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Intuitionistic circular bifuzzy matrices

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ABSTRACT

In this paper, we define the intuitionistic circular fuzzy matrix and introduce the necessary and sufficient conditions for an intuitionistic fuzzy matrix to be circular. Also, we study some properties of intuitionistic circular fuzzy matrices

1. Introduction

The concept of intuitionistic fuzzy matrices was introduced by Pal et al. [1] as a generalization of the well known ordinary fuzzy matrices introduced by Thomason [2] which take its elements from the unit interval [0,1]. An intuitionistic fuzzy matrix is a pair of fuzzy matrices, namely, a membership and non-membership function which represent positive and negative aspects of the given information (see [3,4]). However, intuitionistic fuzzy matrices have been proposed to represent finite intuitionistic fuzzy relations. This concept is a generalization to that of the ordinary fuzzy relations which also is a generalization to the crisp relations (or Boolean relations).

In this paper, we concentrate oure attention on one of the important kind of intuitonstic fuzzy matrices called intuitionistic circular fuzzy matrices. However, a characterization of intuitionistic circular fuzzy matrices is given and some important properties are established.

The paper is organized in three sections. In Section 2, the definitions and operations on intuitionistic fuzzy matrices are briefly introduced. In Section 3, results concerning of intuitionistic circular fuzzy matrices are proved using the operations and notations in the previous section. In Section 4, we exhibit the adjoint of an intuitionistic circular fuzzy matrix throughout its determinant and show that the adjoint of an intuitionistic circular fuzzy matrix is also circular. However, the operations $\vee$ and $\wedge$ play an important role in our work.

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2. Preliminaries and definitions

We give here some definitions and notations which are applied in the paper. Note that an intuitionistic fuzzy matrix A of order $m \times n$ is defined as follows: $A = [a_{ij}]$ where $a_{ij} = \left\langle a'_{ij}, a''_{ij} \right\rangle$ and $a'_{ij}, a''_{ij} \in [0,1]$ maintaining the condition $0 \le a'_{ij} + a''_{ij} \le 1$.

Now, we define some operations on the intuitionistic fuzzy matrices.. For intuitionistic fuzzy matrices $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, and $C = [c_{ij}]_{n\times m}$ the following operations are defined [3,5–7].

$$\begin{aligned}
A \wedge B &= [a_{ij} \wedge b_{ij}] = [\langle \min(a'_{ij}, b'_{ij}), \max(a''_{ij}, b''_{ij}) \rangle],\\
A \vee B &= [a_{ij} \vee b_{ij}] = [\langle \max(a'_{ij}, b'_{ij}), \min(a''_{ij}, b''_{ij}) \rangle],\\
AC &= \left[\langle \bigvee_{k=1}^{n} (a'_{ik} \wedge c'_{kj}), \bigwedge_{k=1}^{n} (a''_{ik} \vee c''_{kj}) \rangle\right],\\
A^k &= \left[a_{ij}^{(k)}\right] = \left[\langle a'^{(k)}_{ij}, a''^{(k)}_{ij} \rangle\right] = A^{k-1}A\\
I_n &= A^0 = \begin{cases} <1,0> & \text{if} \quad i = j,\\ <0,1> & \text{if} \quad i \ne j. \end{cases}\\
A^T &= [a_{ji}] \ \text{(the transpose of } A),\\
\nabla A &= A \wedge A^T
\end{aligned}$$

$A \le B$ if and only if $a_{ij} \le b_{ij}$ .That is if and only if $a'_{ij} \le b'_{ij}$ and $a''_{ij} \ge b''_{ij}$ for all i, j.

We may write **0** instead of $<0, 1>$ and **1** instead of $<1, 0>$.

Definition 2.1. [1,3,8–11] . For an $n \times n$ intuitionistic fuzzy matrix A we have:

(a) A is symmetric if and only if $A^T = A$,
(b) A is idempotent if and only if $A^2 = A$,
(c) A is transitive if and only if $A^2 \le A$,
(d) A is circular if and only if $(A^2)^T \le A$,
(e) A is weakly reflexive if and only if $a_{ii} \ge a_{ij}$ for all $1 \le i, j \le n$,
(f) A is reflexive if and only if $a_{ii} = \mathbf{1}$ for all $1 \le i \le n$,
(g) A is similarity if and only if A is symmetric, transitive and reflexive.

It is noted that $(A^T)^2 = (A^2)^T$ for any $n \times n$ matrix. So, the intuitionistic fuzzy matrix A is circular if and only if $A^2 \le A^T$, i.e., $a_{ik} \wedge a_{kj} \le a_{ji}$ for every $1 \le i, j, k \le n$. Moreover, if A is symmetric, then A is transitive if and only if A is circular.

3. Results

Throughout the next two sections we deal only with $n \times n$ intuitionistic fuzzy matrices. In this section, some properties of intuitionistic circular fuzzy matrices are examined by the definitions in the above section. However, we begin with the following proposition.

Proposition 3.1. *Let A be an $n \times n$ intuitionistic fuzzy matrix and let A_1 denotes the $m \times m$ submatrix of A (where $m < n$) such that*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Then A is circular if and only if $A_1^2 \le A_1^T$, $A_2A_3 \le A_1^T$, $A_3A_1 \le A_2^T$, $A_4A_3 \le A_2^T$, $A_1A_2 \le A_3^T$, $A_2A_4 \le A_3^T$, $A_3A_2 \le A_4^T$ and $A_4^2 \le A_4^T$.

Proof. Suppose that A satisfies all the above conditions and consider

$$A^2 = B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then

$$B_1 = A_1^2 \vee A_2A_3 \le A_1^T \vee A_1^T = A_1^T,$$
$$B_2 = A_1A_2 \vee A_2A_4 \le A_3^T \vee A_3^T = A_3^T,$$
$$B_3 = A_3A_1 \vee A_4A_3 \le A_2^T \vee A_2^T = A_2^T$$

and

$$B_4 = A_3A_2 \vee A_4^2 \le A_4^T \vee A_4^T = A_4^T.$$

Thus, we have $A^2 = B \le A^T$ and A is circular.

Conversely, suppose that A is circular. For $1 \le s \le m$ and $m+1 \le t \le n$, Let $C = A_1$, $D = A_2$, $E = A_3$ and $F = A_4$. Then $c_{st} = a_{st}$ for every $1 \le s, t \le m$, $d_{st} = a_{s(t+m)}$ for every $1 \le s \le m$ and $1 \le t \le n-m$, $e_{st} = a_{(s+m)t}$ for every $1 \le s \le n-m$ and $1 \le t \le m$ and $f_{st} = a_{(s+m)(t+m)}$ for every $1 \le s \le n-m$ and $1 \le t \le n-m$.

1. To show that $A_1^2 \le A_1^T$ and $A_2A_3 \le A_1^T$, let $G = A_1^2$ and $H = A_2A_3$. Then

$$\begin{aligned} g_{st} &= \langle \overset{m}{\underset{k=1}{\vee}} (c'_{sk} \wedge c'_{kt}), \overset{m}{\underset{k=1}{\wedge}} (c''_{sk} \vee c''_{kt}) \rangle \\ &= \langle \overset{m}{\underset{k=1}{\vee}} (a'_{sk} \wedge a'_{kt}), \overset{m}{\underset{k=1}{\wedge}} (a''_{sk} \vee a''_{kt}) \rangle \\ &\le \langle \overset{n}{\underset{k=1}{\vee}} (a'_{sk} \wedge a'_{kt}), \overset{n}{\underset{k=1}{\wedge}} (a''_{sk} \vee a''_{kt}) \rangle = \langle a'^{(2)}_{st}, a''^{(2)}_{st} \rangle \\ &\le \langle a'_{ts}, a''_{ts} \rangle = a_{ts} = c_{ts}. \end{aligned}$$

Thus, $g_{st} \le c_{ts}$ and therefore, $A_1^2 \le A_1^t$.

Also,

$$\begin{aligned} h_{st} &= \langle \overset{n-m}{\underset{k=1}{\vee}} (d'_{sk} \wedge e'_{kt}), \overset{n-m}{\underset{k=1}{\wedge}} (d''_{sk} \vee e''_{kt}) \rangle \\ &= \langle \overset{n-m}{\underset{k=1}{\vee}} (a'_{s(k+m)} \wedge a'_{(k+m)t}), \overset{n-m}{\underset{k=1}{\wedge\wedge}} (a''_{s(k+m)} \vee a''_{(k+m)t}) \rangle \\ &= \langle \overset{n}{\underset{u=m+1}{\vee}} (a'_{su} \wedge a'_{ut}), \overset{n}{\underset{u=m+1}{\wedge}} (a''_{su} \vee a''_{ut}) \rangle \quad (\text{where } u = k+m) \\ &\le \langle \overset{n}{\underset{u=1}{\vee}} (a'_{su} \wedge a'_{ut}), \overset{n}{\underset{u=1}{\wedge}} (a''_{su} \vee a''_{ut}) \rangle = \langle a'^{(2)}_{st}, a''^{(2)}_{st} \rangle \\ &\le \langle a'_{ts}, a''_{ts} \rangle = a_{ts} = c_{ts}. \end{aligned}$$

Thus, $h_{st} \le c_{ts}$ and therefore, $A_2A_3 \le A_1^T$.

2. To show that $A_4A_3 \le A_2^T$ and $A_3A_1 \le A_2^T$, let $Q = A_4A_3$ and $L = A_3A_1$. Then

$$\begin{aligned} q_{st} &= \langle \overset{n-m}{\underset{k=1}{\vee}} (f'_{sk} \wedge e'_{kt}), \overset{n-m}{\underset{k=1}{\wedge}} (f''_{sk} \vee e''_{kt}) \rangle \\ &= \langle \overset{n-m}{\underset{k=1}{\vee}} \left(a'_{(s+m)(k+m)} \wedge a'_{(k+m)t}\right), \overset{n-m}{\underset{k=1}{\wedge}} \left(a''_{(s+m)(k+m)} \vee a''_{(k+m)t}\right) \rangle \\ &= \langle \overset{n}{\underset{u=m+1}{\vee}} (a'_{(s+m)u} \wedge a'_{ut}), \overset{n}{\underset{u=m+1}{\wedge}} (a''_{(s+m)u} \vee a''_{ut}) \rangle \\ &\quad (\text{where } u = k+m) \\ &\le \langle \overset{n}{\underset{u=1}{\vee}} (a'_{(s+m)u} \wedge a'_{ut}), \overset{n}{\underset{u=1}{\wedge}} (a''_{(s+m)u} \vee a''_{ut}) \rangle \\ &= \langle a'^{(2)}_{(s+m)t}, a''^{(2)}_{(s+m)t} \rangle \\ &\le \langle a'_{t(s+m)}, a''_{t(s+m)} \rangle = a_{t(s+m)} = d_{ts}. \end{aligned}$$

Thus, $q_{st} \le d_{ts}$ and therefore, $A_4A_3 \le A_2^T$. Also,

$$\begin{aligned} l_{st} &= \langle \overset{m}{\underset{k=1}{\vee}} (e'_{sk} \wedge c'_{kt}), \overset{m}{\underset{k=1}{\wedge}} (e''_{sk} \vee c''_{kt}) \rangle \\ &= \langle \overset{m}{\underset{k=1}{\vee}} (a'_{(s+m)k} \wedge a'_{kt}), \overset{m}{\underset{k=1}{\wedge}} (a''_{(s+m)k} \vee a''_{kt}) \rangle \\ &\le \langle \overset{n}{\underset{k=1}{\vee}} (a'_{(s+m)k} \wedge a'_{kt}), \overset{n}{\underset{k=1}{\wedge}} (a''_{(s+m)k} \vee a''_{kt}) \rangle \\ &= \langle a'^{(2)}_{(s+m)t}, a''^{(2)}_{(s+m)t} \rangle \le \langle a'_{t(s+m)}, a''_{t(s+m)} \rangle = a_{t(s+m)} = d_{ts}. \end{aligned}$$

i.e., $l_{st} \le d_{ts}$ and therefore, $A_3A_1 \le A_2^T$.

3. To show that $A_1A_2 \le A_3^T$ and $A_2A_4 \le A_3^T$, let $R = A_1A_2$ and $Z = A_2A_4$. Then

$$\begin{aligned} r_{st} &= \langle \overset{m}{\underset{k=1}{\vee}} (c'_{sk} \wedge d'_{kt}), \overset{m}{\underset{k=1}{\wedge}} (c''_{sk} \vee d''_{kt}) \rangle \\ &= \langle \overset{m}{\underset{k=1}{\vee}} (a'_{sk} \wedge a'_{k(t+m)}), \overset{m}{\underset{k=1}{\wedge}} (a''_{sk} \vee a''_{k(t+m)}) \rangle \\ &\le \langle \overset{n}{\underset{k=1}{\vee}} (a'_{sk} \wedge a'_{k(t+m)}), \overset{n}{\underset{k=1}{\wedge}} (a''_{sk} \vee a''_{k(t+m)}) \rangle \\ &= \langle a'^{(2)}_{s(t+m)}, a''^{(2)}_{s(t+m)} \rangle \le \langle a'_{(t+m)s}, a''_{(t+m)s} \rangle = a_{(t+m)s} = e_{ts}. \end{aligned}$$

Therefore, $A_1A_2 \le A_3^T$. Also,

$$\begin{aligned} z_{st} &= \langle \overset{n-m}{\underset{k=1}{\vee}} (d'_{sk} \wedge f'_{kt}), \overset{n-m}{\underset{k=1}{\wedge}} (d''_{sk} \vee f''_{kt}) \rangle \\ &= \langle \overset{n-m}{\underset{k=1}{\vee}} (a'_{s(k+m)} \wedge a'_{(k+m)(t+m)}), \overset{n-m}{\underset{k=1}{\wedge}} (a''_{s(k+m)} \vee a''_{(k+m)(t+m)}) \rangle \\ &= \langle \overset{n}{\underset{u=m+1}{\vee}} (a'_{su} \wedge a'_{u(t+m)}), \overset{n}{\underset{u=m+1}{\wedge}} (a''_{su} \vee a''_{u(t+m)}) \rangle \\ &\le \langle \overset{n}{\underset{u=1}{\vee}} (a'_{su} \wedge a'_{u(t+m)}), \overset{n}{\underset{u=1}{\wedge}} (a''_{su} \vee a''_{u(t+m)}) \rangle \\ &= \langle a'^{(2)}_{s(t+m)}, a''^{(2)}_{s(t+m)} \rangle \le \langle a'_{(t+m)s}, a''_{(t+m)s} \rangle = a_{(t+m)s} = e_{ts}. \end{aligned}$$

Hence, $A_2A_4 \le A_3^T$.

4. To show that $A_3A_2 \le A_4^T$ and $A_4^2 \le A_4^T$, let $P = A_3A_2$ and $W = A_4^2$. Then

$$\begin{aligned} p_{st} &= \langle \overset{n}{\underset{k=m+1}{\vee}} (e'_{sk} \wedge d'_{kt}), \overset{n}{\underset{k=m+1}{\wedge}} (e''_{sk} \vee d''_{kt}) \rangle \\ &= \langle \overset{n}{\underset{k=m+1}{\vee}} (a'_{(s+m)k} \wedge a'_{k(t+m)}), \overset{n}{\underset{k=m+1}{\wedge}} (a''_{(s+m)k} \vee a''_{k(t+m)}) \rangle \\ &\le \langle \overset{n}{\underset{k=1}{\vee}} (a'_{(s+m)k} \wedge a'_{k(t+m)}), \overset{n}{\underset{k=1}{\wedge}} (a''_{(s+m)k} \vee a''_{k(t+m)}) \rangle \\ &= \langle a'^{(2)}_{(s+m)(t+m)}, a''^{(2)}_{(s+m)(t+m)} \rangle \end{aligned}$$

$\leq \langle a'_{(t+m)(s+m)}, a''_{(t+m)(s+m)} \rangle = a_{(t+m)(s+m)} = f_{ts}$. Therefore, $A_3A_2 \leq A_4^T$.

Also,

$$w_{st} = \langle \overset{n-m}{\underset{k=1}{\vee}} (f'_{sk} \wedge f'_{kt}), \overset{n-m}{\underset{k=1}{\wedge}} (f''_{sk} \vee f''_{kt}) \rangle$$
$$= \langle \overset{n-m}{\underset{k=1}{\vee}} (a'_{(s+m)(k+m)} \wedge a'_{(k+m)(t+m)}), \overset{n-m}{\underset{k=1}{\wedge}} (a''_{(s+m)(k+m)} \vee a''_{(k+m)(t+m)}) \rangle$$
$$= \langle \overset{n}{\underset{u=m+1}{\vee}} (a'_{(s+m)u} \wedge a'_{u(t+m)}), \overset{n}{\underset{u=m+1}{\wedge}} (a''_{(s+m)u} \vee a''_{u(t+m)}) \rangle$$
$$\leq \langle \overset{n}{\underset{u=1}{\vee}} (a'_{(s+m)u} \wedge a'_{u(t+m)}), \overset{n}{\underset{u=1}{\wedge}} (a''_{(s+m)u} \vee a''_{u(t+m)}) \rangle$$
$$= \langle a'^{(2)}_{(s+m)(t+m)}, a''^{(2)}_{(s+m)(t+m)} \rangle$$
$$\leq \langle a'_{(t+m)(s+m)}, a''_{(t+m)s} \rangle = a_{(t+m)(s+m)} = f_{ts}.$$

Thus, $A_4^2 \leq A_4^T$. This completets the proof. □

Remark. It is easy to see that the intuitionistic fuzzy matrix A is circular if and only if A^T is circular.

Lemma 3.2. [3] *For intuitionistic fuzzy matrices* $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [c_{ij}]_{n\times p}$ *and* $D = [d_{ij}]_{p\times n}$, *if* $A \leq B$, *then* $AC \leq BC$ *and* $DA \leq DB$.

Proposition 3.3. *An intuitionistic fuzzy matrix A is circular if and only if* $E^{(i,j)}AE^{(i,j)}$ *is circular for every* $1 \leq i, j \leq n$, *where* $E^{(i,j)}$ *is the intuitionistic fuzzy matrix obtained from the identity intuitionistic matrix* I_n *by interchanging the row i and row j.*

Proof. First, we notice that $E^{(i,j)}E^{(i,j)} = I_n$ and $(E^{(i,j)})^T = E^{(i,j)}$ for every $1 \leq i, j \leq n$. Suppose that A is circular. Then

$$(E^{(i,j)}AE^{(i,j)})^2 = (E^{(i,j)}AE^{(i,j)})(E^{(i,j)}AE^{(i,j)})$$
$$= E^{(i,j)}A^2E^{(i,j)} \leq E^{(i,j)}A^TE^{(i,j)} = (E^{(i,j)}AE^{(i,j)})^T$$

and hence $E^{(i,j)}AE^{(i,j)}$ is circular.

Now, suppose that $E^{(i,j)}AE^{(i,j)}$ is circular. That is $(E^{(i,j)}AE^{(i,j)})(E^{(i,j)}AE^{(i,j)}) \leq (E^{(i,j)}AE^{(i,j)})^T = E^{(i,j)}A^TE^{(i,j)}$. Then $E^{(i,j)}A^2E^{(i,j)} \leq E^{(i,j)}A^TE^{(i,j)}$ and so by Lemma 3.2, we get $E^{(i,j)}A^2E^{(i,j)}E^{(i,j)} \leq E^{(i,j)}A^TE^{(i,j)}E^{(i,j)}$. That is $E^{(i,j)}A^2 \leq E^{(i,j)}A^T$. Also, $E^{(i,j)}E^{(i,j)}A^2 \leq E^{(i,j)}E^{(i,j)}A^T$ and so $A^2 \leq A^T$ and A is thus circular. □

Proposition 3.4. *Let* $*$ *be a binary operation on* $[0, 1]$ *satisfies for every* $x, y, u, v \in [0, 1]$, *the following conditions:*

1. $(x * y)\wedge(u * v) \leq (x\wedge u)*(y\wedge v)$,
2. $(x * y)\vee(u * v) \geq (x\vee u)*(y\vee v)$,
3. $x \leq y$, $u \leq v$ *imply* $x * u \leq y * v$.

If A and B are $n \times n$ *intuitionistic circular fuzzy matrices, then* $A*B$ *is circular, where* $A*B$ *is defined as* $A * B = [a_{ij} * b_{ij}] = [\langle a'_{ij} * b'_{ij}, a''_{ij} * b''_{ij} \rangle]$.

Proof. Let $C = A * B$ and $D = C^2$. Then

$$d_{ij} = \langle d'_{ij}, d''_{ij} \rangle = \langle \overset{n}{\underset{k=1}{\vee}} (c'_{ik} \wedge c'_{kj}), \overset{n}{\underset{k=1}{\wedge}} (c''_{ik} \vee c''_{kj}) \rangle$$
$$= \langle \overset{n}{\underset{k=1}{\vee}} ((a'_{ik} * b'_{ik}) \wedge (a'_{kj} * b'_{kj})), \overset{n}{\underset{k=1}{\wedge}} ((a''_{ik} * b''_{ik}) \vee (a''_{kj} * b''_{kj})) \rangle$$
$$= \langle (a'_{il} * b'_{il}) \wedge (a'_{lj} * b'_{lj}), (a''_{ih} * b''_{ih}) \vee (a''_{hj} * b''_{hj}) \rangle$$
$$\text{for some } l \text{ and } h \leq n.$$

By the properties of the operation $*$, we get

$d'_{ij} = (a'_{il} * b'_{il}) \wedge (a'_{lj} * b'_{lj}) \leq (a'_{il} \wedge a'_{lj}) * (b'_{il} \wedge b'_{lj}) \leq a'_{ji} * b'_{ji} = c'_{ji}$ (since A and B are circular) and

$d''_{ij} = (a''_{ih} * b''_{ih}) \vee (a''_{hj} * b''_{hj}) \geq (a''_{ih} \vee a''_{hj}) * (b''_{ih} \vee b''_{hj}) \geq a''_{ji} * b''_{ji} = c''_{ji}$. Thus, $d_{ij} \leq c_{ji}$ and $A*B$ is circular. □

Corollary 3.5. *The intuitionistic fuzzy matrix* $A\wedge B$ *is circular if A and B are circular.*

Proof. Since the operation $\wedge$ satisfies the conditions of operation$*$, then $A\wedge B$ is circular. □

Corollary 3.6. *The intuitionistic fuzzy matrix* $\triangledown A$ *is circular when A is circular.*

Proof. By Corollary 3.5, since A and A^T are circular. □

Corollary 3.7. *Let* $F{:}X \to X$ *be a function such that* $F(x)\wedge F(y) \leq F(x\wedge y)$ *and if* $x \leq y$, *then* $F(x) \leq F(y)$ *for all* $x, y \in X$.*If A is an intuitionistic circular fuzzy matrix, then* $F(A)$ *is circular, where* $F(A) = [F(a_{ij})]$.

Proof. Since A is circular, we have, $a_{ik}\wedge a_{kj} \leq a_{ji}$ for all $1 \leq i, j, k \leq n$.But by the definition of F, we have that $F(a_{ik}\wedge a_{kj}) \leq F(a_{ji})$ and $F(a_{ik})\wedge F(a_{kj}) \leq F(a_{ik}\wedge a_{kj}) \leq F(a_{ji})$. Thus $F(A)$ is circular. □

Proposition 3.8. *A is circular and weakly reflexive if and only if A is symmetric and transitive.*

Proof. Suppose that A is circular and weakly reflexive. Then $a_{ji} = a_{jj} \wedge a_{ji} \leq a_{ij}$. Also, $a_{ij} = a_{ii} \wedge a_{ij} \leq a_{ji}$. So, $a_{ij} = a_{ji}$ and A is symmetric. Also, we have $a^{(2)}_{ij} \leq a_{ji} = a_{ij}$ and A is thus transitive.

Conversely, suppose that A is symmetric and transitive. Then $a^{(2)}_{ij} \leq a_{ij} = a_{ji}$. Hence A is circular. To show that A is weakly reflexive, we have $a_{ij} = a_{ij} \wedge a_{ji} \leq a_{ii}$ (by the circularity and symmetry of A). Thus A is weakly reflexive. □

Corollary 3.9. *If the intuitionistic circular fuzzy matrix A is reflexive, then A is similarity.*

Proposition 3.10. *If A is circular and symmetric intuitionistic fuzzy matrix, then A is idempotent.*

Proof. Since A is symmetric, we have A is transitive. i.e., $A^2 \leq A$, it remains to show that $A^2 \geq A$. Again since A is symmetric, we have $a_{ij} = a_{ji}$ for every $1 \leq i, j \leq n$. Let $a_{ij} = c > \mathbf{0}$. That is $\langle a'_{ij}, a''_{ij} \rangle = \langle c', c'' \rangle > \langle 0, 1 \rangle$. Then by circularity of A we have $a_{ii} \geq a_{ij}\wedge a_{ji}$ or $a'_{ii} \geq c'$ and $a''_{ii} \leq c''$. i.e., $a_{ii} \geq c$. On the other hand

$a^{(2)}_{ij} = \langle a'^{(2)}_{ij}, a''^{(2)}_{ij} \rangle = \langle \overset{n}{\underset{k=1}{\vee}} (a'_{ik} \wedge a'_{kj}), \overset{n}{\underset{k=1}{\wedge}} (a''_{ik} \vee a''_{kj}) \rangle = \langle a'_{il} \wedge a'_{lj}, a''_{ih} \vee a''_{hj} \rangle$ for some $1 \leq l, h \leq n$. But $a'_{il} \wedge a'_{lj} \geq a'_{ii} \wedge a'_{ij} = c'$ and

$a''_{ih} \vee a''_{hj} \leq a''_{ii} \vee a''_{ij} = c''$. Therefore, $a^{(2)}_{ij} \geq c = a_{ij}$ and $A^2 \geq A$. This completes the proof. □

Corollary 3.11. *If A is circular , then* ∇A *is idempotent.*

Proof. By Proposition 3.10 and Corollary 3.6, since ∇A is symmetric. □

Proposition 3.12. *For a circular and weakly reflexive intuitionistic fuzzy matrix A, we have A is idempotent.*

Proof. By Propositions 3.8 and 3.10. □

Corollary 3.13. *Let A be a circular and reflexive Then A is idempotent.*

Proof. By Proposition 3.10 and Corollary 3.9. □

4. Adjoint of an intuitionistic circular fuzzy matrix

Definition 4.1. [5,10]. The determinant $|A|$ of an $n \times n$ intuitionistic fuzzy matrix A is defined as: $|A| = \vee_{\delta\in S_n} (\overset{n}{\underset{t=1}{\wedge}} \langle a'_{t\delta(t)}, a''_{t\delta(t)} \rangle)$, where S_n is the symmetric group of all permutations of the indices (1,2,...,n).

Definition 4.2. [5,10]. The adjoint of A is denoted by adj$A$ and is defined as: $b_{ij} = \left|A_{ji}\right|$ where $|A_{ji}|$ is the determinant of the $(n-1)\times(n-1)$ matrix A_{ji} formed by deleting row j and column i from A and where $B=$ adj$A$.

Remarks:

1. The element of the matrix $D = A_{ij}$ can be written in terms of the elements of the matrix A as follows:

$$d_{uv} = \begin{cases} a_{uv} & \text{if } u<i \text{ and } v<j, \\ a_{(u+1)v} & \text{if } u \ge i \text{ and } v<j, \\ a_{u(v+1)} & \text{if } u<i \text{ and } v \ge j, \\ a_{(u+1)(v+1)} & \text{if } u \ge i \text{ and } v \ge j. \end{cases}$$

2. If $W = A^T$, then $W_{ij} = A^T_{ij} = (A_{ji})^T$ and so, for $G = (A_{ji})^T$ we have:

$$g_{uv} = \begin{cases} a_{vu} & \text{if } u<i \text{ and } v<j, \\ a_{v(u+1)} & \text{if } u \ge i \text{ and } v<j, \\ a_{(v+1)u} & \text{if } u<i \text{ and } v \ge j, \\ a_{(v+1)(u+1)} & \text{if} u \ge i \text{ and} v \ge j. \end{cases}$$

Proposition 4.3. *Let A be an $n \times n$ intuitionistic circular fuzzy matrix. Then $A_{ik}A_{kj} \le (A_{ji})^T$ for every $1 \le i, j, k \le n$.*

Proof. Let $C = A_{ik}, F = A_{kj}, G = (A_{ji})^T$ and $R = CF$. Then $r_{uv} = \langle \overset{n-1}{\underset{m-1}{\vee}} (c'_{um} \wedge f'_{mv}), \overset{n-1}{\underset{m-1}{\wedge}} (c''_{um} \vee f''_{mv})\rangle$ Since we have that A is circular, we have the following cases:

Case 1: If $u < i$, $m < k$ and $v < j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{um} \wedge a'_{mv}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{um} \vee a''_{mv}\right)\rangle \\ &= \langle a'_{up} \wedge a'_{pv}, a''_{ug} \vee a''_{gv}\rangle \le \langle a'_{vu}, a''_{vu}\rangle = a_{vu} = g_{uv} \\ &\quad \text{for some } p, g \le n. \end{aligned}$$

Case 2: If $u < i$, $m < k$ and $v \ge j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{um} \wedge a'_{m(v+1)}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{um} \vee a''_{m(v+1)}\right)\rangle \\ &= \langle a'_{up} \wedge a'_{p(v+1)}, a''_{ug} \vee a''_{g(v+1)}\rangle \le \langle a'_{(v+1)u}, a''_{(v+1)u}\rangle \\ &= a_{(v+1)u} = g_{uv}. \end{aligned}$$

Case 3: If $u < i$, $m \ge k$ and $v < j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{u(m+1)} \wedge a'_{(m+1)v}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{u(m+1)} \vee a''_{(m+1)v}\right)\rangle \\ &= \langle a'_{u(p+1)} \wedge a'_{(p+1)v}, a''_{u(g+1)} \vee a''_{(g+1)v}\rangle \le \langle a'_{vu}, a''_{vu}\rangle = a_{vu} = g_{uv}. \end{aligned}$$

Case 4: If $u \ge i$, $m < k$ and $v \ge j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{(u+1)m} \wedge a'_{m(v+1)}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{(u+1)m} \vee a''_{m(v+1)}\right)\rangle \\ &= \langle a'_{(u+1)p} \wedge a'_{p(v+1)}, a''_{(u+1)g} \vee a''_{g(v+1)}\rangle \\ &\le \langle a'_{(v+1)(u+1)}, a''_{(v+1)(u+1)}\rangle = a_{(v+1)(u+1)} = g_{uv}. \end{aligned}$$

Case 5: If $u \ge i$, $m \ge k$ and $v < j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{(u+1)(m+1)} \wedge a'_{(m+1)v}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{(u+1)(m+1)} \vee a''_{(m+1)v}\right)\rangle \\ &= \langle a'_{(u+1)(p+1)} \wedge a'_{(p+1)v}, a''_{(u+1)(g+1)} \vee a''_{(g+1)v}\rangle \\ &\le \langle a'_{v(u+1)}, a''_{v(u+1)}\rangle = a_{v(u+1)} = g_{uv}. \end{aligned}$$

Case 6: If $u < i$, $m \ge k$ and $v \ge j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{u(m+1)} \wedge a'_{(m+1)(v+1)}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{u(m+1)} \vee a''_{(m+1)(v+1)}\right)\rangle \\ &= \langle a'_{u(p+1)} \wedge a'_{(p+1)(v+1)}, a''_{u(g+1)} \vee a''_{(g+1)(v+1)}\rangle \\ &\le \langle a'_{(v+1)u}, a''_{(v+1)u}\rangle = a_{(v+1)u} = g_{uv}. \end{aligned}$$

Case 7: If $u \ge i$, $m < k$ and $v < j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{(u+1)m} \wedge a'_{mv}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{(u+1)m} \vee a''_{mv}\right)\rangle \\ &= \langle a'_{(u+1)p} \wedge a'_{pv}, a''_{(u+1)g} \vee a''_{gv}\rangle \le \langle a'_{v(u+1)}, a''_{v(u+1)}\rangle \\ &= a_{v(u+1)} = g_{uv}. \end{aligned}$$

Case 8: If $u \ge i$, $m \ge k$ and $v \ge j$, then

$$\begin{aligned} r_{uv} &= \langle \overset{n-1}{\underset{m=1}{\vee}} \left(a'_{(u+1)(m+1)} \wedge a'_{(m+1)(v+1)}\right), \overset{n-1}{\underset{m=1}{\wedge}} \left(a''_{(u+1)(m+1)} \vee a''_{(m+1)(v+1)}\right)\rangle \\ &= \langle a'_{(u+1)(p+1)} \wedge a'_{(p+1)(v+1)}, a''_{(u+1)(g+1)} \vee a''_{(g+1)(v+1)}\rangle \\ &\le \langle a'_{(v+1)(u+1)}, a''_{(v+1)(u+1)}\rangle = a_{(v+1)(u+1)} = g_{uv}. \end{aligned}$$

Thus, we have $r_{uv} \le g_{uv}$ in every case and hence $A_{ik}A_{kj} \le (A_{ji})^T$ for every $1 \le i, j, k \le n$. □

Corollary 4.4. *If A is intuitionistic circular fuzzy matrix, then A_{ii} is circular for every $1 \le i \le n$.*

Proposition 4.5. [6]. *Let A and B be two intuitionistic fuzzy matrices of order $n \times n$. Then we have the followings:*

(i) $\left|A^T\right| = |A|$,
(ii) $|A| \wedge |B| \le |AB|$,
(iii) $|A| = \overset{n}{\underset{j=1}{\vee}} a_{ij}|A_{ij}|$ *where A_{ij} is the intuitionistic fuzzy matrix of order $(n-1)\times(n-1)$ formed by deleting row i and column j from A.*

Corollary 4.6. *Let A and B be two intuitionistic fuzzy matrices of order $n \times n$ such that $A \le B$. Then $|A| \le |B|$.*

Proposition 4.7. *Let A be a circular matrix. Then adjA is circular.*

Proof. Let B=adj$A$. Then $b_{us} = |A_{su}|, b_{lu} = |A_{ul}|$ and $b_{sl} = |A_{ls}|$.
By Propositions 4.1, 4.3 and Corollary 4.4 we get
$|A_{su}| \wedge |A_{ul}| \le |A_{su}A_{ul}| \le \left|A^T_{ls}\right| = |A_{ls}|$. Since $\wedge$ is commutative, then
$|A_{ul}| \wedge |A_{su}| \le |A_{ls}|$. Therefore, $b_{lu} \wedge b_{us} = b_{sl}$ and so adj$A$ is circular. □

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