2.2) from (2.10) and (2.11). This completes the proof. □

Remark 2.1. If we put $\alpha(t) = t$, then Theorem 2.1 reduces to Theorem 2.1 in [16].

Remark 2.2. If we put $\alpha(t) = t$ and $p = 1$, then Theorem 2.1 reduces to Theorem 1.5.1 in [15].

Theorem 2.2. *Let $\omega, f \in \mathcal{C}(J_1, \mathbb{R}_+)$, $c(t)$, $\alpha(t) \in \mathcal{C}^1(J_1, \mathbb{R}_+)$, with $\alpha(t) \le t$, $\alpha(a) = a$, and $k(t,s)$, $\frac{\partial}{\partial t}k(t,s) \in \mathcal{C}(J_1 \times J_1, \mathbb{R}_+)$ and $p \ge 1$ be a constant. And*

$$\omega^p(t) \le c(t) + \int_a^{\alpha(t)} k(t,s)\omega(s)ds + \int_a^b f(s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.12}$$

If

$$E_2 = \int_a^b f(s)\exp\left[\int_a^s m_1\eta(\tau)d\tau\right]ds < 1,$$

then

$$\begin{aligned}\omega^p(t) \le &\left\{c(a) + \int_a^b f(s)\left(\exp\left[\int_a^s m_1\eta(\tau)d\tau\right]\right.\right.\\ &\times \int_a^s\left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta(r)\right]\\ &\left.\left.\times \exp\left[-\int_a^r m_1\eta(\tau)d\tau\right]dr\right)ds\right\}\left\{1 - E_1\right\}^{-1}\\ &\times \exp\left[\int_a^{\alpha(t)} m_1\eta(s)ds\right] + \exp\left[\int_a^{\alpha(t)} m_1\eta(s)ds\right]\\ &\times \int_a^{\alpha(t)}\left[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2\eta(s)\right]\exp\left[-\int_a^s m_1\eta(r)dr\right]ds,\end{aligned} \tag{2.13}$$

for all $t \in J_1$. Where m_1, m_2 are defined as in Lemma 1.4, and

$$\eta(t) = k(t,\alpha(t))\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)ds, \qquad \forall t \in J_1.$$

Proof. Define a function $z_2(t)$ by:

$$z_2(t) = c(t) + \int_a^{\alpha(t)} k(t,s)\omega(s)ds + \int_a^b f(s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.14}$$

We observe, $z_2(t) \ge 0$ nondecreasing on J_1 with

$$z_2(a) = c(a) + \int_a^b f(s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.15}$$

Then from (2.12) and (2.14) and using the monotonicity of $z_2(t)$, we have

$$\omega(t) \le z_2^{\frac{1}{p}}(t), \omega(\alpha(t)) \le z_2^{\frac{1}{p}}(\alpha(t)) \le z_2^{\frac{1}{p}}(t), \qquad \forall t \in J_1. \tag{2.16}$$

By differentiating (2.14) and using (2.16), we obtain

$$\begin{aligned}z_2'(t) &= c'(t) + \Big[k(t,\alpha(t))\omega(\alpha(t))\Big]\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\omega(s)ds\\ &\le c'(t) + \Big[k(t,\alpha(t))z_2^{\frac{1}{p}}(t)\Big]\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)z_2^{\frac{1}{p}}(s)ds\\ &\le c'(t) + \left[k(t,\alpha(t))\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)ds\right]z_2^{\frac{1}{p}}(t)\\ &\le c'(t) + \eta(t)z_2^{\frac{1}{p}}(t), \qquad \forall t \in J_1,\end{aligned} \tag{2.17}$$

from (2.17) and using Lemma (1.4), we get

$$z_2'(t) \le c'(t) + m_1\eta(t)z_2(t) + m_2\eta(t), \qquad \forall t \in J_1. \tag{2.18}$$

The inequality (2.18) can be written as the following

$$z_2'(t) - m_1\eta(t)z_2(t) \le c'(t) + m_2\eta(t), \qquad \forall t \in J_1. \tag{2.19}$$

The inequality (2.19) gives us the following estimation

$$\begin{aligned}z_2(t) \le\ & z_2(a)\exp\left[\int_a^{\alpha(t)} m_1\eta(s)ds\right] + \exp\left[\int_a^{\alpha(t)} m_1\eta(s)ds\right]\\ &\times \int_a^{\alpha(t)} \frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2\eta(s)\Big]\exp\left[-\int_a^s m_1\eta(r)dr\right]ds,\end{aligned} \tag{2.20}$$

for all $t \in J_1$. Using $\omega^p(t) \le z_2(t)$ in (2.20), gives

$$\begin{aligned}\omega^p(t) \le\ & z_2(a)\exp\left[\int_a^{\alpha(t)} m_1\eta(s)ds\right] + \exp\left[\int_a^{\alpha(t)} m_1\eta(s)ds\right]\\ &\times \int_a^{\alpha(t)}\left[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2\eta(s)\right]\exp\left[-\int_a^s m_1\eta(r)dr\right]ds,\end{aligned} \tag{2.21}$$

for all $t \in J_1$. From (2.15) and (2.21), we get

$$\begin{aligned}z_2(a) \le\ & c(a) + \int_a^b f(s)\left(z_2(a)\exp\left[\int_a^s m_1\eta(\tau)d\tau\right]\right.\\ &+ \exp\left[\int_a^s m_1\eta(\tau)d\tau\right]\int_a^s\left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta(r)\right]\\ &\left.\times \exp\left[-\int_a^r m_1\eta(\tau)d\tau\right]dr\right)ds\\ \le\ & c(a) + z_2(a)\int_a^b f(s)\exp\left[\int_a^s m_1\eta(\tau)d\tau\right]ds\\ &+ \int_a^b f(s)\left(\exp\left[\int_a^s m_1\eta(\tau)d\tau\right]\int_a^s\left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta(r)\right]\right.\\ &\left.\times \exp\left[-\int_a^r m_1\eta(\tau)d\tau\right]dr\right)ds\\ \le\ & \left\{c(a) + \int_a^b f(s)\left(\exp\left[\int_a^s m_1\eta(\tau)d\tau\right]\right.\right.\\ &\int_a^s\left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta(r)\right]\\ &\left.\left.\times \exp\left[-\int_a^r m_1\eta(\tau)d\tau\right]dr\right)ds\right\}\\ &\times\left\{1 - \int_a^b f(s)\exp\left[\int_a^s m_1\eta(\tau)d\tau\right]ds\right\}^{-1}, \forall t \in J_1.\end{aligned} \tag{2.22}$$

We obtain the required inequality (2.13) from (2.21) and (2.22). This completes the proof. □

Remark 2.3. If we put $\alpha(t) = t$, then Theorem 2.2 reduces to Theorem 2.3 in [16].

Remark 2.4. If we put $\alpha(t) = t$ and $p = 1$, then Theorem 2.2 reduces to Theorem 1.5.2 part (b_1) in [15].

Theorem 2.3. *Let $\omega \in \mathcal{C}(J_1, \mathbb{R}_+)$, $c(t)$, $\alpha(t) \in \mathcal{C}^1(J_1, \mathbb{R}_+)$, with $\alpha(t) \le t$, $\alpha(a) = a$, and $k_1(t,s), k_2(t,s)$, $\frac{\partial}{\partial t}k_1(t,s)$, $\frac{\partial}{\partial t}k_2(t,s) \in \mathcal{C}(J_1 \times J_1, \mathbb{R}_+)$ and $p \ge 1$ be a constant. And*

$$\omega^p(t) \le c(t) + \int_a^{\alpha(t)} k_1(t,s)\omega(s)ds + \int_a^b k_2(t,s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.23}$$

If

$$E_3 = \int_a^b k_2(a,s)\exp\left[\int_a^s (m_1\eta_1(\tau) + \eta_2(\tau))d\tau\right]ds < 1,$$

then

$$\omega^p(t) \le \left\{c(a) + \int_a^b k_2(a,s)\left(\exp\left[\int_a^s (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]\right.\right.$$
$$\times \int_a^s \left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta(r)\right]$$
$$\left.\left.\times \exp\left[-\int_a^r (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]dr\right)ds\right\}\left\{1-E_3\right\}^{-1}$$
$$\times \exp\left[\int_a^{\alpha(t)} (m_1\eta_1(s)+\eta_2(s))ds\right]$$
$$+\exp\left[\int_a^{\alpha(t)} (m_1\eta_1(s)+\eta_2(s))ds\right]$$
$$\times \int_a^{\alpha(t)} \left[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2\eta_1(s)\right]$$
$$\times \exp\left[-\int_a^s (m_1\eta_1(r)+\eta_2(r))dr\right]ds, \tag{2.24}$$

for all $t \in J_1$, *where* m_1, m_2 *are defined as in Lemma 1.4,*

$$\eta_1 = k_1(t,\alpha(t))\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial}{\partial t} k_1(t,s)ds, \qquad \forall t \in J_1.$$

And

$$\eta_2(t) = \int_a^b \frac{\partial}{\partial t} k_2(t,s)ds, \qquad \forall t \in J_1.$$

Proof. Define a function $z_3(t)$ by:

$$z_3(t) = c(t) + \int_a^{\alpha(t)} k_1(t,s)\omega(s)ds + \int_a^b k_2(t,s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.25}$$

We observe, $z_3(t) \ge 0$ nondecreasing on J_1 with

$$z_3(a) = c(a) + \int_a^b k_2(a,s)\omega^p(s)ds, \quad \forall t \in J_1. \tag{2.26}$$

Then from (2.23) and (2.25) and using the monotonicity of $z_3(t)$, we get

$$\omega(t) \le z_3^{\frac{1}{p}}(t), \omega(\alpha(t)) \le z_3^{\frac{1}{p}}(\alpha(t)) \le z_3^{\frac{1}{p}}(t), \qquad \forall t \in J_1. \tag{2.27}$$

By differentiating (2.25) and using (2.27), we obtain

$$\begin{aligned} z_3'(t) &= c'(t) + \Big[k_1(t,\alpha(t))\omega(\alpha(t))\Big]\alpha'(t) \\ &\quad + \int_a^{\alpha(t)} \frac{\partial}{\partial t}k_1(t,s)\omega(s)ds + \int_a^b \frac{\partial}{\partial t}k_2(t,s)\omega^p(s)ds \\ &\le c'(t) + \Big[k_1(t,\alpha(t))z_3^{\frac{1}{p}}(t)\Big]\alpha'(t) \\ &\quad + \int_a^{\alpha(t)} \frac{\partial}{\partial t}k_1(t,s)z_3^{\frac{1}{p}}(s)ds \\ &\quad + \int_a^b \frac{\partial}{\partial t}k_2(t,s)z_3(s)ds \\ &\le c'(t) + \left[k_1(t,\alpha(t))\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial}{\partial t}k_1(t,s)ds\right]z_3^{\frac{1}{p}}(t) \\ &\quad + \int_a^b \frac{\partial}{\partial t}k_2(t,s)z_3(s)ds \\ &\le c'(t) + \eta_1(t)z_3^{\frac{1}{p}}(t) + \eta_2(t)z_3(t), \qquad \forall t \in J_1, \end{aligned} \tag{2.28}$$

from (2.28) and using Lemma (1.4), we get

$$\begin{aligned} z_3'(t) &\le c'(t) + m_1\eta_1(t)z_3(t) + m_2\eta_1(t) \\ &\quad + \eta_2(t)z_3(t), \qquad \forall t \in J_1. \end{aligned} \tag{2.29}$$

The inequality (2.29) can be written as the following

$$z_3'(t) - (m_1\eta_1(t) + \eta_2(t))z_3(t) \le c'(t) + m_2\eta_1(t), \qquad \forall t \in J_1. \tag{2.30}$$

The inequality (2.29) gives us the following estimation

$$z_3(t) \le z_3(a)\exp\left[\int_a^{\alpha(t)} (m_1\eta_1(s)+\eta_2(s))ds\right]$$
$$+\exp\left[\int_a^{\alpha(t)} (m_1\eta_1(s)+\eta_2(s))ds\right]$$
$$\times \int_a^{\alpha(t)} \left[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2\eta_1(s)\right]$$
$$\times \exp\left[-\int_a^s (m_1\eta_1(r)+\eta_2(r))dr\right]ds, \tag{2.31}$$

for all $t \in J_1$. Using $\omega^p(t) \le z(t)$ in (2.31), gives

$$\omega^p(t) \le z_3(a)\exp\left[\int_a^{\alpha(t)} (m_1\eta_1(s)+\eta_2(s))ds\right]$$
$$+\exp\left[\int_a^{\alpha(t)} (m_1\eta_1(s)+\eta_2(s))ds\right]$$
$$\times \int_a^{\alpha(t)} \left[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} + m_2\eta_1(s)\right]$$
$$\times \exp\left[-\int_a^s (m_1\eta_1(r)+\eta_2(r))dr\right]ds, \tag{2.32}$$

for all $t \in J_1$. From (2.26) and (2.32), we get

$$z_3(a) \le c(a) + \int_a^b k_2(a,s)\left(z_3(a)\exp\left[\int_a^s (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]\right.$$
$$+\exp\left[\int_a^s (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]$$
$$\times \int_a^s \left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta_1(r)\right]$$
$$\left.\times \exp\left[-\int_a^r (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]dr\right)ds$$
$$\le c(a) + z_3(a)\int_a^b k_2(t,s)\exp\left[\int_a^s (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]ds$$
$$+\int_a^b k_2(a,s)\left(\exp\left[\int_a^s (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]\right.$$
$$\times \int_a^s \left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta_1(r)\right]$$
$$\left.\times \exp\left[-\int_a^r (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]dr\right)ds$$
$$\le \left\{c(a) + \int_a^b k_2(a,s)\left(\exp\left[\int_a^s (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]\right.\right.$$
$$\times \int_a^s \left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta_1(r)\right]$$
$$\left.\left.\times \exp\left[-\int_a^r (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]dr\right)ds\right\}$$
$$\times\left\{1 - \int_a^b k_2(a,s)\exp\left[\int_a^s (m_1\eta_1(\tau)+\eta_2(\tau))d\tau\right]ds\right\}^{-1}, \tag{2.33}$$

for all $t \in J_1$. We obtain the required inequality (2.24) from (2.32) and (2.33). This completes the proof. □

Remark 2.5. If we put $\alpha(t) = t$, then Theorem 2.3 reduces to Theorem 2.4 in [16].

Remark 2.6. If we put $\alpha(t)=t,\ c(t)=k$ (*any constant*) and $p=1$, then Theorem 2.3 reduces to Theorem 1.5.2 part (b_2) in [15].

Theorem 2.4. *Let $\omega\in C(J_1,\mathbb{R}_+),\ c(t),\alpha(t)\in C^1(J_1,\mathbb{R}_+)$, with $\alpha(t)\le t,\ \alpha(a)=a$, and $k_1(t,s),\ k_2(t,s),\ k_3(t,s),\ \frac{\partial}{\partial t}k_1(t,s),\ \frac{\partial}{\partial t}k_2(t,s),\ \frac{\partial}{\partial t}k_3(t,s)\in C(J_1\times J_1,\mathbb{R}_+)$ and $p\ge 1$ be a constant. And*

$$\omega^p(t)\le c(t)+\int_a^{\alpha(t)}k_1(t,s)\Big[\omega(s)+\int_a^s k_2(s,\tau)\omega(\tau)d\tau\Big]ds+\int_a^b k_3(t,s)\omega^p(s)ds, \tag{2.34}$$

for all $t\in J_1$. If

$$E_4=\int_a^b k_3(a,s)\exp\Big[\int_a^s(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]ds<1,$$

then

$$\omega^p(t)\le\Big\{c(a)+\int_a^b k_3(a,s)\Big(\exp\Big[\int_a^s(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]\int_a^s\Big[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}+m_2\eta_4(r)\Big]\times\exp\Big[-\int_a^r(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]dr\Big)ds\Big\}$$
$$\times\Big\{1-E_3\Big\}^{-1}\exp\Big[\int_a^{\alpha(t)}(m_1\eta_4(s)+\eta_5(s))ds\Big]+\exp\Big[\int_a^{\alpha(t)}(m_1\eta_4(s)+\eta_5(s))ds\Big]\times\int_a^{\alpha(t)}\Big[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}+m_2\eta_4(s)\Big]\times\exp\Big[-\int_a^s(m_1\eta_4(r)+\eta_5(r))dr\Big]ds,\ \forall t\in J_1, \tag{2.35}$$

where $m_1,\ m_2$ are defined as in Lemma 1.4,

$$\eta_4(t)=k_1(t,\alpha(t))\Big[1+\int_a^{\alpha(t)}\frac{\partial}{\partial t}k_2(s,\tau)d\tau\Big]\alpha'(t)+\int_a^{\alpha(t)}\frac{\partial}{\partial t}k_1(t,s)\Big[1+\int_a^s k_2(s,\tau)d\tau\Big]ds,$$

for all $t\in J_1$, and

$$\eta_5(t)=\int_a^b\frac{\partial}{\partial t}k_3(t,s)ds,\qquad\forall t\in J_1.$$

Proof. Define a function $z_4(t)$ by:

$$z_4(t)=c(t)+\int_a^{\alpha(t)}k_1(t,s)\Big[\omega(s)+\int_a^s k_2(s,\tau)\omega(\tau)d\tau\Big]ds+\int_a^b k_3(t,s)\omega^p(s)ds,\ \forall t\in J_1. \tag{2.36}$$

We observe, $z_4(t)\ge 0$ nondecreasing on J_1 with

$$z_4(a)=c(a)+\int_a^b k_3(a,s)\omega^p(s)ds,\quad\forall t\in J_1. \tag{2.37}$$

Then from (2.34) and (2.36) and using the monotonicity of $z_4(t)$, we have

$$\omega(t)\le z_4^{\frac{1}{p}}(t),\ \omega(\alpha(t))\le z_4^{\frac{1}{p}}(\alpha(t))\le z_4^{\frac{1}{p}}(t),\qquad\forall t\in J_1. \tag{2.38}$$

By differentiating (2.36) and using (2.38), we obtain

$$z_4'(t)=c'(t)+k_1(t,\alpha(t))\Big[\omega(\alpha(t))+\int_a^{\alpha(t)}k_2(s,\tau)\omega(\tau)d\tau\Big]\alpha'(t)+\int_a^{\alpha(t)}\frac{\partial}{\partial t}k_1(t,s)\Big[\omega(s)+\int_a^s k_2(s,\tau)\omega(\tau)d\tau\Big]ds+\int_a^b\frac{\partial}{\partial t}k_3(t,s)\omega^p(s)ds$$
$$\le c'(t)+k_1(t,\alpha(t))\Big[z_4^{\frac{1}{p}}(t)+\int_a^{\alpha(t)}k_2(s,\tau)z_4^{\frac{1}{p}}(\tau)d\tau\Big]\alpha'(t)+\int_a^{\alpha(t)}\frac{\partial}{\partial t}k_1(t,s)\Big[z_4^{\frac{1}{p}}(s)+\int_a^s k_2(s,\tau)z_4^{\frac{1}{p}}(\tau)d\tau\Big]ds+\int_a^b\frac{\partial}{\partial t}k_3(t,s)z_4(s)ds$$
$$\le c'(t)+\Big[k_1(t,\alpha(t))\Big[1+\int_a^{\alpha(t)}\frac{\partial}{\partial t}k_2(s,\tau)d\tau\Big]\alpha'(t)+\int_a^{\alpha(t)}\frac{\partial}{\partial t}k_1(t,s)\Big[1+\int_a^s k_2(s,\tau)d\tau\Big]ds\Big]z_4^{\frac{1}{p}}(t)+\Big[\int_a^b\frac{\partial}{\partial t}k_3(t,s)ds\Big]z_4(t)$$
$$\le c'(t)+\eta_4(t)z_4^{\frac{1}{p}}(t)+\eta_5(t)z_4(t),\qquad\forall t\in J_1, \tag{2.39}$$

from (2.39) and using Lemma (1.4), we get

$$z_4'(t)\le c'(t)+m_1\eta_4(t)z_4(t)+m_2\eta_4(t)+\eta_5(t)z_4(t),\qquad\forall t\in J_1. \tag{2.40}$$

The inequality (2.40) can be written as the following

$$z_4'(t)-(m_1\eta_4(t)+\eta_5(t))z_4(t)\le c'(t)+m_2\eta_4(t),\ \forall t\in J_1. \tag{2.41}$$

The inequality (2.41) gives us the following estimation

$$z_4(t)\le z_4(a)\exp\Big[\int_a^{\alpha(t)}(m_1\eta_4(s)+\eta_5(s))ds\Big]+\exp\Big[\int_a^{\alpha(t)}(m_1\eta_4(s)+\eta_5(s))ds\Big]\times\int_a^{\alpha(t)}\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}+m_2\eta_4(s)\Big]\times\exp\Big[-\int_a^s(m_1\eta_4(r)+\eta_5(r))dr\Big]ds, \tag{2.42}$$

for all $t\in J_1$. Using $\omega^p(t)\le z(t)$ in (2.42), reduces

$$\omega^p(t)\le z_4(a)\exp\Big[\int_a^{\alpha(t)}(m_1\eta_4(s)+\eta_5(s))ds\Big]+\exp\Big[\int_a^{\alpha(t)}(m_1\eta_4(s)+\eta_5(s))ds\Big]\times\int_a^{\alpha(t)}\Big[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}+m_2\eta_4(s)\Big]\times\exp\Big[-\int_a^s(m_1\eta_4(r)+\eta_5(r))dr\Big]ds, \tag{2.43}$$

for all $t\in J_1$. From (2.37) and (2.43), we get

$$z_4(a)\le c(a)+\int_a^b k_3(a,s)\Big(z(a)\exp\Big[\int_a^s(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]+\exp\Big[\int_a^s(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]\int_a^s\Big[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}+m_2\eta_4(r)\Big]\times\exp\Big[-\int_a^r(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]dr\Big)ds$$
$$\le c(a)+z_4(a)\int_a^b k_3(a,s)\exp\Big[\int_a^s(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]ds+\int_a^b k_2(t,s)\Big(\exp\Big[\int_a^s(m_1\eta_4(\tau)+\eta_5(\tau))d\tau\Big]$$

$$\times \int_a^s \left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta_4(r)\right]$$
$$\times \exp\left[-\int_a^r (m_1\eta_4(\tau)+\eta_5(\tau))d\tau\right]dr\bigg)ds$$
$$\le \Bigg\{c(a) + \int_a^b k_3(a,s)\bigg(\exp\left[\int_a^s (m_1\eta_4(\tau)+\eta_5(\tau))d\tau\right]$$
$$\times \int_a^s \left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\eta_4(r)\right]$$
$$\times \exp\left[-\int_a^r (m_1\eta_4(\tau)+\eta_5(\tau))d\tau\right]dr\bigg)ds\Bigg\}$$
$$\times \left\{1 - \int_a^b k_3(a,s)\exp\left[\int_a^s (m_1\eta_4(\tau)+\eta_5(\tau))d\tau\right]ds\right\}^{-1}, \tag{2.44}$$

for all $t \in J_1$. We get the required inequality (2.35) from (2.43) and (2.44). This completes the proof. □

Remark 2.7. If we put $\alpha(t) = t$, then Theorem 2.4 reduces to Theorem 2.5 in [16].

Remark 2.8. If we put $\alpha(t) = t$, $c(t) = k$ (any constant) and $p = 1$, then Theorem 2.4 reduces to Theorem 1.5.3 part (c_1) in [15].

Theorem 2.5. *Let $\omega, f, g, h \in C(J_1, \mathbb{R}_+)$, $\alpha(t) \in C^1(J_1, \mathbb{R}_+)$, with $\alpha(t) \le t$, $\alpha(a) = a$, and $p \ge 1$ and $c \ge 0$ be constants. If*

$$\omega^p(t) \le c + \int_a^{\alpha(t)} g(s)\left[\omega(s) + \int_a^s f(\tau)\omega(\tau)d\tau\right.$$
$$\left.+ \int_a^b h(\tau)\omega^p(\tau)d\tau\right]ds, \forall t \in J_1, \tag{2.45}$$

then

$$\omega^p(t) \le c\exp\left[\int_a^{\alpha(t)} (m_1\eta_6(s)+\eta_7(s))ds\right]$$
$$+ \exp\left[\int_a^{\alpha(t)} (m_1\eta_6(s)+\eta_7(s))ds\right]$$
$$\times \int_a^{\alpha(t)} m_2\eta_6(s)\Big]\exp\left[-\int_a^s (m_1\eta_6(r)+\eta_7(r))dr\right]ds, \tag{2.46}$$

for all $t \in J_1$, where m_1, m_2 are defined as in Lemma 1.4,

$$\eta_6(t) = g(\alpha(t))\left[1 + \int_a^{\alpha(t)} f(\tau)d\tau\right]\alpha'(t), \forall t \in J_1,$$

and

$$\eta_7(t) = \left[g(\alpha(t))\int_a^b h(\tau)d\tau\right]\alpha'(t), \qquad \forall t \in J_1.$$

Proof. Define a function $z_5(t)$ by:

$$z_5(t) = c + \int_a^{\alpha(t)} g(s)\left[\omega(s) + \int_a^s f(\tau)\omega(\tau)d\tau\right.$$
$$\left.+ \int_a^b h(\tau)\omega^p(\tau)d\tau\right]ds, \quad \forall t \in J_1. \tag{2.47}$$

We observe, $z_5(t) \ge 0$ nondecreasing on $\mathbb{R}_+$ with $z(a) = c$. Then from (2.45) and (2.47) using the monotonicity of $z_5(t)$, we get

$$\omega(t) \le z_5^{\frac{1}{p}}(t), \omega(\alpha(t)) \le z_5^{\frac{1}{p}}(\alpha(t)) \le z_5^{\frac{1}{p}}(t), \qquad \forall t \in J_1. \tag{2.48}$$

By differentiating (2.47) and using (2.48), we obtain

$$z_5'(t) = g(\alpha(t))\left[\omega(\alpha(t)) + \int_a^{\alpha(t)} f(\tau)\omega(\tau)d\tau\right.$$
$$\left.+ \int_a^b h(\tau)\omega^p(\tau)d\tau\right]\alpha'(t)$$
$$\le g(\alpha(t))\left[1 + \int_a^{\alpha(t)} f(\tau)d\tau\right]\alpha'(t)z_5^{\frac{1}{p}}$$
$$+\left[g(\alpha(t))\int_a^b h(\tau)d\tau\right]\alpha'(t)z_5(t)$$
$$= \eta_6(t)z_5^{\frac{1}{p}} + \eta_7(t)z_5(t) \qquad \forall t \in J_1, \tag{2.49}$$

from (2.49) and using Lemma (1.4), we get

$$z_5'(t) \le m_1\eta_6(t)z_5(t) + m_2\eta_6(t) + \eta_7(t)z_5(t), \forall t \in J_1. \tag{2.50}$$

The inequality (2.50) can be written as the following

$$z_5'(t) - (m_1\eta_6(t) + \eta_7(t))z_5(t) \le m_2\eta_6(t), \qquad \forall t \in J_1. \tag{2.51}$$

The inequality (2.51) gives us the following estimation

$$z_5(t) \le c\exp\left[\int_a^{\alpha(t)} (m_1\eta_6(s)+\eta_7(s))ds\right]$$
$$+ \exp\left[\int_a^{\alpha(t)} (m_1\eta_6(s)+\eta_7(s))ds\right]$$
$$\times \int_a^{\alpha(t)} m_2\eta_6(s)\Big]\exp\left[-\int_a^s (m_1\eta_6(r)+\eta_7(r))dr\right]ds, \tag{2.52}$$

for all $t \in J_1$. Using $\omega^p(t) \le z_5(t)$ in (2.52), We get the required inequality (2.46). This completes the proof. □

Remark 2.9. If we put $\alpha(t) = t$, then Theorem 2.5 reduces to Theorem 2.6 in [16].

Remark 2.10. If we put $\alpha(t) = t$ and $p = 1$, then Theorem 2.5 reduces to Theorem 1.5.3 part (c_2) in [15].

3. Application

Example 3.1. Consider the following retarded Volterra–Fredholm integral equation:

$$\omega^p(t) = M\left(t, \int_a^t H_1(t, \omega(\alpha(s)))ds, \int_a^b H_2(t, \omega^p(s))ds\right), \forall t \in J_1, \tag{3.1}$$

where $M \in C(J_1 \times \mathbb{R}_+^2, \mathbb{R}_+)$ and $H_i \in C(J_1 \times \mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, satisfy the following hypothesis:

$$|M(t, \omega, u)| \le c(t) + |\omega| + |u|, \qquad \forall t \in J_1, \tag{3.2}$$

$$|H_1(t, \omega)| \le g(t)|\omega|, |H_2(t, \omega)| \le f(t)|\omega|, \qquad \forall t \in J_1, \tag{3.3}$$

where $\omega(t)$, $c(t)$, $g(t)$, $\alpha(t)$ and p are defined as in Theorem 2.1. Using the conditions (3.2) and (3.3), from (3.1), we get

$$|\omega(t)|^p \le c(t) + \int_a^t g(s)|\omega(\alpha(s))|ds + \int_a^b f(t)|\omega^p(s)|ds$$
$$\le c(t) + \int_a^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}|\omega(s)|ds + \int_a^b f(t)|\omega^p(s)|ds,$$

for all $t \in J_1$. Now an application of Theorem 2.1, we have

$$\omega^p(t) \le \Bigg\{c(a) + \int_a^b f(s)\bigg(\exp\left[\int_a^s m_1\frac{g(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))}d\tau\right]$$
$$\times \int_a^s \left[\frac{c'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + m_2\frac{g(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}\right]$$
$$\times \exp\left[-\int_a^r m_1\frac{g(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))}d\tau\right]dr\bigg)ds\Bigg\}$$

$$\times\left\{1-\int_a^b \exp\left[\int_a^s m_1 \frac{g(\alpha^{-1}(\tau))}{\alpha'(\alpha^{-1}(\tau))}d\tau\right]ds\right\}^{-1}$$
$$\times \exp\left[\int_a^{\alpha(t)} m_1 \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right]$$
$$+\exp\left[\int_a^{\alpha(t)} m_1 \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right]$$
$$\times\int_a^{\alpha(t)}\left[\frac{c'(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}+m_2\frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\right]$$
$$\times \exp\left[-\int_a^s m_1 \frac{g(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}dr\right]ds, \quad (3.4)$$

for all $t \in J_1$, where $m_1 = \frac{1}{p}K^{\frac{1-p}{p}}, m_2 = \frac{p-1}{p}K^{\frac{1}{p}}$ and $K > 0$. Thus, the estimation in inequality (3.4), implies the boundedness of the solution $\omega(t)$ of retarded Volterra–Fredholm integral Eq. (3.1).

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