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$\underline{k}$-partial groups

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ABSTRACT

In this paper, we introduce the concept of $\underline{k}-$partial groups and discuss some of their basic properties. We introduce the category kpg of $\underline{k}-$partial groups, which is more convenient than the category Sstg of strong semilattices of topological groups [1] and satisfies the same nice properties of Sstg.

1. Introduction

In [1], we introduced the concept of topological partial groups and discussed some of their basic properties. The category Tpg of topological partial groups, as objects, and the homomorphisms of topological partial groups, as arrows, has the following deficiencies:

(i) Let S be a topological partial group and $a \in S$. Then, the right transformation $r_a: S \to S, x \mapsto x\,a$ and the left transformation $\ell_a: S \to S, x \mapsto a\,x$, may not be open.
(ii) Let S be a topological partial group and $N \trianglelefteq S$. Then, the partial group S/N with the identification topology with respect to the quotient map $\rho_N: S \to S/N, x \mapsto x\,N$, may not be a topological partial group, since $\rho_N \times \rho_N$ may not be an identification map.

If S is a locally compact space, then S/N is also locally compact. Thus, $\rho_N \times \rho_N$ is an identification map [2]. So, S/N is a topological partial group. Therefore, we introduced the category Lcpg, of locally compact partial groups. The category Lcpg modified some of the above deficiencies as (ii) but it does not modify (i) (see Example (4.1) in [1]). Let $S = \mathcal{L}(S_i, Y, \varphi_{i,j})$ be a strong semilattice of groups [3]. Let $(S_i)_{i \in Y}$ be a family of topological groups. Then, S with the final topology with respect to the inclusions $(i_\lambda)_{\lambda \in L}$, which is called the sum topology on S [2], is a topological partial group, denoted by $S = \coprod_{\lambda \in L} S_\lambda$ [1]. The weak product $S \times_W S$, which is $S \times S$ with the final topology with respect to the inclusion maps $(i_\alpha \times i_\beta)_{\alpha, \beta \in L}$, makes S a topological partial group. We proved that every strong semilattice of topological groups is a topological partial group but the converse may not be true [1]. The category Sstg of strong semilattices of topological groups is a convenient category, as we think, because:

(i) If S and T are strong semilattices of topological groups, then $S \times_W T$ is also a strong semilattice of topological groups. That is, the weak product is a categorical product of Sstg.
(ii) A wide subpartial group [4] of a strong semilattice of topological groups with the relative topology is a strong semilattice of topological groups.
(iii) If S is a strong semilattice of topological groups and $N \trianglelefteq S$, then the partial group S/N with the identification topology with respect to the quotient map $\rho_N: S \to S/N$, is a strong semilattice of topological groups.

In this paper, we introduced the category kpg of $\underline{k}-$partial groups, as objects, and continuous partial group homomorphisms, as arrows which is more convenient than the category Sstg and satisfies the same nice properties of Sstg. Let τ be the category of topological spaces and continuous maps and let $\wp$ be a non- empty full subcategory of τ.

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Let S be a topological partial group, then the map $\alpha: C \to S$ is called a $\wp-$ test map if α is continuous and $\alpha^{-1}(S_{e_x})$ is open in C for each $e_x \in E(S)$, where $C \in obj\,(\wp)$. The $\underline{k}-$topology on S is the final topology with respect to all $\wp-$ test maps $\alpha: C \to S, C \in obj\,(\wp)$. S with the $\underline{k}-$topology is denoted by $\underline{k}(S)$. S is called a $\underline{k}-$ space if $\underline{k}\,(S) = S$.

For objects S and T in kpg, let $S \times_{W'} T = \underline{k}\,(S \times T)$ be $S \times T$ with the final topology with respect to all $\wp-$ test maps $\alpha \times \beta: C \times D \to S \times T$. The product $S \times_{W'} T$ is called the weak product of S and T which will be the categorical product of kpg. The topological partial group S with the $\underline{k}-$topology on S is called a $\underline{k}-$partial group, where $\mu : S \times_{W'} S \to S$ is the product map.

The category kpg is a convenient category if and only if the category $\wp$ satisfies the following conditions [5]:

(A) If A is a closed subspace of an object B of $\wp$ then A is a $k\wp-$ space [5].
(B) If B and C are objects in $\wp$ then $B \times C$ is also an object in $\wp$
(C) For objects X in $\wp$ and Y in τ, the evaluation map $e : Y^X \times X \to Y;\ e\,(f, x) = f\,(x),\ f \in Y^X$ and $x \in X$, is continuous, where Y^X has the compact open topology.
(D) If A and B are objects in $\wp$ then the topological sum $A \sqcup B$ is also an object in $\wp$

Examples of such non- empty full subcategories $\wp$ are, the category H of all compact Hausdorff spaces, the category LH of all locally compact Hausdorff spaces and the category L of all locally compact spaces.

2. Preliminaries

We collect for sake of reference the needed definitions and results appeared in the given references.

Let S be a partial group [6]. If $x \in S$, then e_x and x^{-1} are the partial identity and the partial inverse of x, respectively, which are unique [6]. $E(S)$ is the set of all partial identity elements of S, which is also the set of all idempotents of S [4]. The set $S_x = \{y \in S : e_x = e_y\}$ is a maximal subgroup of S and $S = \bigcup\{S_x : x \in S\} = \bigcup\{S_{e_x} : e_x \in E(S)\}$. That is, every partial group is a disjoint union of groups [6]. A semigroup S is a partial group if and only if it is a strong semilattice of groups [6].

Let S be a partial group and τ be a topology on S. Then, S is called a topological partial group if the following maps are continuous [1]:

(i) The product map $\mu: S \times S \to S; (x, y) \mapsto xy$
(ii) The partial inverse map $\gamma : S \to S;\ x \mapsto x^{-1}$
(iii) The partial identity map $e_S: S \to S; x \mapsto e_x$.

Let S be a topological partial group and N be a subpartial group [4] of S. Then N with the relative topology is a topological partial group, called a topological subpartial group [1].

3. $\underline{k}$-partial groups.

Let τ be the category of topological spaces and continuous maps and let $\wp$ be a non- empty full subcategory of τ.

Definition 3.1. Let S be a topological partial group, then the map $\alpha: C \to S$ is called a $\wp-$ test map if α is continuous and $\alpha^{-1}(S_{e_x})$ is open in C for each $e_x \in E(S)$, where $C \in obj\,(\wp)$.

Definition 3.2. The $\underline{k}-$topology on S is the final topology with respect to all $\wp-$ test maps $\alpha: C \to S, C \in obj\,(\wp)$. S with the $\underline{k}-$topology is denoted by $\underline{k}(S)$. S is called a $\underline{k}-$ space if $\underline{k}\,(S) = S$.

From this definition, we have that S_{e_x} is open in S for all $e_x \in E(S)$. So, the $\underline{k}-$space S, which is not a group, is not connected. Also, we have that the identity map $I: \underline{k}(S) \to S$ is continuous.

Theorem 3.1. *There exists a set of $\wp-$ test maps sufficient to define the $\underline{k}-$ topology on S.*

Proof. Let $\mathcal{C}$ be the family of all non open subsets in S. Since S is a set, then $\mathcal{C}$ is a set. So, $\forall U \in \mathcal{C}$, we can choose $F \in obj\,(\wp)$ and $\alpha_U: F \to S$ such that $\alpha_U^{-1}(U)$ is not open in F. Let $L = \{\alpha_U : F \to S,\ U \in \mathcal{C}\}$. Let S' have the final topology with respect to L. Then, the topology onS' is finer than the topology on S. Let V be a non- empty non open subset in S, then $\exists$ a $\wp$-test map $\alpha_V: F' \to S$ such that $\alpha_V^{-1}(V)$ is not open in F'. So, V is not open in S'. Hence, the topology on S' is coarser than the topology on S. □

Definition 3.3. The $\underline{k}-$ topology on $S \times S$ is the final topology with respect to all $\wp-$ test maps of the form $\alpha \times \beta: C \times D \to S \times S$, where $C, D \in obj\,(\wp)$. $S \times S$ with this topology will be denoted by $S \times_{W'} S$.

Theorem 3.2. *Let S and T be topological partial groups. If $f: S \to T$is continuous, then $\underline{k}(f) = f : \underline{k}(S) \to \underline{k}(T)$ is continuous.*

Proof. Let $\alpha: C \to S$ be a $\wp-$ test map, then $f\alpha: C \to T$ is a $\wp-$ test map since $f\alpha$ is continuous and if T_e is a maximal subgroup of T, then $(f\alpha)^{-1}\,(T_e)$ is open in C. Then $f\alpha: C \to \underline{k}\,(T)$ is a $\wp-$ test map, since $\underline{k}(\underline{k}(T)) = \underline{k}(T)$. That means, $\underline{k}(f) = f : \underline{k}(S) \to \underline{k}(T)$ is continuous. □

We note that the following structure maps are continuous from the above theorem, for any $\underline{k}$- space S:

(i) $\mu : S \times_{W'} S \to S$,
(ii) $\gamma: S \to S$,
(iii) $e_S: S \to S$.

So, the partial group S with the $\underline{k}-$topology on S is a topological partial group, called a $\underline{k}-$partial group.

Let kpg be a category of $\underline{k}-$partial groups, as objects, and continuous partial group homomorphisms, as arrows. Then one can define a functor $\underline{k} : Tpg \to kpg, S \mapsto \underline{k}(S), f \mapsto \underline{k}(f)$.

The category kpg is a convenient category if and only if the conditions A, B, C and D in the introduction are satisfied.

Theorem 3.3. *Let S be a $\underline{k}-$partial group, then the maps r_a and ℓ_a are continuous.*

Proof. Since S is a topological partial group, then from Theorem 3.2, the maps r_a and ℓ_a are continuous. □

Theorem 3.4. *The maps r_a and ℓ_a are open.*

Proof. We only prove that r_a is open as follows: Let $U \subseteq S$ be open. Then, $U \bigcap S_{e_x}$ is open in the maximal topological subgroup S_{e_x}. Since, the right transformation $r_a|_{S_{e_x}} : S_{e_x} \to S_{e_x}$ is a homeomorphism, then, $r_a|_{S_{e_x}}\,(U \bigcap S_{e_x})$ is open in S_{e_x}. Since S_{e_x} is open in S, then $r_a|_{S_{e_x}}\,(U \bigcap S_{e_x}) = Ua \bigcap S_{e_x}$ is open in S. That means $r_a(U) = \bigcup_{e_x \in E(S)} r_a|_{S_{e_x}}\,(U \bigcap S_{e_x})$ is open in S. Similarly, ℓ_a is open. □

Theorem 3.5. *Let S be a $\underline{k}-$partial group and $A, B \subseteq S$. Then, if A is open in S, then AB and BA are also open in S, where $AB = \{ab : a \in A,\ b \in B\}$.*

Proof. Since $AB = \bigcup_{b \in B} r_b(A)$, and $r_b(A)$ is open in S , then AB is open in S. Similarly, since $BA = \bigcup_{b \in B} \ell_b(A)$, and $\ell_b(A)$ is open in S, then BA is open in S. □

Theorem 3.6. *Let N be a topological subpartial group of the $\underline{k}-$partial group S, then $\underline{k}\,(N) = N$.*

Proof. Let $U \subseteq N$ be such that $\alpha^{-1}(U)$ is open in C, where $\alpha: C \to N$ is a $\wp$-test map. Let $i: N \to S$ be the inclusion. Since $i\alpha: C \to S$ is continuous and S_{e_x} is open in S, then $(i\alpha)^{-1}\,(S_{e_x})$ is open in C. That means $i\alpha: C \to S$ is a $\wp-$ test map. Now, $(i\alpha)^{-1}\,(U) =$

$\alpha^{-1}(i^{-1}(U)) = \alpha^{-1}(U)$ is open in C. Hence, U is open in S. Since $U \bigcap N = U$, then U is open in N. So, $\underline{k}(N) = N$. □

We will denote the topological subpartial group N of a $\underline{k}-$partial group S by $N \le S$.

Theorem 3.7. *If S is a $\underline{k}-$partial group, then every open topological subpartial group of S is closed.*

Proof. Let N be an open topological subpartial group of S. Then, xN is open in S, $\forall x \in S$. Since, $S - N = \bigcup_{x \notin N} xN$, then $S - N$ is open. Therefore, N is closed. □

Let $\{S_i : i = 1, 2, \cdots, n\}$ be a family of $\underline{k}-$ partial groups and let $S = \overset{n}{\underset{i=1}{\otimes}} S_i = \{x = \langle x_i \rangle : x_i \in S_i \ \ , \forall i = 1, 2, \ldots, n\}$ be the external direct product of the $\underline{k}-$ partial groups S_i. Then, S is a topological partial group [1].

Theorem 3.8. $\underline{k}(S) = S$.

Proof. We have that the identity maps I_i: $S_i \to \underline{k}(S_i)$ and the projections P_i: $S \to S_i$ are continuous. So, the maps $I_i P_i$: $S \to \underline{k}(S_i)$ are continuous. Hence, the identity map $\langle I P_i \rangle$: $S \to \underline{k}(S)$ is continuous. That means $\underline{k}(S) = S$. □

4. Quotients in $\underline{k}$-partial groups and separation axioms

Definition 4.1. If S is a $\underline{k}-$partial group and $N \le S$, then S/N with the identification topology, with respect to the quotient map ρ_N: $S \to S/N$, $x \mapsto xN$, is called the coset space.

Theorem 4.1. *Let S be a $\underline{k}$-partial group and $N \le S$. Then, the quotient map ρ_N: $S \to S/N$ is open.*

Proof. Let $U \subseteq S$ be open. Now,

$$\begin{aligned} \rho_N^{-1}(\rho_N(U)) &= \{x \in S : \rho_N(x) \in \rho_N(U)\} \\ &= \{x \in S : xN \in U/N\} \\ &= UN. \end{aligned}$$

Since U is open in S, then UN is open in S (Theorem 3.5). Therefore, $\rho_N(U)$ is open is S/N. □

Theorem 4.2. *If S is a $\underline{k}$ -partial group and $N \trianglelefteq S$, then S/N is a$\underline{k}$-partial group.*

Proof. Firstly, we show that S/N is a topological partial group. Since ρ_N is an open identification map, then $\rho_N \times \rho_N$ is an identification map [2]. So, the product map μ: $S/N \times S/N \to S/N$ is continuous, since $\mu(\rho_N \times \rho_N) = \rho_N \mu'$, where μ': $S \times S \to S$ is the product map. The partial inverse map γ: $S/N \to S/N$and the partial identity map $e_{S/N}$: $S/N \to S/N$ are continuous, since $\gamma \rho_N = \rho_N \gamma'$ and $e_{S/N} \rho_N = \rho_N e_S$, where γ': $S \to S$, $x \mapsto x^{-1}$ and e_S: $S \to S$, $x \mapsto e_x$ are continuous. Since $I \rho_N = \underline{k}(\rho_N) I'$, then I is continuous, where I: $S/N \to \underline{k}(S/N)$ and I': $S \to \underline{k}(S)$ are the identity maps. □

Definition 4.2. Let S and T be $\underline{k}-$partial groups. Then, φ: $S \to T$ is called a morphism if φ is continuous and partial group homomorphism [4].

Theorem 4.3. *Let φ: $S \to T$ be an idempotent separating [3] surjective morphism and $K = \ker \varphi$. Then, there exists a unique bijective morphism α: $S/K \to T$ such that $\varphi = \alpha \rho_K$.*

Proof. α is continuous since φ is continuous and ρ_K is an identification map. That is all we need. □

Theorem 4.4. *Let S be a $\underline{k}-$partial group and $M, N \trianglelefteq S$ be such that $M \subseteq N$, then*

(i) *$N/M \trianglelefteq S/M$.*
(ii) *There exists a unique bijective morphism $\alpha : (S/M)/(N/M) \to S/N$,* such that $\rho_N = \alpha \rho_{N/M} \rho_M$.

Proof.

(i) See [4]
(ii) Let ρ_N: $S \to S/N$ and ρ_M: $S \to S/M$ be the quotient maps. Since ρ_N is an idempotent separating surjective morphism and $\ker \rho_N = N$, then from the last theorem, there exists a unique bijective morphism φ: $S/M \to S/N, xM \mapsto xN$ such that $\varphi \rho_M = \rho_N$. Since $\ker \varphi = N/M$ [1] is a $\underline{k}-$partial group, then by the last theorem, there exists a unique bijective morphism α: $(S/M)/(N/M) \to S/N$, such that $\rho_N = \alpha \rho_{N/M} \rho_M$. □

Theorem 4.5. *Let S be a $\underline{k}-$partial group. Then, S is a Hausdorff space if and only if S is a T_0-space.*

Proof. Let S be a Hausdorff space. Then, S is a T_0-space. Conversely, let S is a T_0-space. Let $x, y \in S, x \neq y$.

(i) If $x, y \in S_a$, then S_a is a T_2-group, then $\exists$ two open sets U, V in S_a and also open in S such that $U \bigcap V = \phi$ and $x \in U$ and $y \in V$.
(ii) If $x \in S_a$ and $y \in S_b$. Then, we have that S_a and S_b are open in S and $S_a \bigcap S_b = \phi$. So, S is a Hausdorff space. □

Theorem 4.6. *Let S be a Hausdorff $\underline{k}-$partial group. If f, g: $S \to T$ are morphisms of $\underline{k}-$partial groups, then the difference kernel $A = \{x \in T : f(x) = g(x)\}$ is a closed subpartial group.*

Proof. A is closed (see [2]). Let $x, y \in A$, then

$$\begin{aligned} f(xy^{-1}) &= f(x)\, f(y^{-1}) \\ &= f(x)\, f(y)^{-1} \\ &= g(x)\, g(y)^{-1} = g(xy^{-1}), \quad \text{then } xy^{-1} \in A. \end{aligned}$$

So, A is a closed subpartial group. □

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