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Original Article

# Oscillation criteria for higher order quasilinear dynamic equations with Laplacians and a deviating argument

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In this paper, we deal with the oscillation of the solutions of the higher order quasilinear dynamic equation with Laplacians and a deviating argument in the form of

$$(x^{[n-1]})^{\Delta}(t) + p(t)\phi_{\gamma}(x(g(t))) = 0$$

on an above-unbounded time scale, where $n \geq 2$,

$$x^{[i]}(t) := r_i(t)\phi_{\alpha_i}\left[\left(x^{[i-1]}\right)^{\Delta}(t)\right], \quad i = 1, 2, \ldots, n-1, \quad \text{with } x^{[0]} = x.$$

By using a generalized Riccati transformation and integral averaging technique, we establish some new oscillation criteria for the cases when $n$ is even and odd, and when $\alpha > \gamma$, $\alpha = \gamma$, and $\alpha < \gamma$, respectively, with $\alpha = \alpha_1 \cdots \alpha_{n-1}$ and without any restrictions on the time scale.



## 1. Introduction

In this paper we study the oscillatory behavior of the higher order quasilinear dynamic equation with Laplacians and a deviating argument

$$(x^{[n-1]})^{\Delta}(t) + p(t)\phi_{\gamma}(x(g(t))) = 0 \tag{1.1}$$

on an above-unbounded time scale $\mathbb{T}$, where

(i) $n \geq 2$ is an integer and $\gamma > 0$;
(ii) $x^{[i]}(t) := r_i(t)\phi_{\alpha_i}\left[\left(x^{[i-1]}\right)^{\Delta}(t)\right]$, $i = 1, 2, \ldots, n-1$, with $x^{[0]} = x$;
(iii) $\phi_{\theta}(u) := |u|^{\theta}\operatorname{sgn} u$ for $\theta > 0$;

Without loss of generality we assume $t_0 \in \mathbb{T}$. For $A \subset \mathbb{T}$ and $B \subset \mathbb{R}$, we denote by $C_{rd}(A, B)$ the space of right-dense continuous functions from $A$ to $B$ and by $C^1_{rd}(A, B)$ the set of functions in $C_{rd}(A, B)$ with right-dense continuous $\Delta-$derivatives, for an excellent introduction to the calculus on time scales, see Bohner and Peterson [1,2]. Throughout this paper we make the following assumptions:

(iv) For $i = 1, 2, \ldots, n-1$, $\alpha_i > 0$ is a constant and $r_i \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ such that

$$\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(\tau)\Delta\tau = \infty; \tag{1.2}$$

(v) $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$ such that $p \not\equiv 0$;
(vi) $g \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\lim_{t\to\infty} g(t) = \infty$ with $g^*(t) := \min\{t, g(t)\}$ is nondecreasing on $[t_0, \infty)_{\mathbb{T}}$.

By a solution of Eq. (1.1) we mean a function $x \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ for some $T_x \geq 0$ such that $x^{[i]} \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$, $i = 1, 2, \ldots, n-1$, which satisfies Eq. (1.1) on $[T_x, \infty)_{\mathbb{T}}$. A solution $x(t)$ of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is nonoscillatory.

Oscillation criteria for higher order dynamic equations on time scales have been studied by many authors. For instance, Grace, Agarwal, and Zafer [3] established oscillation criteria for the higher order nonlinear dynamic equations on general time scales

$$x^{\Delta^n}(t) + p(t)(x^{\sigma}(g(t)))^{\gamma} = 0,$$

where $\gamma$ is ratios of positive odd integers and where $g(t) \leq t$. In [3], some comparison criteria have been obtained when $g(t) \leq t$ and some oscillation criteria are given when $n$ is even and $g(t) = t$.

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The authors in [3] assumed that

$$\int_{t_0}^{\infty}\int_{t}^{\infty}\int_{s}^{\infty} p(u)\Delta u\Delta s\Delta t = \infty. \tag{1.3}$$

Wu et al [4] established Kamanev-type oscillation criteria for the higher order nonlinear dynamic equation

$$\{r_{n-1}(t)[(r_{n-2}(t)(\ldots(r_1(t)x^{\Delta}(t))^{\Delta}\ldots)^{\Delta})^{\Delta}]^{\alpha}\}^{\Delta} + f(t,x(g(t))) = 0,$$

where $\alpha$ is the quotient of odd positive integers, $g:\mathbb{T}\to\mathbb{T}$ with $g(t) > t$ and $\lim_{t\to\infty} g(t) = \infty$ and there there exists a positive rd-continuous function $p(t)$ such that $\frac{f(t,u)}{u^{\alpha}} \geq p(t)$ for $u \neq 0$. Sun et al [5] presented some criteria for oscillation and asymptotic behavior of dynamic equation

$$\{r_{n-1}(t)[(r_{n-2}(t)(\ldots(r_1(t)x^{\Delta}(t))^{\Delta}\ldots)^{\Delta})^{\Delta}]^{\alpha}\}^{\Delta} + f(t,x(g(t))) = 0,$$

where $\alpha \geq 1$ is the quotient of odd positive integers, $g:\mathbb{T}\to\mathbb{T}$ is an increasing differentiable function with $g(t) \leq t$, $g\circ\sigma = \sigma\circ g$ and $\lim_{t\to\infty} g(t) = \infty$ and there there exists a positive rd-continuous function $p(t)$ such that $\frac{f(t,u)}{u^{\beta}} \geq p(t)$ for $u \neq 0$ and $\beta \geq 1$ is the quotient of odd positive integers. Sun et al [6] considered quasilinear dynamic equation of the form

$$\left\{r_{n-1}(t)\left[(r_{n-2}(t)(\ldots(r_1(t)x^{\Delta}(t))^{\Delta}\ldots)^{\Delta})^{\Delta}\right]^{\alpha}\right\}^{\Delta} + p(t)x^{\beta}(t) = 0,$$

where $\alpha$, $\beta$ are the quotient of odd positive integers.

Also, The results obtained in [4–6] are given when

$$\int_{t_0}^{\infty} \frac{1}{r_{n-2}(t)}\left\{\int_t^{\infty}\left[\frac{1}{r_{n-1}(s)}\int_s^{\infty} p(u)\Delta u\right]^{1/\alpha}\Delta s\right\}\Delta t = \infty. \tag{1.4}$$

Hassan and Kong [7] obtained asymptotics and oscillation criteria for the $n$th-order half-linear dynamic equation with deviating argument

$$(x^{[n-1]})^{\Delta}(t) + p(t)\phi_{\alpha[1,n-1]}(x(g(t))) = 0,$$

where $\alpha[1,n-1] := \alpha_1\cdots\alpha_{n-1}$; and Grace and Hassan [8] further studied the asymptotics and oscillation for the higher order nonlinear dynamic equation with Laplacians and deviating argument

$$(x^{[n-1]})^{\Delta}(t) + p(t)\phi_{\gamma}(x^{\sigma}(g(t))) = 0.$$

However, the establishment of the results in [8] requires the restriction on the time scale $\mathbb{T}$ that $g^*\circ\sigma = \sigma\circ g^*$ where $g^*(t) = \min\{t, g(t)\}$ (though it is missed in most places) which is hardly satisfied. For more results on dynamic equations, we refer the reader to the papers [9–14,16,15,17–26].

In this paper, we will discuss the higher order nonlinear dynamic equation (1.1) with Laplacians and deviating argument on a general time scale without any restrictions on $g(t)$ and $\sigma(t)$ and also without the conditions (1.3) and (1.4). Some asymptotics and oscillation criteria will be derived for the cases when $n$ is even and odd, and when $\alpha \geq \gamma$ and $\alpha \leq \gamma$, respectively, with $\alpha = \alpha_1\cdots\alpha_{n-1}$. The results in this paper improve the results in [3–8] on the oscillation of various dynamic equations.

## 2. Main results

We introduce the following notation:

$$\alpha[h,k] := \begin{cases}\alpha_h\cdots\alpha_k & h \leq k,\\ 1, & h > k,\end{cases} \tag{2.1}$$

with $\alpha = \alpha[1,n-1]$. For any $t,s\in\mathbb{T}$ and for a fixed $m \in \{0,1,\ldots,n-1\}$, define the functions $R_{m,j}(t,s)$ and $p_j(t)$, $j = 0,1,\ldots,m$, by the following recurrence formulas:

$$R_{m,j}(t,s) := \begin{cases}1, & j=0,\\ \int_s^t\left[\frac{R_{m,j-1}(\tau,s)}{r_{m-j+1}(\tau)}\right]^{1/\alpha_{m-j+1}}\Delta\tau, & j=1,2,\ldots,m,\end{cases} \tag{2.2}$$

and

$$p_j(t) := \begin{cases}p(t), & j=0,\\ \left[\frac{1}{r_{n-j}(t)}\int_t^{\infty} p_{j-1}(\tau)\Delta\tau\right]^{1/\alpha_{n-j}}, & j=1,2,\ldots,n-1,\end{cases} \tag{2.3}$$

provided the improper integrals involved are convergent.

In order to prove the main results, we need the following lemmas. The first one is an extension of Lemma 2.1 in [7] to the nonlinear Eq. (1.1) with exactly the same proof.

**Lemma 2.1.** *Assume Eq. (1.1) has an eventually positive solution $x(t)$. Then there exists an integer $m \in \{0,1,\ldots,n-1\}$ with $m+n$ odd such that*

$$x^{[k]}(t) > 0 \quad \text{for} \quad k = 0,1,\ldots,m \tag{2.4}$$

*and*

$$(-1)^{m+k}x^{[k]}(t) > 0 \quad \text{for} \quad k = m,m+1,\ldots,n-1 \tag{2.5}$$

*eventually.*

**Remark 2.1.** If $n = 2$ in Lemma 2.1 then $m = 1$, whereas if $n = 3$ then $m = 2$ or $m = 0$.

**Remark 2.2.** If $n \geq 4$ in Lemma 2.1 and

$$\int_{t_0}^{\infty} p_2(\tau)\Delta\tau = \infty, \tag{2.6}$$

then

$$m := \begin{cases}n-1, & if \quad n\in 2\mathbb{N},\\ n-1 or 0, & if \quad n \in 2\mathbb{N}-1.\end{cases} \tag{2.7}$$

**Proof.** From Lemma 2.1 that there exists an integer number $m \in \{0,1,\ldots,n-1\}$ such that (2.4) and (2.5) hold for $t \geq t_1 \in [t_0,\infty)_{\mathbb{T}}$.

(I) $n \in 2\mathbb{N}$. We claim that (2.6) implies that $m = n-1$. In fact, if $1 \leq m \leq n-3$, then for $t \geq t_1$

$$x^{[n]}(t) < 0,\; x^{[n-1]}(t) > 0,\; x^{[n-2]}(t) < 0,\; x^{[n-3]}(t) > 0.$$

Since $x(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$ then for sufficiently large $t_2 \in [t_1,\infty)_{\mathbb{T}}$, we have $x(g(t)) \geq l > 0$ for $t \geq t_2$. It follows that

$$\phi_{\gamma}(x(g(t))) \geq l^{\gamma} \geq L \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}},$$

Eq. (1.1) can be written as

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = p(t)\phi_{\gamma}(x(g(t)) \geq Lp(t) = L\, p_0(t).$$

Integrating the above inequality from $t$ to $v \in [t,\infty)_{\mathbb{T}}$ and letting $v\to\infty$ and using (2.5), we get

$$x^{[n-1]}(t) \geq L\int_t^{\infty} p_0(s)\Delta s,$$

which implies

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq L^{1/\alpha_{n-1}}\left[\frac{1}{r_{n-1}(t)}\int_t^{\infty} p_0(s)\Delta s\right]^{1/\alpha_{n-1}} = L^{1/\alpha_{n-1}}\, p_1(t).$$

By integrating the above inequality from $t$ to $v \in [t,\infty)_{\mathbb{T}}$ and then taking limits as $v\to\infty$ and using the fact $x^{[n-2]} < 0$ eventually, we get

$$-x^{[n-2]}(t) > L^{1/\alpha_{n-1}}\int_t^{\infty} p_1(s)\Delta s,$$

which implies

$$-\left(x^{[n-3]}(t)\right)^{\Delta} > L^{1/\alpha[n-2,n-1]}\left[\frac{1}{r_{n-2}(t)}\int_t^{\infty} p_1(s)\Delta s\right]^{1/\alpha_{n-2}}$$
$$= L^{1/\alpha[n-2,n-1]}p_2(t).$$

Again, integrating the above inequality from $t_2$ to $t \in [t_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(t_2) - x^{[n-3]}(t) \geq L^{1/\alpha[n-2,n-1]}\int_{t_2}^{t} p_2(s)\Delta s.$$

As a result, $\lim_{t\to\infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Thus $1 \leq m \leq n-3$ is false so (recall $m$ is odd in this case) $m = n-1$.

(II) $n \in 2\mathbb{N} - 1$. Thus from Lemma 2.1, we get either $m = 0$ or $m \geq 2$. As shown in Case (I) when $m \geq 2$, (2.6) implies that $m = n-1$. □

**Lemma 2.2.** *(see [27, Lemma 2.3.]) Let $\omega(u) = au - bu^{1+1/\beta}$, where $a$, $u \geq 0$ and $b$, $\beta > 0$. Then*

$$\omega(u) \leq \left(\frac{\beta}{b}\right)^{\beta}\left(\frac{a}{1+\beta}\right)^{1+\beta}.$$

**Lemma 2.3.** *Assume Eq. (1.1) has an eventually positive solution $x(t)$ and $m \in \{0, 1, \ldots, n-1\}$ is given in Lemma 2.1 such that (2.4) and (2.5) hold for $t \in [t_1, \infty)_{\mathbb{T}}$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then the following hold for $t \in (t_1, \infty)_{\mathbb{T}}$:*

(a) for $i = 0, 1, \ldots, m$

$$\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)} \text{ is strictly decreasing;} \tag{2.8}$$

(b) for $i \in \{0, 1, \ldots, m\}$ and $j = 0, 1, \ldots, m-i$

$$x^{[j]}(t) \geq \phi^{-1}_{\alpha[j+1,m-i]}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right]R_{m,m-j}(t,t_1). \tag{2.9}$$

**Proof.**

(a) We show it by induction. From (2.4) and (2.5), (2.8) holds for $i = 0$ since $R_{m,0}(t,t_1) = 1$. Assume (2.8) holds for some $i \in \{0, 1, \ldots, m-1\}$. Then

$$\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)} \text{ is strictly decreasing on } (t_1, \infty)_{\mathbb{T}}.$$

This implies that

$$x^{[m-i-1]}(t) = x^{[m-i-1]}(t_1) + \int_{t_1}^{t}\phi^{-1}_{\alpha_{m-i}}\left[\frac{x^{[m-i]}(\tau)}{r_{m-i}(\tau)}\right]\Delta\tau$$
$$> \int_{t_1}^{t}\phi^{-1}_{\alpha_{m-i}}\left[\frac{x^{[m-i]}(\tau)}{R_{m,i}(\tau,t_1)}\right]\left(\frac{R_{m,i}(\tau,t_1)}{r_{m-i}(\tau)}\right)^{1/\alpha_{m-i}}\Delta\tau$$
$$> \phi^{-1}_{\alpha_{m-i}}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right]\int_{t_1}^{t}\left(\frac{R_{m,i}(\tau,t_1)}{r_{m-i}(\tau)}\right)^{1/\alpha_{m-i}}\Delta\tau$$
$$= \phi^{-1}_{\alpha_{m-i}}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right]R_{m,i+1}(t,t_1).$$

Since

$$\left[\frac{x^{[m-i-1]}(t)}{R_{m,i+1}(t,t_1)}\right]^{\Delta}$$
$$= \frac{R_{m,i+1}(t,t_1)\left(x^{[m-i-1]}(t)\right)^{\Delta} - (R_{m,i+1}(t,t_1))^{\Delta}x^{[m-i-1]}(t)}{R_{m,i+1}(t,t_1)R_{m,i+1}(\sigma(t),t_1)}$$
$$= \frac{(R_{m,i}(t,t_1)/r_{m-i}(t))^{1/\alpha_{m-i}}}{R_{m,i+1}(t,t_1)R_{m,i+1}(\sigma(t),t_1)}\cdot$$
$$\left[R_{m,i+1}(t,t_1)\phi^{-1}_{\alpha_{m-i}}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right] - x^{[m-i-1]}(t)\right] < 0,$$

then $\frac{x^{[m-i-1]}(t)}{R_{m,i+1}(t,t_1)}$ is strictly decreasing on $(t_1, \infty)_{\mathbb{T}}$. This shows that (2.8) holds for $i+1$. Therefore, (2.8) holds for all $i = 0, 1, \ldots, m$.

(b) We show it by a backward induction. Note from (2.1) that $\phi^{-1}_{\alpha[m-i+1,m-i]} = I$, the identity operator, then (2.9) holds for $j = m-i$ as an identity. Assume (2.9) holds for some $j \in \{1, 2, \ldots, m-i\}$. Then for $t \in (t_1, \infty)_{\mathbb{T}}$,

$$\left[x^{[j-1]}(t)\right]^{\Delta} = \phi^{-1}_{\alpha_j}\left[\frac{x^{[j]}(t)}{r_j(t)}\right]$$
$$> \phi^{-1}_{\alpha[j,m-i]}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right]\left(\frac{R_{m,m-j}(t,t_1)}{r_j(t)}\right)^{1/\alpha_j}.$$

Replacing $t$ by $\tau$ in the above, integrating it from $t_1$ to $t \in (t_1, \infty)_{\mathbb{T}}$, and then using Part (a), we have

$$x^{[j-1]}(t) > x^{[j-1]}(t) - x^{[j-1]}(t_1)$$
$$\geq \int_{t_1}^{t}\phi^{-1}_{\alpha[j,m-i]}\left[\frac{x^{[m-i]}(\tau)}{R_{m,i}(\tau,t_1)}\right]\left(\frac{R_{m,m-j}(\tau,t_1)}{r_j(\tau)}\right)^{1/\alpha_j}\Delta\tau$$
$$> \phi^{-1}_{\alpha[j,m-i]}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right]\int_{t_1}^{t}\left(\frac{R_{m,m-j}(\tau,t_1)}{r_j(\tau)}\right)^{1/\alpha_j}\Delta\tau$$
$$= \phi^{-1}_{\alpha[j,m-i]}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right]R_{m,m-j+1}(t,t_1).$$

This shows that (2.9) holds for $j-1$. Therefore, (2.9) holds for all $j = 0, 1, \ldots, m-i$. □

**Lemma 2.4.** *Assume Eq. (1.1) has an eventually positive solution $x(t)$ and $m$ is given in Lemma 2.1 such that $m \in \{1, 2, \ldots, n-1\}$, (2.4) and (2.5) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then for $t \in [t_1, \infty)_{\mathbb{T}}$ and $j = m, m+1, \ldots, n-1$,*

$$\int_t^{\infty} p_{n-j-1}(\tau)\Delta\tau < \infty; \tag{2.10}$$

*and*

$$(-1)^{m+j}x^{[j]}(t) \geq \phi^{-1}_{\alpha[j+1,n-1]}\{\phi_\gamma(x(g^*(t)))\}\int_t^{\infty} p_{n-j-1}(\tau)\Delta\tau. \tag{2.11}$$

**Proof.** We show it by a backward induction. By Lemma 2.1 with $m \geq 1$ we see that $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Hence from (1.1) we have that for $t \in [t_2, \infty)_{\mathbb{T}}$

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = p(t)\phi_\gamma(x(g(t))) \geq p_0(t)\ \phi_\gamma(x(g^*(t))). \tag{2.12}$$

Replacing $t$ by $\tau$ in (1.1), integrating from $t \in [t_1, \infty)_{\mathbb{T}}$ to $s \in [t, \infty)_{\mathbb{T}}$, and using (2.5), we have

$$x^{[n-1]}(t) > -x^{[n-1]}(s) + x^{[n-1]}(t) \geq \int_t^{s} p_0(\tau)\ \phi_\gamma(x(g^*(\tau)))\Delta\tau$$
$$\geq \phi_\gamma(x(g^*(t)))\int_t^{s} p_0(\tau)\Delta\tau.$$

By Lemma 2.1 with $m \geq 1$ we see that $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$ and Hence by taking limits as $s \to \infty$ we obtain that

$$x^{[n-1]}(t) \geq \phi_\gamma(x(g^*(t)))\int_t^{\infty} p_0(\tau)\Delta\tau.$$

This shows that $\int_t^{\infty} p_0(\tau)\Delta\tau < \infty$ and (2.11) holds for $j = n-1$. Assume $\int_t^{\infty} p_{n-j-1}(\tau)\Delta\tau < \infty$ and (2.11) holds for some $j \in \{m+1, m+2, \ldots, n-1\}$. Then for (2.11)

$$(-1)^{m+j}\left[x^{[j-1]}(t)\right]^{\Delta}$$

$$= (-1)^{m+j}\phi_{\alpha_j}^{-1}\left[\frac{x^{[j]}(t)}{r_j(t)}\right]$$
$$\geq \phi_{\alpha[j,n-1]}^{-1}\left\{\phi_\gamma(x(g^*(t)))\right\}\left[\frac{1}{r_j(t)}\int_t^\infty p_{n-j-1}(\tau)\Delta\tau\right]^{1/\alpha_j}$$
$$= \phi_{\alpha[j,n-1]}^{-1}\left\{\phi_\gamma(x(g^*(t)))\right\}\ p_{n-j}(t).$$

Replacing $t$ by $\tau$ in the above and then integrating it from $t\in[t_2,\infty)_{\mathbb{T}}$ to $s\in[t,\infty)_{\mathbb{T}}$, we have

$$(-1)^{m+j-1}x^{[j-1]}(t) > (-1)^{m+j}(x^{[j-1]}(s)-x^{[j-1]}(t))$$
$$\geq \int_t^s \phi_{\alpha[j,n]}^{-1}\left\{\phi_\gamma(x(g^*(\tau)))\right\}p_{n-j}(\tau)\ \Delta\tau$$
$$\geq \phi_{\alpha[j,n]}^{-1}\left\{\phi_\gamma(x(g^*(t)))\right\}\int_t^s p_{m,n-j}(\tau)\ \Delta\tau.$$

Taking limits as $s\to\infty$ we obtain that

$$(-1)^{m+j-1}x^{[j-1]}(t) \geq \phi_{\alpha[j,n]}^{-1}\left\{\phi_\gamma(x(g^*(t)))\right\}\int_t^\infty p_{n-j}(\tau)\ \Delta\tau.$$

This shows that $\int_t^\infty p_{n-j}(\tau)\ \Delta\tau<\infty$ and (2.11) holds for $j-1$. Therefore, the conclusion holds. □

In the sequel, we will present conditions which guarantee the following conclusions hold:

**(C)** (i) every solution of Eq. (1.1) is oscillatory if $n$ is even;
(ii) every solution of Eq. (1.1) either is oscillatory or tends to zero eventually if $n$ is odd.

We may now state and prove our main results. In these, we will consider the two cases $\alpha\geq\gamma$ and $\alpha\leq\gamma$.

**Theorem 2.1.** *Let $\alpha\geq\gamma$. Assume for each $i\in\{1,2,\ldots,n-1\}$ and sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$ there exists a $\rho_i\in C_{rd}^1([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and a constant $k_i>0$ such that*

$$\limsup_{t\to\infty}\int_{T_1}^t\left[\rho_i(\tau)P_i(\tau,T) -\left(\frac{\alpha}{\gamma\psi_i(\tau,T)}\right)^{\alpha[1,i]}\left[\frac{(\rho_i^\Delta(\tau))_+}{\alpha[1,i]+1}\right]^{\alpha[1,i]+1}\right]\Delta\tau=\infty, \tag{2.13}$$

*where $g^*(t)>T$ for $t\geq T_1$ and*

$$P_i(\tau,T):=p_{n-i-1}(\tau)\left[\frac{R_{i,i}(g^*(\tau),T)}{R_{i,i}(\tau,T)}\right]^{\gamma_i}, \tag{2.14}$$

*and*

$$\psi_i(\tau,T):=\frac{\rho_i(\tau)}{k_i^{1-\gamma/\alpha}\ R_{i,i}^{1-\gamma/\alpha}(\sigma(\tau),T)}\left[\frac{R_{i,i-1}(\tau,T)}{r_1(\tau)}\right]^{1/\alpha_1}. \tag{2.15}$$

*with $\gamma_i:=\dfrac{\gamma}{\alpha[i+1,n-1]}$. Moreover, for the case when $n$ is odd, assume*

$$\int_T^\infty p_{n-1}(\tau)\ \Delta\tau=\infty. \tag{2.16}$$

*Then conclusions (C) hold.*

**Proof.** Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t)>0$ and $x(g(t))>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an integer $m\in\{0,1,\ldots,n-1\}$ with $m+n$ odd such that (2.4) and (2.5) hold for $t\in[t_1,\infty)_{\mathbb{T}}$ for some $t_1\in[t_0,\infty)_{\mathbb{T}}$.

(i) Assume $m\geq1$. Define

$$w_m(t):=\rho_m(t)\frac{x^{[m]}(t)}{x^{\gamma_m}(t)}. \tag{2.17}$$

By the product rule and the quotient rule we have

$$w_m^\Delta(t)=\frac{\rho_m(t)}{x^{\gamma_m}(t)}\left(x^{[m]}(t)\right)^\Delta+\left(\frac{\rho_m(t)}{x^{\gamma_m}(t)}\right)^\Delta\left(x^{[m]}(t)\right)^\sigma$$
$$=\rho_m(t)\frac{\left(x^{[m]}(t)\right)^\Delta}{x^{\gamma_m}(t)}+\left[\frac{\rho_m^\Delta(t)}{(x^{\gamma_m}(t))^\sigma}-\frac{\rho_m(t)(x^{\gamma_m}(t))^\Delta}{x^{\gamma_m}(t)(x^{\gamma_m}(t))^\sigma}\right]\times\left(x^{[m]}(t)\right)^\sigma.$$

Using the definition of $w_m(t)$ we obtain

$$w_m^\Delta(t)=\rho_m(t)\frac{\left(x^{[m]}(t)\right)^\Delta}{x^{\gamma_m}(t)}+\rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma -\rho_m(t)\frac{(x^{\gamma_m}(t))^\Delta}{x^{\gamma_m}(t)}\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma. \tag{2.18}$$

From Lemma 2.4 with $j=m+1$, we have

$$-x^{[m+1]}(t)\geq\phi_{\alpha[m+2,n-1]}^{-1}\left\{\phi_\gamma(x(g^*(t)))\right\}\int_t^\infty p_{n-m-2}(\tau)\ \Delta\tau. \tag{2.19}$$

which, together with (2.3), implies that for $t\in[t_1,\infty)_{\mathbb{T}}$,

$$-\left(x^{[m]}(t)\right)^\Delta\geq\phi_{\alpha[m+1,n-1]}^{-1}\left\{\phi_\gamma(x(g^*(t)))\right\}\times\left[\frac{1}{r_{m+1}(t)}\int_t^\infty p_{n-m-2}(\tau)\ \Delta\tau\right]^{1/\alpha_{m+1}}$$
$$=\phi_{\gamma_m}(x(g^*(t)))\ p_{n-m-1}(t). \tag{2.20}$$

Subtituting (2.20) into (2.18) we obtain

$$w_m^\Delta(t)\leq-\rho_m(t)p_{n-m-1}(t)\left[\frac{x(g^*(t))}{x(t)}\right]^{\gamma_m}+\rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma -\rho_m(t)\frac{(x^{\gamma_m}(t))^\Delta}{x^{\gamma_m}(t)}\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma.$$

Let $t\in[t_1,\infty)_{\mathbb{T}}$ be fixed. In view of Lemma 2.3, Part (a) we see that for $i=m$,

$$x(g^*(t))\geq\frac{R_{m,m}(g^*(t),t_1)}{R_{m,m}(t,t_1)}x(t)\quad\text{ for } t\in[t_2,\infty)_{\mathbb{T}},$$

where $g^*(t)>t_1$ for $t\geq t_2$. From the definition of $P_m(t,t_1)$ we have that for $t\in[t_2,\infty)_{\mathbb{T}}$,

$$w_m^\Delta(t)\leq-\rho_m(t)P_m(t,t_1)+\rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma -\rho_m(t)\frac{(x^{\gamma_m}(t))^\Delta}{x^{\gamma_m}(t)}\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma.$$

By the Pötzsche chain rule ([1, Theorem 1.90]) we obtain

$$(x^{\gamma_m}(t))^\Delta=\gamma_m\int_0^1[(1-h)x(t)+h\,x^\sigma(t)]^{\gamma_m-1}dh\ x^\Delta(t)$$
$$\geq\begin{cases}\gamma_m\ [x^\sigma(t)]^{\gamma_m-1}x^\Delta(t), & 0<\gamma_m\leq1,\\ \gamma_m\ [x(t)]^{\gamma_m-1}x^\Delta(t), & \gamma_m\geq1.\end{cases}$$

If $0<\gamma_m\leq1$, we have

$$w_m^\Delta(t)\leq-\rho_m(t)P_m(t,t_1)+\rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma -\gamma_m\rho_m(t)\frac{x^\Delta(t)}{x^\sigma(t)}\left[\frac{x^\sigma(t)}{x(t)}\right]^{\gamma_m}\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma;$$

and if $\gamma_m \geq 1$, we have

$$w_m^\Delta(t) \leq -\rho_m(t)P_m(t,t_1) + \rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\rho_m(t)\frac{x^\Delta(t)}{x^\sigma(t)}\frac{x^\sigma(t)}{x(t)}\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma.$$

Using the fact that $x^\Delta(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ we see that for $\gamma_m > 0$,

$$w_m^\Delta(t) \leq -\rho_m(t)P_m(t,t_1) + \rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\rho_m(t)\frac{x^\Delta(t)}{x^\sigma(t)}\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma. \tag{2.21}$$

By using Lemma 2.3, Part (b) with $i = 0$ and $j = 1$ we see that

$$x^{[1]}(t) \geq \phi^{-1}_{\alpha[2,m]}\left(x^{[m]}(t)\right)R_{m,m-1}(t,t_1)$$

which implies

$$x^\Delta(t) \geq \phi^{-1}_{\alpha[1,m]}\left(x^{[m]}(t)\right)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} \geq \phi^{-1}_{\alpha[1,m]}\left(\left[x^{[m]}(t)\right]^\sigma\right)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} = \phi^{-1}_{\alpha[1,m]}\left(\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma\right)(x^\sigma(t))^{\gamma/\alpha}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}. \tag{2.22}$$

Then, from (2.21) and (2.22), we get

$$w_m^\Delta(t) \leq -\rho_m(t)P_m(t,t_1) + \rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\rho_m(t)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma\right]^{1+1/\alpha[1,m]} \times \frac{1}{(x^\sigma(t))^{1-\gamma/\alpha}}. \tag{2.23}$$

By (2.5) there is a positive constant $k_0$ such that

$$x^{[m]}(t) \leq k_0 = k_0 R_{m,0}(t,t_1) \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

which gives

$$\left(x^{[m-1]}(t)\right)^\Delta \leq k_0^{1/\alpha_m}\left[\frac{R_{m,0}(t,t_1)}{r_m(t)}\right]^{1/\alpha_m} \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Integrating the above from $t_2$ to $t \in [t_2,\infty)_{\mathbb{T}}$ we have

$$x^{[m-1]}(t) \leq x^{[m-1]}(t_2) + k_0^{1/\alpha_m}\int_{t_2}^t\left[\frac{R_{m,0}(\tau,t_1)}{r_m(\tau)}\right]^{1/\alpha_m}\Delta\tau = x^{[m-1]}(t_2) + k_0^{1/\alpha_m}R_{m,1}(t,t_1).$$

Hence from (1.2), there exists a positive constant $k_1$ such that for $t \in [t_2,\infty)_{\mathbb{T}}$,

$$x^{[m-1]}(t) \leq R_{m,1}(t,t_1)\left(k_0^{1/\alpha_m} + \frac{x^{[m-1]}(t_2)}{R_{m,1}(t,t_1)}\right) \leq k_1 R_{m,1}(t,t_1).$$

Continuing this process, we obtain for some a positive constant $k_m$,

$$x(t) \leq k_m R_{m,m}(t,t_1) \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Since $\gamma < \alpha$ we have

$$(x^\sigma(t))^{1-\gamma/\alpha} \leq k_m^{1-\gamma/\alpha}\, R_{m,m}^{1-\gamma/\alpha}(\sigma(t),t_1) \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \tag{2.24}$$

Substituting (2.24) into (2.23) and using the definition of $\psi_m(t,t_1)$ we get

$$w_m^\Delta(t) \leq -\rho_m(t)P_m(t,t_1) + \rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\psi_m(t,t_1)\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma\right]^{1+1/\alpha[1,m]} \leq -\rho_m(t)P_m(t,t_1) + (\rho_m^\Delta(t))_+\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\psi_m(t,t_1)\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma\right]^{1+1/\alpha[1,m]}. \tag{2.25}$$

Using Lemma 2.2 with

$$a := (\rho_m^\Delta(t))_+,\ \ b := \gamma_m\psi_m(t,t_1), \quad \beta := \alpha[1,m] \text{ and } u := \left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma,$$

we obtain

$$(\rho_m^\Delta(t))_+\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\psi_m(t,t_1)\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma\right]^{1+1/\alpha[1,m]} \leq \frac{(\alpha[1,m])^{\alpha[1,m]}}{(\alpha[1,m]+1)^{\alpha[1,m]+1}}\frac{\left((\rho_m^\Delta(t))_+\right)^{\alpha[1,m]+1}}{\gamma_m^{\alpha[1,m]}(\psi_m(t,t_1))^{\alpha[1,m]}} = \left(\frac{\alpha}{\gamma\psi_m(t,t_1)}\right)^{\alpha[1,m]}\left[\frac{(\rho_m^\Delta(t))_+}{\alpha[1,m]+1}\right]^{\alpha[1,m]+1}.$$

From this and (2.25) we have

$$w_m^\Delta(t) \leq -\rho_m(t)P_m(t,t_1) + \left(\frac{\alpha}{\gamma\psi_m(t,t_1)}\right)^{\alpha[1,m]}\left[\frac{(\rho_m^\Delta(t))_+}{\alpha[1,m]+1}\right]^{\alpha[1,m]+1}.$$

Integrating both sides from $t_2$ to $t$ we get

$$\int_{t_2}^t\left[\rho_m(\tau)P_m(\tau,t_1) - \left(\frac{\alpha}{\gamma\psi_m(\tau,t_1)}\right)^{\alpha[1,m]}\left[\frac{(\rho_m^\Delta(\tau))_+}{\alpha[1,m]+1}\right]^{\alpha[1,m]+1}\right]\Delta\tau \leq w_m(t_2) - w_m(t) \leq w_m(t_2),$$

which contradicts (2.13).

(ii) We show that if $m = 0$, then $\lim_{t\to\infty}x(t) = 0$. In fact, from Lemma 2.1 we see that it is only possible when $n$ is odd. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k = 0, 1, \ldots, n.$$

This implies that $x(t)$ is positive and strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty}x(t) = l \geq 0$. Assume $l > 0$. Then $x(g(t)) \geq l$ for $t \in [t_2,\infty)_{\mathbb{T}}$. It follows that

$$\phi_\gamma(x(g(t))) \geq l^\gamma =: L \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Then from (1.1), we obtain

$$-\left(x^{[n-1]}(t)\right)^\Delta = p(t)\phi_\gamma(x(g(t))) \geq L\ p(t) = L\ p_0(t).$$

Integrating the above from $t$ to $s \in [t,\infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(s) + x^{[n-1]}(t) \geq L\int_t^s p_0(\tau)\Delta\tau,$$

and by (2.5) we see that $x^{[n-1]}(s) > 0$. Hence by taking limits as $v \to \infty$ we have

$$x^{[n-1]}(t) \geq L \int_t^\infty p_0(\tau)\Delta\tau$$

which implies

$$\left(x^{[n-2]}(t)\right)^\Delta \geq L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^\infty p_0(\tau)\Delta\tau\right]^{1/\alpha_{n-1}} = L^{1/\alpha_{n-1}}\ p_1(t).$$

Integrating the above from $t$ to $s \in [t, \infty)_{\mathbb{T}}$ and letting $s \to \infty$, by (2.5) we get

$$-x^{[n-2]}(t) \geq L^{1/\alpha_{n-1}} \int_t^\infty p_1(\tau)\ \Delta\tau.$$

Continuing this process, we get

$$-x^{[1]}(t) \geq L^{1/\alpha[2,n-1]} \int_t^\infty p_{n-2}(\tau)\ \Delta\tau$$

which implies that

$$-x^\Delta(t) \geq L^{1/\alpha[1,n-1]} \left[\frac{1}{r_1(t)} \int_t^\infty p_{n-2}(\tau)\Delta\tau\right]^{1/\alpha_1} = L^{1/\alpha[1,n-1]}\ p_{n-1}(t).$$

Again integrating the above from $t_2$ to $t \in [t_2, \infty)_{\mathbb{T}}$, we get

$$-x(t) + x(t_2) \geq L^{1/\alpha[1,n-1]} \int_{t_2}^t p_{n-1}(\tau)\ \Delta\tau.$$

Hence by (2.16), $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the assumption that $x(t) > 0$ eventually. This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

**Theorem 2.2.** *Let $\alpha \leq \gamma$. Assume for each $i \in \{1, 2, \ldots, n-1\}$ and sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ there exists a $\rho_i \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ and a constant $C > 0$ such that*

$$\limsup_{t\to\infty} \int_{T_1}^t \left[\rho_i(\tau)P_i(\tau, T) - \left(\frac{\alpha}{\gamma\bar{\psi}_i(\tau, T)}\right)^{\alpha[1,i]} \left[\frac{(\rho_i^\Delta(\tau))_+}{\alpha[1,i]+1}\right]^{\alpha[1,i]+1}\right]\Delta\tau = \infty, \tag{2.26}$$

*where $g^*(t) > T$ for $t \geq T_1$, and $P_i(\tau, T)$ is defined by (2.14) and*

$$\bar{\psi}_i(\tau, T) := \rho_i(\tau)C^{\gamma/\alpha-1}\left[\frac{R_{i,i-1}(\tau, T)}{r_1(\tau)}\right]^{1/\alpha_1}. \tag{2.27}$$

*Moreover, for the case when n is odd, assume (2.16) holds. Then conclusions (C) hold.*

**Proof.** Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an integer $m \in \{0, 1, \ldots, n-1\}$ with $m + n$ odd such that (2.4) and (2.5) hold for $t \in [t_1, \infty)_{\mathbb{T}}$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(i) Assume $m \geq 1$. Proceeding as in the proof of Theorem 2.1 we get that (2.23) holds. i.e.,

$$w_m^\Delta(t) \leq -\rho_m(t)P_m(t, t_1) + \rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\rho_m(t)\left[\frac{R_{m,m-1}(t, t_1)}{r_1(t)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma\right]^{1+1/\alpha[1,m]} \times (x^\sigma(t))^{\gamma/\alpha-1}.$$

Since $\alpha \leq \gamma$ and $x(t)$ is increasing on $[t_1, \infty)_{\mathbb{T}}$, then $x^\sigma(t) \geq x^\sigma(t_1)$ for $t \in [t_2, \infty)_{\mathbb{T}}$ and so

$$(x^\sigma(t))^{\gamma/\alpha-1} \geq (x^\sigma(t_1))^{\gamma/\alpha-1} =: C^{\gamma/\alpha-1} > 0 \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}},$$

where $g^*(t) > t_1$ for $t \geq t_2$. Consequently,

$$w_m^\Delta(t) \leq -\rho_m(t)P_m(t, t_1) + \rho_m^\Delta(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma - \gamma_m\bar{\psi}_m(t, t_1)\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^\sigma\right]^{1+1/\alpha[1,m]}.$$

The rest of proof is similar to the first case of Theorem 2.1 and hence can be omitted.

(ii) With essentially the same proof as that of Theorem 2.1, Part (ii), we can show that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. We omit the details. □

**Example 2.1.** Consider the higher order nonlinear dynamic equation

$$(x^{[n-1]})^\Delta(t) + \frac{1}{t^{\alpha+1}}\phi_\gamma(x(g(t))) = 0, \quad g(t) \geq t, \tag{2.28}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $n \geq 2$ is odd, $\alpha_i \geq 1$, $\alpha \leq \gamma$ and

$$r_1(t) := \frac{1}{\alpha_1}, \quad r_i(t) := \frac{t^{\alpha_i}}{\alpha[1,i]}, \quad i = 2, \ldots, n-1$$

It is clear that conditions (1.2) hold, since

$$\int_{t_0}^\infty r_1^{-1/\alpha_1}(\tau)\Delta\tau = \alpha_1^{1/\alpha_1}\int_{t_0}^\infty \Delta\tau = \infty,$$

and

$$\int_{t_0}^\infty r_i^{-1/\alpha_i}(\tau)\Delta\tau = (\alpha[1,i])^{1/\alpha_i}\int_{t_0}^\infty \frac{\Delta\tau}{\tau} = \infty, \quad i = 2, \ldots, n-1,$$

by [1, Example 5.60]. Also,

$$p_0(t) = \frac{\beta}{t^{\alpha+1}} = \frac{\beta}{t^{\alpha[1,n-1]+1}}.$$

By the Pötzsche chain rule, we get

$$\begin{aligned} p_1(t) &= \left[\frac{1}{r_{n-1}(t)}\int_t^\infty p_0(\tau)\Delta\tau\right]^{1/\alpha_{n-1}} \\ &= \beta^{1/\alpha_{n-1}}\left[\frac{\alpha[1,n-1]}{t^{\alpha_{n-1}}}\int_t^\infty \frac{1}{\tau^{\alpha[1,n-1]+1}}\Delta\tau\right]^{1/\alpha_{n-1}} \\ &\geq \beta^{1/\alpha_{n-1}}\left[\frac{1}{t^{\alpha_{n-1}}}\int_t^\infty \left(\frac{-1}{\tau^{\alpha[1,n-1]}}\right)^\Delta \Delta\tau\right]^{1/\alpha_{n-1}} \\ &= \frac{\beta^{1/\alpha_{n-1}}}{t^{\alpha[1,n-2]+1}} = \frac{\beta^{1/\alpha[n-1,n-1]}}{t^{\alpha[1,n-2]+1}}. \end{aligned}$$

It is easy to see that

$$p_i(t) \geq \frac{\beta^{1/\alpha[n-i,n-1]}}{t^{\alpha[1,n-i-1]+1}}, \quad i = 0, 1, \ldots, n-2,$$

and

$$\begin{aligned} p_{n-1}(t) &= \left[\frac{1}{r_1(t)}\int_t^\infty p_{n-2}(\tau)\Delta\tau\right]^{1/\alpha_1} \\ &\geq \beta^{1/\alpha}\left[\alpha_1\int_t^\infty \frac{1}{\tau^{\alpha[1,1]+1}}\Delta\tau\right]^{1/\alpha_1} \\ &\geq \beta^{1/\alpha}\left[\int_t^\infty \left(\frac{-1}{\tau^{\alpha_1}}\right)^\Delta \Delta\tau\right]^{1/\alpha_1} = \frac{\beta^{1/\alpha}}{t}. \end{aligned}$$

Hence

$$\int_T^\infty p_{n-1}(\tau)\ \Delta\tau = \beta^{1/\alpha}\int_T^\infty \frac{\Delta\tau}{\tau} = \infty,$$

so that condition (2.16) holds. Therefore, we can find $T_1 \geq T$ such that $R_{i,i-1}(t, T) \geq 1$ and $g(t) > T$ for $t \geq T_1$. Let us take $\rho_i(t) = t^{\alpha[1,i]}$, then, by the Pö tzsche chain rule

$$\rho_i^\Delta(t) = \left(t^{\alpha[1,i]}\right)^\Delta = \alpha[1,i]\int_0^1 (t + h\mu(t))^{\alpha[1,i]-1}dh$$

$$\leq \alpha[1,i](\sigma(t))^{\alpha[1,i]-1}.$$

Now, we assume $\mathbb{T}$ is a time scale satisfying $\sigma(t) \leq Kt$, for some $K > 0$, $t \geq T_K > T_1$. Now

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left[ \rho_i(\tau) P_i(\tau, T) - \left( \frac{\alpha}{\gamma \bar{\psi}_i(\tau, T)} \right)^{\alpha[1,i]} \left[ \frac{(\rho_i^{\Delta}(\tau))_+}{\alpha[1,i]+1} \right]^{\alpha[1,i]+1} \right] \Delta\tau$$
$$\geq \limsup_{t\to\infty} \int_{T_K}^{t} \left[ \rho_i(\tau) P_i(\tau, T) - \left( \frac{\alpha}{\gamma \bar{\psi}_i(\tau, T)} \right)^{\alpha[1,i]} \left[ \frac{(\rho_i^{\Delta}(\tau))_+}{\alpha[1,i]+1} \right]^{\alpha[1,i]+1} \right] \Delta\tau$$
$$\geq \limsup_{t\to\infty} \int_{T_K}^{t} \left[ \frac{\beta^{1/\alpha[i+1,n-1]}}{\tau} - \left( \frac{\alpha}{\gamma} \right)^{\alpha[1,i]} \left[ \frac{\alpha[1,i]}{\alpha[1,i]+1} \right]^{\alpha[1,i]+1} \frac{K^{(\alpha[1,i])^2-1}}{(C^{\gamma/\alpha-1})^{\alpha[1,i]}} \frac{1}{\tau} \right] \Delta\tau$$
$$\geq \left[ \beta^{1/\alpha[i+1,n-1]} - \left( \frac{\alpha}{\gamma} \right)^{\alpha[1,i]} \left[ \frac{\alpha[1,i]}{\alpha[1,i]+1} \right]^{\alpha[1,i]+1} \frac{K^{(\alpha[1,i])^2-1}}{(C^{\gamma/\alpha-1})^{\alpha[1,i]}} \right]$$
$$\limsup_{t\to\infty} \int_{T_K}^{t} \frac{\Delta\tau}{\tau} = \infty,$$

if

$$\beta^{1/\alpha[i+1,n-1]} > \left( \frac{\alpha}{\gamma} \right)^{\alpha[1,i]} \left[ \frac{\alpha[1,i]}{\alpha[1,i]+1} \right]^{\alpha[1,i]+1} \frac{K^{(\alpha[1,i])^2-1}}{(C^{\gamma/\alpha-1})^{\alpha[1,i]}}$$

and hence (2.26) holds. We conclude that if $[T,\infty)_{\mathbb{T}}$ is a time scale interval where $\sigma(t) \leq Kt$, for some $K > 0$, $t \geq T_K$, then, by Theorem 2.2, every solution of (2.28) is oscillatory or tends to zero if

$$\beta^{1/\alpha[i+1,n-1]} > \left( \frac{\alpha}{\gamma} \right)^{\alpha[1,i]} \left[ \frac{\alpha[1,i]}{\alpha[1,i]+1} \right]^{\alpha[1,i]+1} \frac{K^{(\alpha[1,i])^2-1}}{(C^{\gamma/\alpha-1})^{\alpha[1,i]}}.$$

**Remark 2.3.** When $n$ is odd, if the assumption (2.16) is not satisfied, we have some sufficient conditions which ensure that every solution $x$ of (1.1) oscillates or $\lim_{t\to\infty} x(t)$ exists (finite).

**Remark 2.4.** By using Remarks 2.1 and 2.2 we get the further oscillation criteria for Eq. (1.1), see [7].

## 3. Conclusions

1. In this paper, some oscillation criteria are presented that can be applied to gerenerlized quasilinear dynamic Eq. (2.16) for both cases $g(t) \geq t$ and $g(t) \leq t$. Contrary to [3–6] we do not need to assume restrictive conditions (1.3) and (1.4) and do not impose restrctive condition on time scale $\mathbb{T}$ as in [5,8] in our oscillation results.
2. Our results extend and improve related contributions to second and third orders dynamic equations; see the following results.
   (a) When $n = 2$. In this case $i = 1$:
      (1) Let $\alpha_1 = \gamma > 1$ be an odd number and $g(t) = t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 reduces to [28, Theorem 3.1];
      (2) Let $\alpha_1 = \gamma$ be a quotient of odd positive integers and $g(t) \leq t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 improves the results in [29,30] since the conditions $r_1^{\Delta}(t) \geq 0$ and $\int_{t_0}^{\infty} q(\tau) g^{\gamma}(\tau)\Delta\tau = \infty$ are not needed;
      (3) Let $\alpha_1 = \gamma$ be a quotient of odd positive integers and $g(t) \leq t$ on $[t_0, \infty)$, then Theorems 2.1 and 2.2 reduces to [31, Theorem 2.1].
   (b) When $n = 3$. In this case $i = 2$:
      (1) Let $\alpha_1 = \alpha_2 = \gamma = 1$ and $g(t) = t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 reduces to [32, Theorem 1];
      (2) Let $\alpha_1 = 1$, $\alpha_2 = \gamma \geq 1$ be a quotient of odd positive integers and $g(t) = t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 reduces to[33, Theorem 1];
      (3) Let $\alpha = \gamma = 1$ and $g(t) = t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 reduces to[34, Theorem 2.1];
      (4) Let $\alpha_1 = 1$, $\alpha_2 = \gamma$ be a quotient of odd positive integers and $g(t) \leq t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 improves the results in [35] since the condition $\sigma \circ g = g \circ \sigma$ is not needed;
      (5) Let $\alpha_1 = \alpha_2 = \gamma = 1$ and $g(t) \leq t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 improves the results in [36] since the conditions $r_1^{\Delta}(t) < 0$ and $\int_{t_0}^{\infty} q(\tau) g(\tau)\Delta\tau = \infty$ are not needed;
      (6) Let $\alpha_1 = 1$, $\alpha_2 = \gamma$ be a quotient of odd positive integers and $g(t) \leq t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 improves the results in [37] since the conditions $r_1^{\Delta}(t) < 0$ and $\int_{t_0}^{\infty} q(\tau) g^{\gamma}(\tau)\Delta\tau = \infty$ are not needed;
      (7) Let $\alpha = \gamma$ be a quotient of odd positive integers and $g(t) \leq t$ on $[t_0, \infty)_{\mathbb{T}}$, then Theorems 2.1 and 2.2 improves the results in [38] since the condition $\sigma \circ g = g \circ \sigma$ is not needed.

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