Journal of the Egyptian Mathematical Society 25 (2017) 206–211

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original Article

A hybrid numerical method for solving system of high order boundary value problems

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ARTICLE INFO

Article history:
Received 29 April 2016
Revised 30 September 2016
Accepted 4 October 2016
Available online 15 October 2016

2010 MSC:
65M70
65N35
35C10
42C106

Keywords:
Chebyshev wavelet finite difference method
System of ordinary differential equations
Boundary value problems

ABSTRACT

Higher order system of boundary value problems arise in several areas of applications. In this paper, we employ the Chebyshev wavelet finite difference method to solve such system of higher order boundary value problems. Numerical experiments are conducted to show the feasibility of the proposed method.

1. Introduction

Mathematical models of certain problems and phenomena in science and engineering require the solution of system of higher order (≥ 2) boundary value problems (BVPs) involving ordinary differential equations (ODEs). Several new approximate analytical and numerical methods have been developed and applied for various types of problems involving differential equations. Solving system of boundary value problems involving ODEs using new methods has attracted the attention of many researchers. Islam et al. [1] proposed a non polynomial spline approach for the approximate solution of a system of third-order boundary-value problems. Momani [2] employed a modified decomposition method for solving a system of second order obstacle problems. The sinc-collocation method, non-polynomial spline method and variational iteration method were, respectively used in [3–6]. He's homotopy perturbation [7] and B-spline method [8] have also been employed to solve a system of BVPs. Noor et al. [9,10] proposed variational method and the modified variation of parameters method which is a combination of variation of parameters method and Adomian's decomposition method for solving system of second-order and third-order nonlinear boundary value problem. In 2014, Arqub et al. [11] and Chen et al. [12] applied continuous genetic algorithm and deficient discrete cubic spline methods to obtain the solution to a system of second order boundary value problems. Kazemi Nasab et al. [13,14] solved singular boundary value problems of different types using wavelet analysis method. A composite Chebyshev finite difference method was used for solving singular boundary value problems in [15]. Scalar boundary value problems have been solved by Kazemi Nasab et al. [16,17] using the CWFD method. The operational matrix of fractional integration for shifted Chebyshev polynomials was derived in [18]. Bhrawy et al. [19] proposed a new formula for fractional integral of Chebyshev polynomials. Shifted fractional-order Jacobi orthogonal functions was employed for solving a system of fractional differential equations [20]. Chen et al. [21] applied Legendre wavelets to solve system of nonlinear fractional differential equations. Shifted Jacobi spectral approximations was used for solving fractional differential equations [22]. The question we wish to pursue in this paper is whether CWFD method can be extended for nonlinear higher order systems.

In this paper, we employ CWFD of numerical solution of a system of higher order BVPs of the form:

$$\begin{cases} f_1(x, \mathbf{u}(x), \mathbf{u}^{(1)}(x), \cdots, \mathbf{u}^{(\omega-1)}(x), \mathbf{u}^{(\omega)}(x)) = 0, \\ f_2(x, \mathbf{u}(x), \mathbf{u}^{(1)}(x), \cdots, \mathbf{u}^{(\omega-1)}(x), \mathbf{u}^{(\omega)}(x)) = 0, \\ \vdots \\ f_r(x, \mathbf{u}(x), \mathbf{u}^{(1)}(x), \cdots, \mathbf{u}^{(\omega-1)}(x), \mathbf{u}^{(\omega)}(x)) = 0, \end{cases} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.joems.2016.10.002

subject to the conditions

$B_i(u_q(0), \cdots, u_q^{(\omega-1)}(0), u_q(1), \cdots, u_q^{(\omega-1)}(1)),$
$q = 1, 2, \ldots, r, i = 1, 2, \ldots, r\omega,$

where $\mathbf{u}^{(\alpha)}(x) = [u_1^{(\alpha)}(x), u_2^{(\alpha)}(x), \cdots, u_r^{(\alpha)}(x)]$, $\alpha = 0, 1, 2, \ldots, \omega$ and f_j are nonlinear functions of $u_j, u_j', \ldots,$ and $u_j^{(\omega-1)}$, $j = 1, 2, \ldots, r$.

2. Wavelets and Chebyshev wavelets

The attractive properties of wavelets in certain situations have persuaded many researchers to consider them as a mathematical tool to solve different kinds of problems arising in mathematics, physics, and engineering. Wavelet analysis can overcome certain shortcomings of Fourier analysis whilst retaining the advantages. The multiresolution analysis aspect of wavelets allows to decomposition of a function or signal into elementary waveforms at different positions and scales to detect important information such as peaks or singularities.

Hence wavelets can be an important tool in the solution of problems involving peaks or singularities. Varying the dilation parameter a and the translation parameter b give rise to the following family of continuous wavelets [23]:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}} \psi\left(\frac{x-b}{a}\right), \quad a, b \in R, a \neq 0. \tag{2.1}$$

Chebyshev wavelets $\psi_{n,m} = \psi(k, n, m, x)$, have five arguments, $n = 1, \ldots, 2^{k-1}$, m is degree of Chebyshev polynomials of the first kind, x denotes the time and are defined on [0, 1) as,

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} p_m T_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

where p_m, $m = 0, 1, \ldots, M$ are defined as,

$$p_m = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}}, & m \geq 1, \end{cases} \tag{2.3}$$

and T_m are Chebyshev polynomials of the first kind of degree m defined as,

$$T_m(x) = \cos m\beta, \quad \beta = arc\cos x, \tag{2.4}$$

which are orthogonal with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$.

A function $f \in L^2[0, 1)$ may be approximated as

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x) = C^T \Psi(x), \tag{2.5}$$

where C and $\Psi(x)$ are $2^k(M+1) \times 1$ matrices given by

$$C = [c_{1,0}, \ldots, c_{1,M}, c_{2,0}, \ldots, c_{2,M}, \ldots, c_{2^{k-1},1}, \ldots, c_{2^{k-1},M}]^T,$$
$$\Psi(x) = [\psi_{1,0}, \ldots, \psi_{1,M}, \psi_{2,0}, \ldots, \psi_{2,M}, \ldots, \psi_{2^{k-1},1}, \ldots, \psi_{2^{k-1},M}]^T. \tag{2.6}$$

There are a variety of orthogonal basis functions but some of them have received considerable attention including piecewise constant basis functions, polynomials, and sincosine functions in Fourier series. Chebyshev polynomials are employed to minimize approximation error [24]. Chebyshev wavelets have good characteristics of both Chebyshev polynomials and wavelets. They are very well localized functions so can effectively approximate functions. The multiresolution property of wavelets enables us to adjust the parameters M and k in a proper way to improve the accuracy of solution. Another advantage of Chebyshev wavelets is to convert a given problem to a set of algebraic equations which can be solved easier. With the benefit of sparsity of coefficient matrix, computation performs faster.

3. Chebyshev wavelet finite difference method

A function f can be approximated in terms of a basis of Chebyshev polynomials as follows [25],

$$(P_M f)(x) = \sum_{m=0}^{M}{}'' f_m T_m(x), \tag{3.1}$$

$$f_m = \frac{2}{M}\sum_{k=0}^{M}{}'' f(x_k)T_m(x_k) = \frac{2}{M}\sum_{k=0}^{M}{}'' f(x_k)\cos\left(\frac{mk\pi}{M}\right),$$

where the summation symbol with double primes denotes a sum with both the first and last terms halved. Moreover, the well known Chebyshev-Gauss-Lobatto interpolated points x_m are the extrema of the Mth-order Chebyshev polynomial $T_M(x)$ and defined as

$$x_m = \cos\left(\frac{m\pi}{M}\right), \quad m = 0, 1, 2, \ldots, M. \tag{3.2}$$

The first three derivatives of the function $f(x)$ at the points x_m, $m = 0, 1, \ldots, M$ are given by Elbarbary et al. [26]–[27] as

$$f^{(n)}(x_m) = \sum_{j=0}^{M} d_{m,j}^{(n)} f(x_j), \quad n = 1, 2, 3 \tag{3.3}$$

where

$$d_{m,j}^{(1)} = \frac{4\gamma_j}{M}\sum_{k=1}^{M}\sum_{\substack{l=0 \\ (k+l)\ odd}}^{k-1}\frac{k\gamma_k}{c_l}T_k(x_j)T_l(x_m),$$
$$= \frac{4\gamma_j}{M}\sum_{k=1}^{M}\sum_{\substack{l=0 \\ (k+l)\ odd}}^{k-1}\frac{k\gamma_k}{c_l}\cos\left(\frac{kj\pi}{M}\right)\cos\left(\frac{lm\pi}{M}\right), \tag{3.4}$$

$$d_{m,j}^{(2)} = \frac{2\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=0 \\ (k+l)\ even}}^{k-2}\frac{k(k^2-l^2)\gamma_k}{c_l}T_k(x_j)T_l(x_m),$$
$$= \frac{2\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=0 \\ (k+l)\ even}}^{k-2}\frac{k(k^2-l^2)\gamma_k}{c_l}\cos\left(\frac{kj\pi}{M}\right)\cos\left(\frac{lm\pi}{M}\right), \tag{3.5}$$

with $\gamma_0 = \gamma_M = \frac{1}{2}, \gamma_j = 1$ for $j = 1, 2, \ldots M-1$, and

$$d_{m,j}^{(3)} = \frac{4\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=1 \\ (k+l)even}}^{k-2}\sum_{\substack{r=0 \\ (l+r)odd}}^{l-1}\frac{kl(k^2-l^2)\gamma_k}{c_l c_r}T_k(x_j)T_r(x_m),$$
$$= \frac{4\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=1 \\ (k+l)even}}^{k-2}\sum_{\substack{r=0 \\ (l+r)odd}}^{l-1}\frac{kl(k^2-l^2)\gamma_k}{c_l c_r}\cos(\frac{kj\pi}{M})\cos(\frac{rm\pi}{M}). \tag{3.6}$$

We are now ready to set up the main idea of this work. Consider $x_{nm}, n = 1, 2, \ldots, 2^{k-1}, m = 0, 1, \ldots, M,$ as the corresponding Chebyshev-Gauss-Lobatto collocation points at the nth subinterval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ such that,

$$x_{nm} = \frac{1}{2^k}(x_m + 2n - 1). \tag{3.7}$$

On the other hand, a function $f(x)$ can be written in terms of Chebyshev wavelet basis functions as follows

$$(P_M f)(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M}{}'' c_{nm}\psi_{nm}(x), \tag{3.8}$$

where $c_{nm}, n = 1, 2, \ldots, 2^{k-1}, m = 0, 1, \ldots, M,$ are the expansion coefficients of the function $f(x)$ at the subinterval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ and

$\psi_{n,m}(x), n=1,2,\ldots,2^{k-1}, m=0,1,\ldots,M,$ are defined in Eq. (2.2). In view of Eq. (3.1), we can obtain the coefficients c_{nm} as

$$c_{nm} = \frac{1}{(2^{k/2}p_m)^2} \times \frac{2}{M}\sum_{p=0}^{M}{}'' f(x_{np})\psi_{nm}(x_{np}) = \frac{1}{2^{k/2}p_m} \times \frac{2}{M}\sum_{p=0}^{M}{}'' f(x_{np})\cos\left(\frac{mp\pi}{M}\right). \tag{3.9}$$

Using Eqs. (3.3)–(3.5), the first three derivatives of the function f(x) at the points $x_{nm}, n=1,2,\ldots,2^{k-1}, m=0,1,\ldots,M,$ can be obtained as

$$f^{(r)}(x_{nm}) = \sum_{j=0}^{M} d^{(r)}_{n,m,j} f(x_{nj}), \quad r=1,2,3. \tag{3.10}$$

where

$$d^{(1)}_{n,m,j} = \frac{4\gamma_j}{M}\sum_{k=1}^{M}\sum_{\substack{l=0\\(k+l)\ odd}}^{k-1}\frac{k\gamma_k}{c_l p_k p_l}\psi_{nk}(x_{nj})\psi_{nl}(x_{nm}), = \frac{4\gamma_j}{M}\sum_{k=1}^{M}\sum_{\substack{l=0\\(k+l)\ odd}}^{k-1}\frac{2^k k\gamma_k}{c_l}\cos\left(\frac{kj\pi}{M}\right)\cos\left(\frac{lm\pi}{M}\right), \tag{3.11}$$

$$d^{(2)}_{n,m,j} = \frac{2\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=0\\(k+l)\ even}}^{k-2}\frac{2^k k(k^2-l^2)\gamma_k}{c_l p_k p_l}\psi_{nk}(x_{nj})\psi_{nl}(x_{nm}), = \frac{2\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=0,\\(k+l)\ even}}^{k-2}\frac{4^k k(k^2-l^2)\gamma_k}{c_l}\cos\left(\frac{kj\pi}{M}\right)\cos\left(\frac{lm\pi}{M}\right),$$

and

$$d^{(3)}_{n,m,j} = \frac{2^{2k+2}\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=1\\(k+l)even}}^{k-2}\sum_{\substack{r=0\\(l+r)odd}}^{l-1}\frac{kl(k^2-l^2)\gamma_k}{c_l c_r p_k p_r}\psi_{nk}(x_{nj})\psi_{nr}(x_{nm}) = \frac{2^{3k+2}\gamma_j}{M}\sum_{k=2}^{M}\sum_{\substack{l=1\\(k+l)even}}^{k-2}\sum_{\substack{r=0\\(l+r)odd}}^{l-1}\frac{kl(k^2-l^2)\gamma_k}{c_l c_r}\cos\left(\frac{kj\pi}{M}\right)\cos\left(\frac{rm\pi}{M}\right).$$

4. Discretization of problem

In this section, the Chebyshev wavelet finite difference (CWFD) method is applied for solving the system (1.1). We suppose the interval [0, 1) is divided into 2^{k-1} subintervals $I_n = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$, $n=1,2,\ldots,2^{k-1}$. We also consider the Chebyshev-Gauss-Lobatto collocation points $\frac{n-1}{2^{k-1}} = x_{n0} < x_{n1} < \ldots < x_{n,M-1} < x_{n,M} = \frac{n}{2^{k-1}}$ on the nth subinterval I_n, $n=1,2,\ldots,2^{k-1}$, where x_{ns} are defined as following:

$$x_{ns} = \frac{1}{2^k}(x_s + 2n - 1), \quad s=1,2,\ldots,M-1. \tag{4.1}$$

In order to obtain the solutions $u_j(x)$, $j=1,2,..,r$ in Eq. (1.1), we first approximate $u_j(x)$ in terms of Chebyshev wavelet finite difference basis functions as follows

$$u_j(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M}{}'' c^j_{nm}\psi_{nm}(x), \quad j=1,2,\ldots,r \tag{4.2}$$

where $c^j_{nm}, n=1,2,3,\ldots,2^{k-1}, m=0,1,\ldots,M$ are defined in Eq. (3.9). We now collocate Eq. (1.1) at the Chebyshev-Gauss-Lobatto points $x_{ns}, n=1,2,3,\ldots,2^{k-1}, s=1,\ldots,M-\omega+1$ as

$$\begin{cases} f_1(x_{ns}, \mathbf{u}(x_{ns}), \mathbf{u}^{(1)}(x_{ns}), \cdots, \mathbf{u}^{(\omega-1)}(x_{ns}), \mathbf{u}^{(\omega)}(x_{ns})) = 0, \\ f_2(x_{ns}, \mathbf{u}(x_{ns}), \mathbf{u}^{(1)}(x_{ns}), \cdots, \mathbf{u}^{(\omega-1)}(x_{ns}), \mathbf{u}^{(\omega)}(x_{ns})) = 0, \\ \vdots \\ f_r(x_{ns}, \mathbf{u}(x_{ns}), \mathbf{u}^{(1)}(x_{ns}), \cdots, \mathbf{u}^{(\omega-1)}(x_{ns}), \mathbf{u}^{(\omega)}(x_{ns})) = 0. \end{cases} \tag{4.3}$$

Table 1
The absolute errors in solutions $y_1(x)$ (light gray shade) and $y_2(x)$ (unshaded) for different values of M and k for Example 5.1.

x	$M=8,\ k=2$	$M=10,\ k=2$	$M=12,\ k=2$
0.1	2.9×10^{-11}	8.8×10^{-15}	4.5×10^{-18}
	9.9×10^{-12}	3.4×10^{-15}	1.9×10^{-18}
0.2	9.1×10^{-11}	2.5×10^{-14}	1.8×10^{-17}
	2.3×10^{-11}	9.9×10^{-15}	9.4×10^{-18}
0.3	1.8×10^{-10}	4.8×10^{-14}	3.8×10^{-17}
	4.4×10^{-11}	1.9×10^{-14}	2.2×10^{-17}
0.4	3.1×10^{-10}	7.7×10^{-14}	6.8×10^{-17}
	7.6×10^{-11}	3.1×10^{-14}	3.9×10^{-17}
0.5	4.6×10^{-10}	1.2×10^{-13}	1.0×10^{-16}
	1.3×10^{-10}	4.5×10^{-14}	6.1×10^{-17}
0.6	6.0×10^{-10}	1.6×10^{-13}	1.3×10^{-16}
	1.8×10^{-10}	5.4×10^{-14}	7.6×10^{-17}
0.7	6.2×10^{-10}	1.7×10^{-13}	1.3×10^{-16}
	2.1×10^{-10}	5.2×10^{-14}	7.6×10^{-17}
0.8	5.3×10^{-10}	1.4×10^{-13}	1.1×10^{-16}
	1.8×10^{-10}	4.2×10^{-14}	6.3×10^{-17}
0.9	3.2×10^{-10}	8.8×10^{-14}	6.5×10^{-17}
	1.1×10^{-10}	2.5×10^{-14}	3.9×10^{-17}

In view of Eqs. (3.10)–(3.11), we obtain

$$u^{(\gamma)}_j(x_{ns}) = \sum_{i=0}^{M} d^{(\gamma)}_{n,s,i,j} u_j(x_{ni}), \quad \gamma = 1,2,\ldots,\omega-1. \tag{4.4}$$

Besides, it is necessary that $u_j(x)$ and it's first ω derivatives be continuous at the interface of subintervals,

$$u^{(d)}_j\left(\frac{n}{2^{k-1}}\right) = \lim_{x\to(\frac{n}{2^{k-1}})^-} u^{(d)}_j(x), \quad d=0,1,\ldots,\omega-1, n=1,2,\ldots,2^{k-1}-1. \tag{4.5}$$

As it is assumed in formula (1.1), we have $r\omega$ boundary conditions. According to a given problem and with the help of formulas (4.2) and (4.4), we collocate $u_j(x)$ and/or its derivatives at the endpoints 0 and 1 to obtain other $r\omega$ equations. Therefore we have a system of $2^{k-1}r(M+1)$ algebraic equations, which can be solved for the unknowns $u_j(x_{ns})$. Consequently, we obtain the solution $u_j(x), j=1,2,\ldots,r$ to the given system (1.1) on the interval [0, 1).

5. Illustrative examples

In this section, some examples of different types are given to demonstrate the applicability and the accuracy of the CWFD method to solve system of BVPs involving ODE.

Example 5.1. As the first example, consider the nonlinear singular system of boundary value problems

$$\begin{cases} y_1'''(x) + \dfrac{1}{5x-2}y_1^2(x) + y_2(x) = f(x), \\ \dfrac{1}{10x-8}y_2'''(x) + y_1''(x)y_2(x) + y_2'(x) = g(x), \end{cases} \tag{5.1}$$

with the following boundary conditions

$$y_1(0)=1, y_1'(0)=1, y_1(1)=e$$
$$y_2(0)=0, y_2'(0)=1, y_2(1)=\sin 1,$$

where $f(x) = e^x + \frac{e^{2x}}{5x-2} + \sin x$ and $g(x) = \frac{-\cos x}{10x-8} + e^x\sin x + \cos x$. The exact solutions of this problem are $y_1(x)=e^x$ and $y_2(x)=\sin x$. This problem has singularity at two points $x_1=0.4$ and $x_2=0.8$. We solve this problem for $M=8, 10, 12$ and k = 2 and report the absolute errors in solution between the approximate solutions $y_1(x)$ and $y_2(x)$ and their corresponding exact solutions in Table 1. For further investigation, the graph of absolute errors in solutions are shown in Fig. 1.

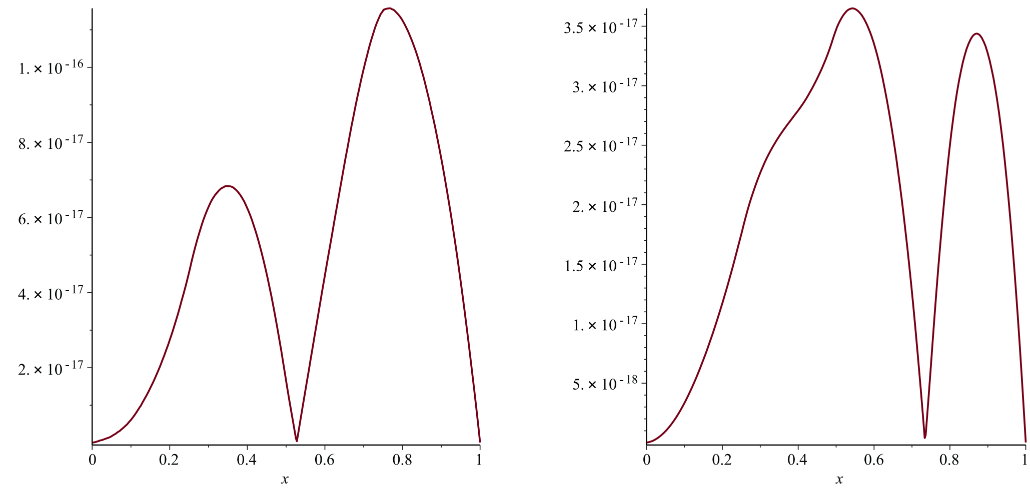

Fig. 1. The graph of absolute errors between approximate solution $\tilde{y_1}(x)$ and exact solution for $M = 12$ and $k = 3$ (right) the graph of absolute errors between approximate solution $\tilde{y_2}(x)$ and exact solution for $M = 12$ and $k = 3$ for Example 5.1.

Table 2
Absolute errors for $y_1(x)$ and $y_2(x)$ at different points for Example 5.2.

x	*Absolute error* for $y_1(x)$	*Absolute error* for $y_2(x)$
0.1	2.7×10^{-17}	1.0×10^{-17}
0.2	5.6×10^{-17}	2.3×10^{-17}
0.3	8.4×10^{-17}	4.1×10^{-17}
0.4	1.3×10^{-16}	5.1×10^{-17}
0.5	1.2×10^{-16}	5.4×10^{-17}
0.6	1.0×10^{-16}	4.3×10^{-17}
0.7	7.9×10^{-17}	3.2×10^{-17}
0.8	5.3×10^{-17}	2.0×10^{-17}
0.9	3.0×10^{-17}	9.3×10^{-18}

Example 5.2. Consider the nonlinear system of boundary value problems [7]

$$\begin{cases} y_1''(x) + xy_2(x) + xy_1^2(x) = f(x), \\ y_2''(x) + xy_1'(x) + y_2(x) = g(x), \end{cases} \tag{5.2}$$

with the following boundary conditions

$$y_1(0) = y_1(1) = 0, \quad y_2(0) = y_2(1) = 0,$$

where $f(x) = -\pi^2 \sin(\pi x) + x\sin^2(\pi x) + x^4 - 3x^3 + 2x^2$ and $g(x) = \pi x\cos(\pi x) + x^3 - 3x^2 + 8x - 6$. The exact solutions of this problem are $y_1(x) = \sin(\pi x)$ and $y_2(x) = x^3 - 3x^2 + 2x$. This example is solved with $M = 10, k = 5$. A comparison between the approximate solutions $\tilde{y_1}(x)$, $\tilde{y_2}(x)$ and the exact solutions is given in Table 2.

Example 5.3. Consider the nonlinear system of boundary value problems

$$\begin{cases} y_1''(x) + y_1'(x) = f_1(x), \\ y_2''(x) - y_1'(x) + y_2^2(x) = f_2(x), \\ y_3''(x) - y_2^2(x) = f_3(x), \end{cases} \tag{5.3}$$

with the following boundary conditions

$$y_1(0) = y_1(1) = 0, \quad y_2(0) = y_2(1) = 0, \quad y_3(0) = y_3(1) = 0,$$

where $f_1(x) = (-x^2 - 3x + 2)\sin x + (-x^2 + 3x + 1)\cos x$, $f_2(x) = (1 - 2x)\cos x + x(x - 1)\sin x + (x^2 - 5x + 4)e^{-x} + (x^4 - 2x^3 + x^2)e^{-2x}$, and $f_3(x) = 2\cos x - (x - 1)\sin x - x^2(x - 1)^2 e^{-2x}$. The exact solutions of this problem are $y_1(x) = x(x - 1)\cos x$, $y_2(x) = x(x - 1)e^{-x}$, and $y_3(x) = (x - 1)\sin x$. This example is solved for variety of values of M and k. The logarithms of absolute errors in solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ are plotted in Fig. 2.

Table 3
Absolute errors for $y_1(x)$, $y_2(x)$, and $y_3(x)$ at singular point with different values of M and k for Example 5.4.

$x = 0.8$	M = 12, k = 2	M = 16, k = 2	M = 16, k = 3
Absolute errors for $y_1(x)$	1.17×10^{-9}	3.44×10^{-14}	7.79×10^{-16}
Absolute errors for $y_2(x)$	1.24×10^{-11}	5.54×10^{-16}	2.90×10^{-16}
Absolute errors for $y_3(x)$	7.35×10^{-11}	2.59×10^{-15}	1.56×10^{-15}

Example 5.4. Consider the nonlinear singular system of boundary value problems

$$\begin{cases} y_1'''(x) + 2y_1''(x) + {y_1'}^2(x) + \dfrac{1}{5x - 4} y_1(x)y_3(x) = f_1(x), \\ \dfrac{1}{5} y_2'''(x) + y_1'(x)y_2''(x)y_3^2(x) + y_2'(x) + y_3(x) = f_2(x), \\ y_3'''(x) + y_3'(x) + y_1(x)y_3^3(x) + y_1''(x)y_2'(x)y_3''(x) = f_3(x), \end{cases} \tag{5.4}$$

with the following boundary conditions

$$y_1(0) = 1, y_1'(0) = 0, y_1(1) = e,$$
$$y_2(0) = 0, y_2'(0) = 1, y_2(1) = \cos 1,$$
$$y_3(0) = 0, y_3'(0) = 2, y_3(1) = 1 + \sin 1,$$

where $f_1(x) = \frac{e^{x^2}}{5x-4}(20x^3e^{x^2} + 40x^4 - 16x^2e^{x^2} + 8x^3 + 28x^2 + \sin x - 27x - 16)$, $f_2(x) = 2x^2e^{x^2}\cos^3(x) - 2x^2e^{x^2}\cos x - 4x^3e^{x^2}\sin x\cos x - 2x^4e^{x^2}\cos x + 4xe^{x^2}\sin x\cos^2(x) - 4xe^{x^2}\sin x + 8x^2e^{x^2}\cos^2(x) - 8x^2e^{x^2} - 4x^3e^{x^2}\sin x - \frac{4}{5}x\sin x + \sin x + \frac{2}{5}\cos x + x$, and $f_3(x) = -4x^3e^{x^2}\cos^2(x) - 4x^2e^{x^2}\sin x\cos x - e^{x^2}\sin x\cos^2(x) + 3x^2e^{x^2}\sin x - 5xe^{x^2}\cos^2(x) + 5x^3e^{x^2} - 2e^{x^2}\sin x\cos x + e^{x^2}\sin x + 5xe^{x^2} + 1$. The exact solutions of this problem are $y_1(x) = e^{x^2}$, $y_2(x) = x\cos x$, and $y_3(x) = x + \sin x$. This problem is singular the point at x = 0.8 with strong nonlinearity. The approximate solutions at singular point are given in Table 3 for different values of M and k.

Example 5.5. Consider the nonlinear singular system of boundary value problems

$$\begin{cases} y_1''(x) + \dfrac{1}{3x - 2} y_1'(x) + y_2(x) = f_1(x), \\ y_2''(x) + y_1(x)y_2(x) + \dfrac{1}{4x - 3} y_3(x) = f_2(x), \\ y_3''(x) + y_2'(x) + y_4(x) = f_3(x), \\ y_4''(x) + 2y_3'(x) + y_1(x)y_2(x)y_3(x) = f_4(x), \end{cases} \tag{5.5}$$

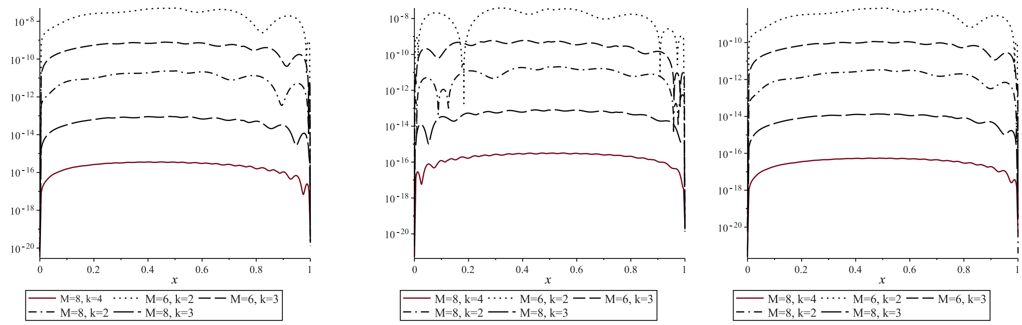

Fig. 2. The logarithm of absolute errors between approximate solutions $\tilde{y_1}(x)$ (left), $\tilde{y_2}(x)$ (middle), $\tilde{y_3}(x)$ (right) and exact solutions for Example 5.3.

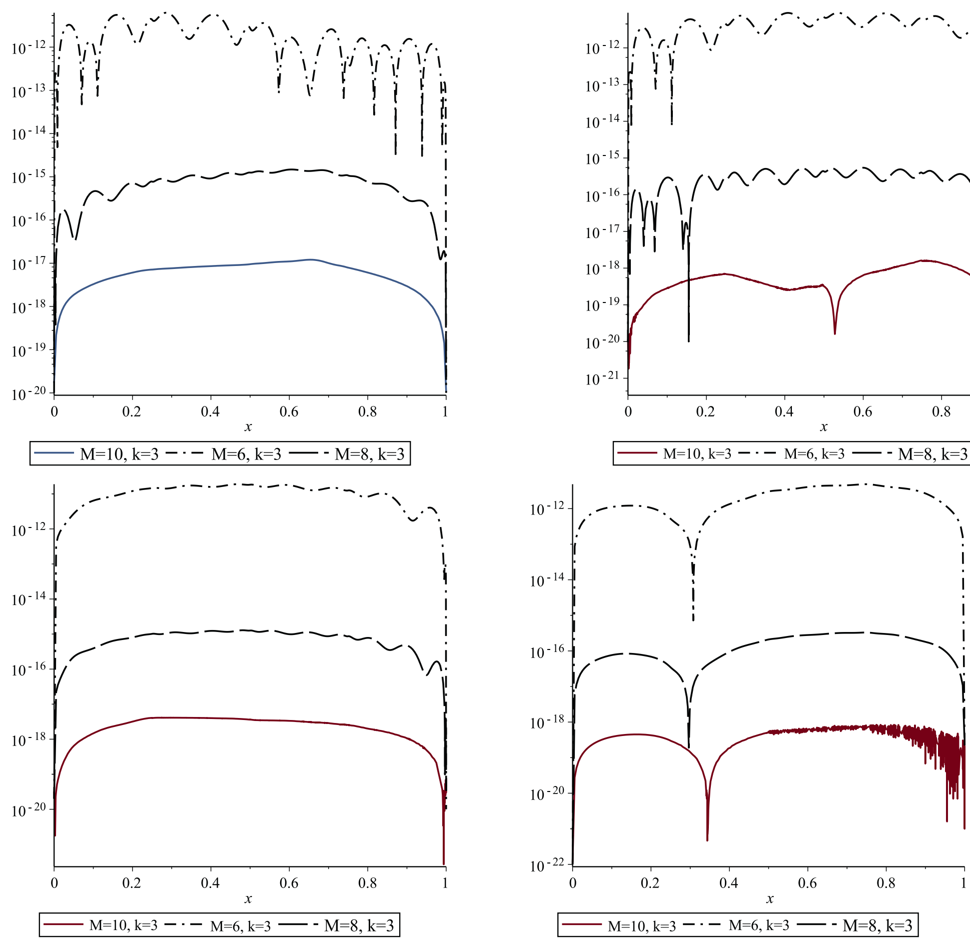

Fig. 3. The logarithm of absolute errors between the approximate solutions $\tilde{y_1}(x)$ (upper left), $\tilde{y_2}(x)$ (upper right), $\tilde{y_3}(x)$ (down left), $\tilde{y_4}(x)$ (down right) and the exact solutions for Example 5.5.

Table 4
Absolute errors for $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ at singular points with $M = 10, k = 3$ for Example 5.5.

M = 10, k = 3	$x = \frac{2}{3}$	$x = \frac{3}{4}$
Absolute errors for $y_1(x)$	1.19×10^{-17}	7.89×10^{-18}
Absolute errors for $y_2(x)$	9.40×10^{-19}	1.55×10^{-18}
Absolute errors for $y_3(x)$	3.02×10^{-18}	2.55×10^{-18}
Absolute errors for $y_4(x)$	6.57×10^{-19}	7.00×10^{-19}

with the following boundary conditions

$$y_1(0) = 1,\ y_1(1) = e^{-1},\quad y_2(0) = 0,$$
$$y_2(1) = \sin 1,\quad y_3(0) = 1,\ y_3(1) = \cos 1,$$
$$y_4(0) = 0,\ y_4(1) = 0,$$

where $f_1(x) = e^{-x} - \frac{e^{-x}}{3x-2} + \sin x$, $f_2(x) = -\sin x + \sin x\, e^{-x} + \frac{\cos x}{4x-3}$, $f_3(x) = x^4 - x^3$, and $f_4(x) = 12x^2 - 6x - 2\sin x + \sin x\, e^{-x}\cos x$. The exact solutions of this problem are $y_1(x) = e^{-x}$, $y_2(x) = \sin x$, $y_3(x) = \cos x$, and $y_4(x) = x^3(x-1)$. This example is solved for different values of M and k. The logarithms of absolute errors in solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ are plotted in Fig. 3. For further research, the values of approximate solutions at singular points are tabulated in Table 4.

6. Conclusion

A Chebyshev wavelet finite difference method was applied for solving higher order systems of nonlinear singular and nonsingular boundary value problems. The main advantages of the current method are simplicity, applicability to a wide variety of problems and the effectiveness of CWFD basis functions in the approximation of smooth and particularly piecewise smooth functions. Numerical examples confirm that we can improve the accuracy of the results either by increasing the values of M or k.

Acknowledgements

The authors would like to thank the anonymous reviewers for their helpful and constructive comments that greatly contributed to improving the final version of the paper.

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