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Original Article

Fuzzy soft connected sets in fuzzy soft topological spaces II

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ABSTRACT

In this paper, we introduce some different types of fuzzy soft connected components related to the different types of fuzzy soft connectedness and based on an equivalence relation defined on the set of fuzzy soft points of X. We have investigated some very interesting properties for fuzzy soft connected components. We show that the fuzzy soft C_5-connected component may be not exists and if it exists, it may not be fuzzy soft closed set. Also, we introduced some very interesting properties for fuzzy soft connected components in discrete fuzzy soft topological spaces which is a departure from the general topology.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh [15] in his classical paper of 1965. In 1968, Chang [2] gave the definition of fuzzy topology. Since Chang applied fuzzy set theory into topology many topological notions were investigated in a fuzzy setting.

In 1999, the Russian researcher Molodtsov [9] introduced the soft set theory which is a completely new approach for modeling uncertainty. He established the fundamental results of this new theory and successfully applied the soft set theory into several directions. Maji et al. [8] defined and studied several basic notions of soft set theory in 2003. Shabir and Naz [12] introduced the concept of soft topological space.

Maji et al. [7] initiated the study involving both fuzzy sets and soft sets. In this paper, Maji et al. combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. In 2011, Tanay Kandemir [14] gave the topological structure of fuzzy soft sets.

The notions of fuzzy soft connected sets and fuzzy soft connected components are very important in fuzzy soft topological spaces which in turn reflect the intrinsic nature of it that is in fact its peculiarity. In fuzzy soft setting, connectedness has been introduced by Mahanta and Das [6] and Karataş et al. [5]. Recently, Kandil et al. [4] introduced some types of separated sets and some types of connected sets. They studied the relationship between these types.

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In this paper, we extend the notion of connected components of fuzzy topological space to fuzzy soft topological space. In Section 3, we introduce and investigate some very interesting properties for fuzzy soft connected components. We define an equivalence relation on the set of fuzzy soft points. The union of equivalence classes turns out to be a maximal fuzzy soft connected set which is called a fuzzy soft connected component. There are many types of connected components deduced from the many types of connected sets due to Kandil et al. [4]. Furthermore, we show that some of these connected components may be not exists and the some if exists, it may not be fuzzy soft closed set. Moreover, we introduced some very interesting properties for fuzzy soft connected components in discrete fuzzy soft topological spaces which is a departure from the general topology.

2. Preliminaries

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X. In this section, we present the basic definitions and results of fuzzy soft set theory which will be needed in the sequel.

Definition 2.1. [2] A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. Let I^X denotes the family of all fuzzy sets on X.

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Definition 2.2. [9] Let A be a non-empty subset of E. A pair (F, A) denoted by F_A is called a soft set over X , where F is a mapping given by F: $A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X . For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$ i.e. $F = \{F(e) : e \in A \subseteq E, F : A \to P(X)\}$.

Aktaş and Çağman [1] showed that every fuzzy set may be considered as a soft set. That is, fuzzy sets are a special class of soft sets.

Definition 2.3. [7] Let $A \subseteq E$. A pair (f, A), denoted by f_A, is called fuzzy soft set over X , where f is a mapping given by $f : A \longrightarrow I^X$ defined by $f_A(e) = \mu^e_{f_A}$; where $\mu^e_{f_A} = \overline{0}$ if $e \notin A$, and $\mu^e_{f_A} \neq \overline{0}$ if $e \in A$, where $\overline{0}(x) = 0 \forall\ x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_E$.

Definition 2.4. [3,7,10,11,13,14] The complement of a fuzzy soft set (f, A) , denoted by $(f, A)^c$, and defined by $(f, A)^c = (f^c, A)$, $f^c_A : A \longrightarrow I^X$ is a mapping given by $\mu^e_{f^c_A} = 1 - \mu^e_{f_A}\ \forall e \in A$. Clearly, $(f^c_A)^c = f_A$.

Definition 2.5. [7,10,11,13,14] A fuzzy soft set f_E over X is said to be a null-fuzzy soft set, denoted by $\widetilde{0}_E$, if for all $e \in E$, $f_E(e) = \overline{0}$.

Definition 2.6. [7,10,11,13,14] A fuzzy soft set f_E over X is said to be an absolute fuzzy soft set, denoted by $\widetilde{1}_E$, if $f_E(e) = \overline{1}\ \forall e \in E$. Clearly, we have $(\widetilde{0}_E)^c = \widetilde{1}_E$ and $(\widetilde{1}_E)^c = \widetilde{0}_E$.

Definition 2.7. [3,7,10,11,13,14] Let $f_A, g_B \in FSS(X)_E$. Then f_A is fuzzy soft subset of g_B, denoted by $f_A \widetilde{\subseteq} g_B$, if $A \subseteq B$ and $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x) \forall x \in X, \forall e \in E$. Also, g_B is called fuzzy soft superset of f_A denoted by $g_B \widetilde{\supseteq} f_A$. If f_A is not fuzzy soft subset of g_B, we written as $f_A \widetilde{\nsubseteq} g_B$.

Definition 2.8. [3,7,10,11,13,14] Two fuzzy soft sets f_A and g_B on X are called equal if $f_A \widetilde{\subseteq} g_B$ and $g_B \widetilde{\subseteq} f_A$.

Definition 2.9. [7,10,11,13,14] The union of two fuzzy soft sets f_A and g_B over the common universe X, denoted by $f_A \sqcup g_B$, is also a fuzzy soft set h_C , where $C = A \cup B$ and for all $e \in C, h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \vee \mu^e_{g_B} \forall e \in E$.

Definition 2.10. [7,10,11,13,14] The intersection of two fuzzy soft sets f_A and g_B over the common universe X, denoted by $f_A \sqcap g_B$, is also a fuzzy soft set h_C , where $C = A \cap B$ and for all $e \in C, h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \wedge \mu^e_{g_B} \forall e \in E$.

Definition 2.11. [14] Let $FSS(X)_E$ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E. Then $\tau \subseteq FSS(X)_E$ is called fuzzy soft topology on X if

1. $\widetilde{0}_E$, $\widetilde{1}_E \in \tau$, where $\widetilde{0}_E(e) = \overline{0}$ and $\widetilde{1}_E(e) = \overline{1} \forall e \in E$,
2. The union of any members of τ belongs to τ.
3. The intersection of any two members of τ belongs to τ.

The triplet (X, τ, E) is called fuzzy soft topological space over X. Also, each member of τ is called fuzzy soft open set in (X, τ, E).

Definition 2.12. [14] Let (X, τ, E) be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be fuzzy soft closed set in X, if its relative complement f^c_A is fuzzy soft open set.

Definition 2.13. [10,11] Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. The fuzzy soft closure of f_A, denoted by $Fcl(f_A)$ is the intersection of all fuzzy soft closed supersets of f_A, i.e. $Fcl(f_A) = \sqcap\{h_C; h_C \in \tau^c$ and $f_A \widetilde{\subseteq} h_C\}$. Clearly, $Fcl(f_A)$ is the smallest fuzzy soft closed set over X which contains f_A, and $Fcl(f_A)$ is fuzzy soft closed set.

Definition 2.14. [11,13] The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha$; $(0 \leq \alpha \leq 1)$ and $\mu^e_{f_A}(y) = 0\ \forall y \in X - \{x\}$ and this fuzzy soft point is denoted by x^e_α or f_e. The class of all fuzzy soft points of X, denoted by $FSP(X)_E$.

Definition 2.15. [6] The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set f_A, denoted by $x^e_\alpha \widetilde{\in} f_A$, if for the element $e \in A$, $\alpha \leq \mu^e_{f_A}(x)$. If x^e_α is not belong to f_A, we write $x^e_\alpha \widetilde{\notin} f_A$ and implies that $\alpha > \mu^e_{f_A}(x)$.

Definition 2.16. [11,13] A fuzzy soft point x^e_α is said to be a quasi-coincident with a fuzzy soft set f_A, denoted by $x^e_\alpha\ q\ f_A$, if $\alpha + \mu^e_{f_A}(x) > 1$. Otherwise, x^e_α is non-quasi-coincident with f_A and denoted by $x^e_\alpha\ \overline{q}\ f_A$.

Definition 2.17. [11,13] A fuzzy soft set f_A is said to be quasi-coincident with g_B, denoted by $f_A\ q\ g_B$, if there exists $x \in X$ such that $\mu^e_{f_A}(x) + \mu^e_{g_B}(x) > 1$, for some $e \in A \cap B$. If this is true we can say that f_A and g_B are quasi-coincident at x. Otherwise, f_A and g_B are not quasi-coincident and denoted by $f_A\ \overline{q}\ g_B$.

Proposition 2.1. *[11, 13] Let f_A and g_B be two fuzzy soft sets. Then, $f_A \widetilde{\subseteq} g_B$ if and only if $f_A\ \overline{q}\ (g_B)^c$. In particular, $x^e_\alpha \widetilde{\in} f_A$ if and only if $x^e_\alpha\ \overline{q}\ (f_A)^c$.*

Definition 2.18. [10] Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Then the map f_{pu} is called fuzzy soft mapping from $FSS(X)_E$ to $FSS(Y)_K$, denoted by f_{pu}: $FSS(X)_E \longrightarrow FSS(Y)_K$, such that:

1. If $g_B \in FSS(X)_E$, then the image of g_B under the fuzzy soft mapping f_{pu} is a fuzzy soft set over Y defined by $f_{pu}(g_B)$ where $\forall k \in p(E)$, $\forall y \in Y$,

$$f_{pu}(g_B)(k)(y) = \bigvee_{u(x)=y} [\bigvee_{p(e)=k} (g_B(e))](x) \quad \text{if } x \in u^{-1}(y), 0$$

2. If $h_C \in FSS(Y)_K$, then the pre-image of h_C under the fuzzy soft mapping f_{pu}, $f^{-1}_{pu}(h_C)$ is a fuzzy soft set over X defined by $\forall e \in p^{-1}(K)$, $\forall x \in X$,

$$f^{-1}_{pu}(h_C)(e)(x) = h_C(p(e))(u(x)) \qquad \text{for } p(e) \in C, 0$$

Definition 2.19. [10] The fuzzy soft mapping f_{pu} is called surjective (resp. injective) if p and u are surjective (resp. injective), also f_{pu} is said to be constant if p and u are constant.

Definition 2.20. [10] Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces and f_{pu}: $FSS(X)_E \longrightarrow FSS(Y)_K$ be a fuzzy soft mapping. Then f_{pu} is called:

1. Fuzzy soft continuous if $f^{-1}_{pu}(h_C) \in \tau_1\ \forall\ h_C \in \tau_2$.
2. Fuzzy soft open if $f_{pu}(g_B) \in \tau_2\ \forall\ g_B \in \tau_1$.

Definition 2.21. [5] Two non-null fuzzy soft sets f_E and g_E are said to be fuzzy soft Q-separated in a fuzzy soft topological space (X, τ, E) if $Fcl(f_E) \sqcap g_E = f_E \sqcap Fcl(g_E) = \widetilde{0}_E$.

Definition 2.22. [5] Let (X, τ, E) be a fuzzy soft topological space and $f_E \in FSS(X)_E$. Then, f_E is called:

FSC_1-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \tilde{\subseteq} h_E \sqcup s_E$, $h_E \sqcap s_E \tilde{\subseteq} f_E^c$, $f_E \sqcap h_E \neq \widetilde{0}_E$, and $f_E \sqcap s_E \neq \widetilde{0}_E$.

FSC_2-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \tilde{\subseteq} h_E \sqcup s_E$, $f_E \sqcap h_E \sqcap s_E = \widetilde{0}_E$, $f_E \sqcap h_E \neq \widetilde{0}_E$, and $f_E \sqcap s_E \neq \widetilde{0}_E$.

FSC_3-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \tilde{\subseteq} h_E \sqcup s_E$, $h_E \sqcap s_E \tilde{\subseteq} f_E^c$, $h_E \tilde{\nsubseteq} f_E^c$, and $s_E \tilde{\nsubseteq} f_E^c$.

FSC_4-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \tilde{\subseteq} h_E \sqcup s_E$, $f_E \sqcap h_E \sqcap s_E = \widetilde{0}_E$, $h_E \tilde{\nsubseteq} f_E^c$, and $s_E \tilde{\nsubseteq} f_E^c$.

Otherwise, f_E is called FSC_i-disconnected set for $i = 1, 2, 3, 4$.

In the above definition, if we take $\widetilde{1}_E$ instead of f_E, then the fuzzy soft topological space (X, τ, E) is called FSC_i -connected space $(i = 1, 2, 3, 4)$.

Remark 2.1. [5] The relationship between FSC_i-connectedness $(i = 1, 2, 3, 4)$ can be described by the following diagram:

$$\begin{array}{ccc} FSC_1 & \Longrightarrow & FSC_2 \\ \Downarrow & & \Downarrow \\ FSC_3 & \Longrightarrow & FSC_4 \end{array}$$

Definition 2.23. [4] Two non-null fuzzy soft sets f_E and g_E are said to be:

1. Weakly separated sets in a fuzzy soft topological space (X, τ, E) if $Fcl(f_E)\ \overline{q}\ g_E$ and $f_E\ \overline{q}\ Fcl(g_E)$.
2. Separated sets in a fuzzy soft topological space (X, τ, E) if there exist non-null fuzzy soft open sets h_E and s_E such that $f_E \tilde{\subseteq} h_E$, $g_E \tilde{\subseteq} s_E$ and $f_E \sqcap s_E = g_E \sqcap h_E = \widetilde{0}_E$.

Definition 2.24. [4] Let $f_E \in FSS(X)_E$. The support of $f_E(e)$, denoted by $S(f_E(e))$, is the set, $S(f_E(e)) = \{x \in X; f_E(e)(x) > 0\}$.

Definition 2.25. [4] Two fuzzy soft sets f_E and g_E are said to be quasi-coincident with respect to f_E if $\mu^e_{f_E}(x) + \mu^e_{g_E}(x) > 1$ for every $x \in S(f_E(e))$.

Definition 2.26. [4] Two non-null fuzzy soft sets f_E and g_E are said to be fuzzy soft strongly separated in a fuzzy soft topological space (X, τ, E) if there exist h_E and $s_E \in \tau$ such that $f_E \tilde{\subseteq} h_E$, $g_E \tilde{\subseteq} s_E$, $f_E \sqcap s_E = g_E \sqcap h_E = \widetilde{0}_E$, f_E, h_E are fuzzy soft quasi-coincident with respect to f_E, and g_E, s_E are fuzzy soft quasi-coincident with respect to g_E.

Remark 2.2. [4] In fuzzy soft topological space (X, τ, E) the relationship between different notions of fuzzy soft separated sets can be described by the following diagram.

$$\begin{array}{ccc} & & \text{fuzzy soft strongly separated} \\ & & \Downarrow \\ & & \text{fuzzy soft separated} \\ & & \Downarrow \\ \text{fuzzy soft } Q\text{-separated} & \Longrightarrow & \text{fuzzy soft weakly separated} \end{array}$$

Definition 2.27. [4] A fuzzy soft set f_E in a fuzzy soft topological space (X, τ, E) is called:

1. FSC_M-disconnected set if there exist two non-null fuzzy soft Q-separated sets h_E, s_E in X such that $f_E = h_E \sqcup s_E$. Otherwise, f_E is called FSC_M-connected set.
2. FSC_S-disconnected set if there exist two non-null fuzzy soft weakly-separated sets h_E, s_E in X such that $f_E = h_E \sqcup s_E$. Otherwise, f_E is called FSC_S-connected set.
3. FSO-disconnected (respectively, FSO_q-disconnected) set if there exist two non-null fuzzy soft separated (respectively, strongly separated) sets h_E, s_E in X such that $f_E = h_E \sqcup s_E$. Otherwise, f_E is called FSO-connected (respectively, FSO_q-connected) set.
4. FSC_5-connected set in X if there does not exist any non-null proper fuzzy soft clopen set in (f_E, τ_{f_E}, E). Note that, this kind of fuzzy soft connectedness was studied by Mahanta and Das [6], Shabir and Naz [12].

In the above definitions, if we take $\widetilde{1}_E$ instead of f_E, then the fuzzy soft topological space (X, τ, E) is called FSC_M-connected (respectively, FSC_S-connected, FSO-connected, FSO_q-connected, FSC_5-connected) space.

Remark 2.3. [4] In a fuzzy soft topological space (X, τ, E). The classes of FSO-connected, FSO_q-connected, and FSC_i-connected sets for $i = 1, 2, 3, 4, S, M$ can be described by the following diagram.

$$\begin{tikzcd} & FSC_1 \arrow[d] & & \\ & FSC_S \arrow[dl] \arrow[dr] & & \\ FSC_3 \arrow[d] \arrow[dr] & & FSC_2 \arrow[dl] \arrow[r, leftrightarrow] & FSO \\ FSC_M & FSC_4 \arrow[d] & & \\ & FSO_q & & \end{tikzcd}$$

3. Equivalence relations and components

In disconnected fuzzy soft topological space (X, τ, E), the universe fuzzy soft set $\widetilde{1}_E$ can be decomposed into several pieces of fuzzy soft sets, each of which is connected. As in general topological space, the whole space is decomposed into components.

In fuzzy soft setting, this decomposition is obtained in form of unions of equivalence classes of a certain equivalence relation, defined on the set of fuzzy soft points in X. The union of equivalence classes turns out to be a maximal fuzzy soft connected set. Accordingly, we have many types of notions of components in fuzzy soft setting.

Proposition 3.1. *For fuzzy soft points $x^{e_1}_\alpha$ and $y^{e_2}_\beta$ in X define a relation E_i as follows:*

$E_i = \{(x^{e_1}_\alpha, y^{e_2}_\beta); x^{e_1}_\alpha, y^{e_2}_\beta \tilde{\in} FSP(X)_E$ and there exists a FSC_i -connected set f_A such that $x^{e_1}_\alpha \tilde{\in} f_A$ and $y^{e_2}_\beta \tilde{\in} f_A$ for $i = 1$, 2, S, M, O, $O_q\}$

Then, E_i is an equivalence relation on $FSP(X)_E$.

Proof. As a sample we will prove the case of $i = 1$. Reflexivity follows from the fact that for each fuzzy soft point x^e_α in X, there exists a fuzzy soft point x^e_1 in X, which is a FSC_1-connected and obviously contains x^e_α. Symmetry is obvious. To show transitivity, let $x^{e_1}_\alpha$, $y^{e_2}_\beta$ and $z^{e_3}_\gamma$ be fuzzy soft points in X such that $(x^{e_1}_\alpha, y^{e_2}_\beta) \in E_1$ and $(y^{e_2}_\beta, z^{e_3}_\gamma) \in E_1$. Then, there exist FSC_1-connected sets f_A and g_B in X such that $x^{e_1}_\alpha \tilde{\in} f_A$, $y^{e_2}_\beta \tilde{\in} f_A$ and $y^{e_2}_\beta \tilde{\in} g_B$, $z^{e_3}_\gamma \tilde{\in} g_B$. Therefore, $\beta \leq \mu^{e_2}_{f_A}(y)$ and $\beta \leq \mu^{e_2}_{g_B}(y)$. Hence, $f_A \sqcap g_B \neq \widetilde{0}_E$. So by Theorem 4.10 in [9], $f_A \sqcup g_B$ is a FSC_1-connected. Also, we have $x^{e_1}_\alpha \tilde{\in} f_A \sqcup g_B$ and $z^{e_3}_\gamma \tilde{\in} f_A \sqcup g_B$. Therefore, E_1 is an equivalence relation. Similarly, E_i is an equivalence relation for $i = 2, S, M, O, O_q$.

Let x_α^e be a fuzzy soft point in X and E_i be the equivalence relation on $FSP(X)_E$, described as above. Then the equivalence class determined by x_α^e, is denoted by $E_i(x_\alpha^e)$ for $i = 1, 2, S, M, O, O_q$. □

Definition 3.1. The union $\sqcup E_i(x_\alpha^e)$ of all fuzzy soft points contained in the equivalence class $E_i(x_\alpha^e)$ is called a C_i-component of the universe fuzzy soft set $\widetilde{1}_E$, determined by x_α^e. We denoted it by $C_i(x_\alpha^e)$ for $i = 1, 2, S, M, O, O_q$.

Theorem 3.1. *For each fuzzy soft point $x_\alpha^e \widetilde{\in} FSP(X)_E$, the component $C_i(x_\alpha^e)$ is the maximal FSC_i-connected (respectively, FSO-connected, FSO_q-connected) set in X containing x_α^e for $i = 1, 2, S, M$.*

Proof. As a sample we will prove the case of $C_1(x_\alpha^e)$. Let $x_\alpha^e \widetilde{\in} FSP(X)_E$ and let $\{(f_A)_i;\ i \in I\}$ be the family of FSC_1-connected sets in X, containing x_α^e. We claim $C_1(x_\alpha^e) = \sqcup_{i\in I}(f_A)_i$.

Firstly, we show that $\sqcup_{i\in I}(f_A)_i \widetilde{\subseteq} C_1(x_\alpha^e)$. Let $y \in X$, $e \in E$, $\mu^e_{(f_A)_i}(y) = \beta_i$ for each $i \in I$ and $\sup_{i\in I}\beta_i = \beta$. Then, $\mu^e_{\sqcup_{i\in I}(f_A)_i}(y) = \sup_{i\in I}\beta_i = \beta$.

Now, if $\beta = 0$, we have nothing to prove. Suppose $\beta \neq 0$. Then for every real number $\epsilon > 0$, there exists $i \in I$ such that $\mu^e_{(f_A)_i}(y) = \beta_i > \beta - \epsilon$. Therefore, for each fuzzy soft point $y^e_{\beta-\epsilon}$ where $0 < \epsilon < \beta$, there exists a fuzzy soft set $(f_A)_i$ such that $y^e_{\beta-\epsilon} \widetilde{\in} (f_A)_i$. Since $(f_A)_i$ is a FSC_1-connected set, containing x_α^e, it follows that $(x_\alpha^e, y^e_{\beta-\epsilon}) \in E_1$ and hence $y^e_{\beta-\epsilon} \widetilde{\in} E_1(x_\alpha^e)$ for every $0 < \epsilon < \beta$.

Now, let $\{(y^e_\beta)_j;\ j \in J\}$ be the family of all fuzzy soft points in X with support y which are E_1-related to x_α^e. Then $\{y^e_{\beta-\epsilon}\}_{0<\epsilon<\beta} \subseteq \{(y^e_\beta)_j;\ j \in J\} \subseteq E_1(x_\alpha^e)$. Therefore, $\sqcup_{0<\epsilon<\beta} y^e_{\beta-\epsilon} \widetilde{\in} \sqcup_{j\in J}(y^e_\beta)_j \widetilde{\in} E_1(x_\alpha^e)$. But $\sqcup_{0<\epsilon<\beta} y^e_{\beta-\epsilon} = y^e_\beta$. Hence, $y^e_\beta \widetilde{\in} \sqcup E_1(x_\alpha^e) = C_1(x_\alpha^e)$ and so $\sqcup_{i\in I}(f_A)_i \widetilde{\subseteq} C_1(x_\alpha^e)$.

Conversely, we show that $C_1(x_\alpha^e) \widetilde{\subseteq} \sqcup_{i\in I}(f_A)_i$. Let $y \in X$, $e \in E$ and $\{(y^e_\beta)_j;\ j \in J\}$ be the family of all fuzzy soft points in X with support y such that $(y^e_\beta)_j \widetilde{\in} E_1(x_\alpha^e)$. Suppose, $\sup_{j\in J}\beta_j = \beta$. Then, $\mu^e_{\sqcup E_1(x_\alpha^e)}(y) = \mu^e_{\sqcup_{j\in J}(y^e_\beta)_j}(y) = \beta$.

Now, since $(y^e_\beta)_j \widetilde{\in} E_1(x_\alpha^e)$, there exists for every $j \in J$ a FSC_1 -connected set $(f_A)_j$ such that $x_\alpha^e \widetilde{\in} (f_A)_j$ and $(y^e_\beta)_j \widetilde{\in} (f_A)_j$. Hence, the family of fuzzy soft sets $\{(f_A)_j;\ j \in J\} \subseteq \{(f_A)_i;\ i \in I\}$. Therefore, $\sup_{j\in J}\{\mu^e_{(f_A)_j}(y)\} = \mu^e_{\sqcup_{j\in J}(f_A)_j}(y) \le \mu^e_{\sqcup_{i\in I}(f_A)_i}(y)$. But, $\beta = \sup_{j\in J}\beta_j \le \sup_{j\in J}\{\mu^e_{(f_A)_j}(y)\}$. Therefore, $\beta \le \mu^e_{\sqcup_{i\in I}(f_A)_i}(y)$. Hence, $C_1(x_\alpha^e) \widetilde{\subseteq} \sqcup_{i\in I}(f_A)_i$.

That $C_1(x_\alpha^e)$ is the maximal FSC_1-connected set containing x_α^e, now follows from the fact that, if g_B is any FSC_1-connected set in X containing x_α^e, then $g_B \in \{(f_A)_i;\ i \in I\}$ and hence $g_B \widetilde{\subseteq} \sqcup_{i\in I}(f_A)_i = C_1(x_\alpha^e)$. □

Theorem 3.2. *In a fuzzy soft topological space (X, τ, E), the universe fuzzy soft set $\widetilde{1}_E$ is the disjoint union of its C_i-components for $i = 1, 2, S, M, O, O_q$.*

Proof. As a sample we will prove the case of C_1-component. Let $\{C_1^i(x_\alpha^e); i \in I\}$ be the family of C_1-components of $\widetilde{1}_E$ in X. Then $\sqcup_{i\in I} C_1^i(x_\alpha^e) \widetilde{\subseteq} \widetilde{1}_E$. Since each fuzzy soft point $x_1^e \widetilde{\in} E_1(x_\alpha^e) \widetilde{\subseteq} C_1^i(x_\alpha^e)$, then $\widetilde{1}_E \widetilde{\subseteq} \sqcup_{i\in I} C_1^i(x_\alpha^e)$. Moreover, if two C_1-components $C_1(x_\alpha^e)$ and $C_1(y^t_\beta)$ are intersecting, then $C_1(x_\alpha^e) \sqcup C_1(y^t_\beta)$ is a FSC_1-connected set in X. Hence $C_1(x_\alpha^e)$ and $C_1(y^t_\beta)$ are identical in view of Theorem 3.1 □

In analogy with the general topological spaces, in an indiscrete fuzzy soft topological space, $\widetilde{1}_E$ is the only C_1-components (C_2-components). In a discrete general topological space, singletons are connected sets and hence components. This feature is too is retained in the fuzzy soft setting but with an interesting departure in the case of FSC_1-connectedness, as reflected in the following results.

Theorem 3.3. *In a discrete fuzzy soft topological space, the only FSC_1-connected sets are fuzzy soft points with value one.*

Proof. Let x_1^e be a fuzzy soft point in a discrete fuzzy soft topological space (X, τ, E). Let h_C and s_D be fuzzy soft open sets in X such that $x_1^e \widetilde{\in} h_C \sqcup s_D$, $h_C \sqcap s_D \widetilde{\subseteq} (x_1^e)^c$. Then we have either ($\mu^e_{h_C}(x) = 1$ and $\mu^e_{s_D}(x) = 0$) or ($\mu^e_{h_C}(x) = 0$ and $\mu^e_{s_D}(x) = 1$). Therefore, $x_1^e \sqcap h_C = \widetilde{0}_E$ or $x_1^e \sqcap s_D = \widetilde{0}_E$. Hence, x_1^e is a FSC_1-connected.

Next, to show that each fuzzy soft point x_α^e, where $0 < \alpha < 1$, has a FSC_1-disconnection, what is required, is the construction of two fuzzy soft open sets u_N and j_L in X satisfying $x_\alpha^e \widetilde{\in} u_N \sqcup j_L$, $u_N \sqcap j_L \widetilde{\subseteq} (x_\alpha^e)^c$ and $x_\alpha^e \sqcap u_N \neq \widetilde{0}_E \neq x_\alpha^e \sqcap j_L$. Now, consider any fuzzy soft sets u_N and j_L in X such that $\mu^e_{u_N}(x) = \max\{\alpha, 1-\alpha\}$ and $\mu^e_{j_L}(x) = \min\{\alpha, 1-\alpha\}$. Then, u_N and j_L are FSC_1-disconnection of x_α^e.

Finally, we construct a FSC_1-disconnection for any fuzzy soft set in X, which is not a fuzzy soft point. Let f_A be any fuzzy soft set which takes non-zero values at least at two distinct points y and z in X. Suppose $\mu^e_{f_A}(y) = \alpha$ and $\mu^e_{f_A}(z) = \beta$. Now, define fuzzy soft sets h_C and s_D, as follows:

$$\mu^e_{h_C}(y) = \alpha, \mu^e_{h_C}(z) = 0 \text{ and } \mu^e_{h_C}(x) = \mu^e_{f_A}(x)\ \forall x \in X - \{y, z\}$$

$$\mu^e_{s_D}(y) = 0, \mu^e_{s_D}(z) = \beta \text{ and } \mu^e_{s_D}(x) = 1 - \mu^e_{f_A}(x)\ \forall x \in X - \{y, z\}$$

It is clear that, h_C and s_D form FSC_1-disconnection of f_A. □

Theorem 3.4. *In a discrete fuzzy soft topological space, fuzzy soft points are only FSC_2-connected sets.*

Proof. Let x_α^e be a fuzzy soft point in discrete fuzzy soft topological space. Let h_C and s_D be fuzzy soft open sets in X such that $x_\alpha^e \widetilde{\in} h_C \sqcup s_D$, $x_\alpha^e \sqcap h_C \sqcap s_D = \widetilde{0}_E$. Then, we have either ($\mu^e_{h_C}(x) \ge \alpha, \mu^e_{s_D}(x) = 0$) or ($\mu^e_{h_C}(x) = 0, \mu^e_{s_D}(x) \ge \alpha$). Therefore, $x_\alpha^e \sqcap h_C = \widetilde{0}_E$ or $x_\alpha^e \sqcap s_D = \widetilde{0}_E$. Hence, x_α^e is a FSC_2 -connected.

Next, we construct a FSC_2-disconnection for any fuzzy soft in X, which is not a fuzzy soft point. Let f_A be any fuzzy soft set which takes non-zero values at least at two distinct points y and z in X. Suppose $\mu^e_{f_A}(y) = \beta$ and $\mu^e_{f_A}(z) = \gamma$. Now, define fuzzy soft sets h_C and s_D, as follows:

$$\mu^e_{h_C}(y) = \beta, \mu^e_{h_C}(z) = 0 \text{ and } \mu^e_{h_C}(x) = \mu^e_{f_A}(x)\ \forall x \in X - \{y, z\}$$

$$\mu^e_{s_D}(y) = 0, \mu^e_{s_D}(z) = \gamma \text{ and } \mu^e_{s_D}(x) = 0\ \forall x \in X - \{y, z\}$$

It is clear that, h_C and s_D form FSC_2-disconnection of f_A. □

Corollary 3.1. *In a discrete fuzzy soft topological space, fuzzy soft points with value 1 are the only C_1-components (C_2-components).*

Let $\beta_{\frac{1}{2}}$ be the set of all fuzzy soft points in a fuzzy soft topological space (X, τ, E), whose values are greater than $\frac{1}{2}$.

Proposition 3.2. *For fuzzy soft points x_α^e and y^t_β in $\beta_{\frac{1}{2}}$ define a relation $E_3^*(E_4^*)$ as follows:*

$E_3^*(E_4^*) = \{(x_\alpha^e,\ y^t_\beta);$ there exists a FSC_3-connected (FSC_4-connected) set f_A in X such that $x_\alpha^e \widetilde{\in} f_A$ and $y^t_\beta \widetilde{\in} f_A\}$

Then $E_3^*(E_4^*)$ is an equivalence relation on $\beta_{\frac{1}{2}}$.

Proof. Reflexivity follows from the fact that each fuzzy soft point x^e_α in $\beta_{\frac{1}{2}}$ is a FSC_3-connected set and obviously contains x^e_α. Symmetry is obvious. To show transitivity, let $x^{e_1}_\alpha$, $y^{e_2}_\beta$ and $z^{e_3}_\gamma$ be fuzzy soft points in $\beta_{\frac{1}{2}}$ such that $(x^{e_1}_\alpha, y^{e_2}_\beta) \in E^*_3$ and $(y^{e_2}_\beta, z^{e_3}_\gamma) \in E^*_3$. Then, there exist FSC_3-connected sets f_A and g_B in X such that $x^{e_1}_\alpha \tilde{\in} f_A$, $y^{e_2}_\beta \tilde{\in} f_A$ and $y^{e_2}_\beta \tilde{\in} g_B$, $z^{e_3}_\gamma \tilde{\in} g_B$. Therefore, $\beta \le \mu^{e_2}_{f_A}(y)$ and $\beta \le \mu^{e_2}_{g_B}(y)$. Hence, f_A and g_B are overlapping at y. So by Theorem 4.12 in [9], $f_A \sqcup g_B$ is a FSC_3-connected. Also, we have $x^{e_1}_\alpha \tilde{\in} f_A \sqcup g_B$ and $z^{e_3}_\gamma \tilde{\in} f_A \sqcup g_B$. Therefore, E^*_3 is an equivalence relation. Similarly, E^*_4 is an equivalence relation.

Here, the equivalence relation $E^*_3(E^*_4)$ partitions the set of fuzzy soft points $\beta_{\frac{1}{2}}$ into equivalence classes. As usual, we shall denote an equivalence class containing the fuzzy soft point x^e_α by $E^*_3(x^e_\alpha)(E^*_4(x^e_\alpha))$. □

Definition 3.2. Let x^e_α be a fuzzy soft point in $\beta_{\frac{1}{2}}$ and $\{(f_A)_i;\ i \in I\}$ be the family of FSC_3-connected (FSC_4-connected) sets in X containing x^e_α. Then, the union $\underset{i\in I}{\sqcup}(f_A)_i$ is called a C_3-quasicomponent (respectively, C_4-quasicomponent) of $\tilde{1}_E$ containing x^e_α and is denoted by $C^*_3(x^e_\alpha)$ (respectively, $C^*_4(x^e_\alpha)$).

Theorem 3.5. *For each fuzzy soft point x^e_α in $\beta_{\frac{1}{2}}$, the quasicomponent $C^*_3(x^e_\alpha)(C^*_4(x^e_\alpha))$ is a FSC_3-connected (FSC_4-connected) set in X, containing the union $\sqcup E^*_3(x^e_\alpha)(\sqcup E^*_4(x^e_\alpha))$.*

Proof. In view of Corollary 4.2 in [9], $C^*_3(x^e_\alpha)$ is a FSC_3-connected set in X, since $x^e_\alpha \tilde{\in} \underset{i\in I}{\sqcap}(f_A)_i$. Now, $E^*_3(x^e_\alpha) \tilde{\sqsubseteq} C^*_3(x^e_\alpha)$ follows from the fact that if $y^t_\beta\ E^*_3\ x^e_\alpha$, there is a FSC_3-connected set $(f_A)_{i_\circ}$ in X containing x^e_α and y^t_β. □

Theorem 3.6. *In a fuzzy soft topological space (X, τ, E), the universe fuzzy soft set $\tilde{1}_E$ is the overlapping union of its C_3-quasicomponents (C_4-quasicomponents).*

Proof. Let $\{C^{*i}_3(x^e_\alpha); i \in I\}$ be the family of C_3-quasicomponents of $\tilde{1}_E$ in X. Then $\underset{i\in I}{\sqcup} C^{*i}_3(x^e_\alpha) \tilde{\sqsubseteq} \tilde{1}_E$. Since each fuzzy soft point $x^e_1 \tilde{\in} E^*_3(x^e_\alpha) \tilde{\sqsubseteq} C^{*i}_3(x^e_\alpha)$, then $\tilde{1}_E \tilde{\sqsubseteq} \underset{i\in I}{\sqcup} C^{*i}_3(x^e_\alpha)$. Moreover, let $C^*_3(x^e_\alpha)$ be the C_3-quasicomponents of $\tilde{1}_E$ containing x^e_α, and y^t_β be a fuzzy soft point in $\beta_{\frac{1}{2}}$ such that $y^t_\beta \tilde{\notin} E^*_3(x^e_\alpha)$. Now, if the quasicomponent $C^*_3(x^e_\alpha)$ and $C^*_3(y^t_\beta)$ are overlapping, then $C^*_3(x^e_\alpha) \sqcup C^*_3(y^t_\beta)$ is a FSC_3-connected set in X, by Theorem 3.6 and Theorem 4.12 in [9]. Hence, $y^t_\beta\ E^*_3\ x^e_\alpha$ and so $y^t_\beta \tilde{\in} E^*_3(x^e_\alpha)$ which is a contradiction. □

Now, in order to introduce the concept of C_3-components (respectively, C_4-components), we begin with the following notions. Let φ be the family of C_3-quasicomponents (C_4-quasicomponents) of $\tilde{1}_E$ and let ψ be the family of arbitrary unions of members of φ.

Then, we prove the following proposition.

Proposition 3.3. *For any fuzzy soft points x^e_α and y^t_β in $FSP(X)_E$, define a relation $E_3(\ E_4)$, as follows: $x^e_\alpha\ E_3 y^t_\beta\ (x^e_\alpha\ E_4 y^t_\beta)$ iff there exists a FSC_3-connected (FSC_4-connected) set f_A in ψ such that $x^e_\alpha \tilde{\in} f_A$ and $y^t_\beta \tilde{\in} f_A$. Then $E_3(E_4)$ is an equivalence relation on $FSP(X)_E$.*

Proof. Let x^e_α be a fuzzy soft point in X. Then there exists a C_3-component, in particular $C^*_3(x^e_1)$, which contains x^e_α. Hence, the relation E_3 is reflexive. Symmetry is obvious. Next, let x^e_α, y^t_β and z^s_γ be fuzzy soft points in $\beta_{\frac{1}{2}}$ such that $x^e_\alpha\ E_3 y^t_\beta$ and $y^t_\beta\ E_3\ z^s_\gamma$. Then, there exist FSC_3-connected sets f_A and g_B in ψ such that $x^e_\alpha \tilde{\in} f_A$, $y^t_\beta \tilde{\in} f_A$ and $y^t_\beta \tilde{\in} g_B$, $z^s_\gamma \tilde{\in} g_B$. Now, two cases arise:

Case I. $\beta > \frac{1}{2}$. Then, the fuzzy soft sets overlap at y. Hence, by Theorem 4.12 in [9], $f_A \sqcup g_B$ is a FSC_3 -connected set in ψ such that $x^e_\alpha \tilde{\in} f_A \sqcup g_B$ and $z^s_\gamma \tilde{\in} f_A \sqcup g_B$. So, $x^e_\alpha\ E_3 z^s_\gamma$.

Case II. $\beta \le \frac{1}{2}$. Choose the C_3-quasicomponent $C^*_3(y^t_1)$, which contains y^t_1, hence also y^t_β. Now, the fuzzy soft sets g_B and $C^*_3(y^t_1)$ overlap at y, so $g_B \sqcup C^*_3(y^t_1)$ is a FSC_3-connected set. By the same argument $f_A \sqcup g_B \sqcup C^*_3(y^t_1)$ is also FSC_3-connected set in ψ, containing both x^e_α and z^s_γ. Hence, E_3 is an equivalence relation.

Now, we attain the desired objective of decomposing $\tilde{1}_E$ into disjoint, maximal FSC_3-connected (FSC_4-connected) sets via the equivalence classes defected by the equivalence relation $E_3(\ E_4)$, as defined in Proposition 3.2 Let $E_3(x^e_\alpha)(E_4(x^e_\alpha))$ denoted the equivalence class containing the fuzzy soft point x^e_α. □

Definition 3.3. The union $\sqcup\ E_3(x^e_\alpha)(\sqcup E_4(x^e_\alpha))$ of all fuzzy soft points contained in the equivalence class $E_3(x^e_\alpha)(E_4(x^e_\alpha))$ is called C_3-component (C_4-component) of $\tilde{1}_E$, and is denoted by $C_3(x^e_\alpha)(C_4(x^e_\alpha))$.

Theorem 3.7. *For each fuzzy soft point $x^e_\alpha \tilde{\in} FSP(X)_E$, the C_3- component (C_4-component) $C_3(x^e_\alpha)(C_4(x^e_\alpha))$ is the maximal FSC_3- connected (FSC_4-connected) set in X, containing x^e_α.*

Proof. We claim that, for each fuzzy soft point x^e_α, $C_3(x^e_\alpha) = \underset{i\in I}{\sqcup}(f_A)_i$, where $\{(f_A)_i;\ i \in I\}$ is the family of those members of ψ which are FSC_3-connected, and contains the fuzzy soft point x^e_1. The family $\{(f_A)_i;\ i \in I\}$ is non-empty since C_3-quasicomponent $C^*_3(x^e_\alpha) \in \psi$.

Firstly, we show that $\underset{i\in I}{\sqcup}(f_A)_i \tilde{\sqsubseteq} C_3(x^e_\alpha)$. Let $y \in X$, $t \in E$ and suppose $\mu^t_{\underset{i\in I}{\sqcup}(f_A)_i}(y) = \beta$. If $\beta = 0$, we have nothing to prove. If $\beta \neq 0$, suppose $\mu^t_{(f_A)_i}(y) = \beta_i$ for each $i \in I$. Now, the fuzzy soft point $y^t_{\beta_i} \tilde{\in} (f_A)_i$ for each $i \in I$. Therefore, $y^t_{\beta_i}\ E_3\ x^e_1$, since the C_3-quasicomponent $C^*_3(x^e_1)$ is a FSC_3-connected set such that $x^e_\alpha \tilde{\in} C^*_3(x^e_1) \in \psi$. Therefore, $y^t_{\beta_i}\ E_3\ x^e_\alpha$ for each $i \in I$. Hence, $y^t_{\beta_i} \tilde{\in} C_3(x^e_\alpha) = \sqcup\ E_3(x^e_\alpha)$, for each $i \in I$ and so $\beta_i \le \mu^t_{C_3(x^e_\alpha)}(y)$ for each $i \in I$. Then $\beta = \underset{i\in I}{\sup}\{\beta_i\} \le \mu^t_{C_3(x^e_\alpha)}(y)$ implies $y^t_\beta \tilde{\in} C_3(x^e_\alpha)$. Hence $\underset{i\in I}{\sqcup}(f_A)_i \tilde{\sqsubseteq} C_3(x^e_\alpha)$.

Conversely, we show that $C_3(x^e_\alpha) \tilde{\sqsubseteq} \underset{i\in I}{\sqcup}(f_A)_i$. Let $y \in X$, $t \in E$ and suppose $\mu^t_{C_3(x^e_\alpha)}(y) = \gamma$. Again if $\gamma = 0$, we have nothing to prove. Suppose $\gamma \neq 0$, and $\{y^t_{\gamma_i}; i \in I_1\}$ be the family of fuzzy soft points such that $y^t_{\gamma_i}\ E_3\ x^e_\alpha$. Then, clearly, $\mu^t_{C_3(x^e_\alpha)}(y) = \mu^t_{\underset{i\in I_1}{\sqcup} y^t_{\gamma_i}}(y) = \underset{i\in I_1}{\sup}\{\gamma_i\} = \gamma$. Since $y^t_{\gamma_i}\ E_3\ x^e_\alpha$ for each $i \in I_1$, there exists a FSC_3-connected set $(f_A)_{\gamma_i} \in \psi$ such that $x^e_\alpha \tilde{\in} (f_A)_{\gamma_i}$ and $y^t_{\gamma_i} \tilde{\in} (f_A)_{\gamma_i}$. Now, for each $i \in I_1$, consider the fuzzy soft set $(f_A)_i$, define as follows: $(f_A)_i = (f_A)_{\gamma_i} \sqcup C^*_3(x^e_1)$. Then, $(f_A)_i \in \psi$ is a FSC_3-connected set, such that $x^e_1 \tilde{\in} (f_A)_i$ and $y^t_{\gamma_i} \tilde{\in} (f_A)_i$. Therefore, $\underset{i\in I_1}{\sqcup} y^t_{\gamma_i} \tilde{\in} \underset{i\in I_1}{\sqcup}(f_A)_i$, but $\{(f_A)_i;\ i \in I_1\} \tilde{\sqsubseteq} \{(f_A)_i;\ i \in I\}$ and so, we have $\underset{i\in I_1}{\sqcup} y^t_{\gamma_i} \tilde{\in} \underset{i\in I}{\sqcup}(f_A)_i$. Now, $\gamma = \mu^t_{\underset{i\in I_1}{\sqcup} y^t_{\gamma_i}}(y) \le \mu^t_{\underset{i\in I}{\sqcup}(f_A)_i}(y)$. Therefore, $\mu^t_{C_3(x^e_\alpha)}(y) \le \mu^t_{\underset{i\in I}{\sqcup}(f_A)_i}(y)$ and so, $C_3(x^e_\alpha) \tilde{\sqsubseteq} \underset{i\in I}{\sqcup}(f_A)_i$.

In view of Corollary 4.2 in [9], the C_3-component $C_3(x^e_\alpha)$ is a FSC_3-connected set, since $x^e_1 \tilde{\in} \underset{i\in I}{\sqcap}(f_A)_i$. To show that $C_3(x^e_\alpha)$ is a maximal FSC_3-connected set containing x^e_α, let g_B be any FSC_3-connected set containing x^e_α, such that $C_3(x^e_\alpha) \tilde{\sqsubseteq} g_B$. Then, the fuzzy soft set, defined as $g_B \sqcup C^*_3(y^e_1)$ for every $y \in S(g_B(e))$. Therefore, g_B

$\widetilde{\sqsubseteq} \sqcup E_3^*(x_1^e) \widetilde{\sqsubseteq} C_3^*(x_1^e)$. But, $C_3^*(x_1^e) \widetilde{\sqsubseteq} C_3(x_\alpha^e)$ as $C_3^*(x_1^e) \in \psi$ is a FSC_3-connected set, containing x_1^e. Thus, we have $g_B = C_3(x_\alpha^e)$. □

Theorem 3.8. *In a fuzzy topological space (X, τ, E), the universe fuzzy soft set $\widetilde{1}_E$ is the disjoint union of its C_3-components (respectively, C_4-components).*

Proof. As a sample, we prove the case of C_3-components. Let $\{C_3^i(x_\alpha^e); i \in I\}$ be the family of C_3-components of $\widetilde{1}_E$ in X. Then, it can be verified that $\sqcup_{i \in I} C_3^i(x_\alpha^e) = \widetilde{1}_E$. Next, suppose the C_3-components $C_3(x_\alpha^e)$ and $C_3(y_\beta^t)$ intersect at a point z. Then $\mu^s_{C_3(x_\alpha^e) \sqcap C_3(y_\beta^t)}(z) \neq 0$. Hence, $\mu^s_{C_3(x_\alpha^e)}(z) = \gamma_1$ and $\mu^s_{C_3(y_\beta^t)}(z) = \gamma_2$ where $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$. Now, consider the C_3-quasicomponent $C_3^*(z_1^s)$, containing the fuzzy soft point z_1^s, which overlaps with C_3-component $C_3(x_\alpha^e)$ and $C_3(y_\beta^t)$ at z. Therefore, $C_3(x_\alpha^e) \sqcup C_3^*(z_1^s)$ and $C_3(y_\beta^t) \sqcup C_3^*(z_1^s)$ are FSC_3-connected sets, containing the fuzzy soft points x_α^e and y_β^t respectively, and also the fuzzy soft point z_1^s. Since $C_3(z_1^s)$ is the maximal FSC_3-connected set containing the fuzzy soft point z_1^s. Therefore, $C_3(x_\alpha^e) \sqcup C_3^*(z_1^s) \widetilde{\sqsubseteq} C_3(z_1^s)$ and $C_3(y_\beta^t) \sqcup C_3^*(z_1^s) \widetilde{\sqsubseteq} C_3(z_1^s)$ so that $C_3(x_\alpha^e) \widetilde{\sqsubseteq} C_3(z_1^s)$ and $C_3(y_\beta^t) \widetilde{\sqsubseteq} C_3(z_1^s)$. Now, as $C_3(x_\alpha^e)$ and $C_3(y_\beta^t)$ are maximal FSC_3-connected sets containing the fuzzy soft points x_α^e and y_β^t respectively, $C_3(x_\alpha^e)$ and $C_3(y_\beta^t)$ are identical. □

Theorem 3.9. *For each fuzzy soft point x_α^e in X, the C_3-component (C_4-component) $C_3(x_\alpha^e)$ ($C_4(x_\alpha^e)$) is a fuzzy soft closed set in X.*

Proof. In view of Theorem 4.15 in [9], $Fcl(C_3(x_\alpha^e))$ is a FSC_3-connected set in X. Moreover, x_α^e is contained in $Fcl(C_3(x_\alpha^e))$, as $x_\alpha^e \widetilde{\in} C_3(x_\alpha^e) \widetilde{\sqsubseteq} Fcl(C_3(x_\alpha^e))$. Since $C_3(x_\alpha^e)$ is the maximal FSC_3-connected set containing x_α^e, it follows that $C_3(x_\alpha^e)$ and $Fcl(C_3(x_\alpha^e))$ are identical. □

Again, it is obvious that an indiscrete fuzzy soft topological space, $\widetilde{1}_E$ is the only C_3-component (C_4-component). Moreover, when the fuzzy soft topological space is discrete, we state the following result:

Theorem 3.10. *In a discrete fuzzy soft topological space, fuzzy soft points are the only FSC_3-connected (FSC_3-connected) sets.*

Proof. Immediate. □

Therefore, in a discrete fuzzy soft topological space, the C_3-component (C_4-component) are only the fuzzy soft points with value 1.

Definition 3.4. Let f_A be a fuzzy soft set in a fuzzy soft topological space (X, τ, E). The maximal FSC_5-connected set containing f_A is called the C_5-component of f_A.

Remark 3.1. The C_5-component of a fuzzy soft set may not exist as shown by the following example:

Example 3.1. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{\frac{1}{2}}\}), (e_2, \{b_{\frac{1}{2}}\})\}, \{(e_1, \{b_{\frac{1}{2}}\}), (e_2, \{a_{\frac{1}{2}}\})\}, \{(e_1, \{a_{\frac{1}{2}}, b_{\frac{1}{2}}\}), (e_2, \{a_{\frac{1}{2}}, b_{\frac{1}{2}}\})\}\}$ be a fuzzy soft topology defined on X. Let $f_A = \{(e_1, \{b_{0.7}\})\}$. Since $\{(e_1, \{b_{\frac{1}{2}}\})\}$ is a non-null proper fuzzy soft clopen set in f_A, then f_A is not a FSC_5-connected set. Also, there does not exist any FSC_5-connected set containing f_A. So, f_A has no C_5-component.

Definition 3.5. For any fuzzy soft points x_α^e and y_β^t in $FSP(X)_E$, define a relation E_5, as follows: $x_\alpha^e \; E_5 y_\beta^t$ iff there exists a FSC_5-connected set f_A such that $x_\alpha^e \widetilde{\in} f_A$ and $y_\beta^t \widetilde{\in} f_A$.

Example 3.1 shows that E_5 may not be reflexive. E_5 is obviously symmetric. By using Theorem 4.10 in [9], it can be readily verified that E_5 is transitive.

Theorem 3.11. *Let (X, τ, E) be a fuzzy soft topological space and $x_\alpha^e \widetilde{\in} FSP(X)_E$. Then, x_α^e is not a FSC_5-connected iff there exists a $0 \neq \beta < \alpha$, f_A and $g_B \in \tau$ such that $\mu^e_{f_A}(x) = \beta$ and $\mu^e_{g_B}(x) = 1 - \beta$.*

Proof. Let x_α^e be not FSC_5-connected. Then, x_α^e contains a non-null proper fuzzy soft clopen set x_β^e (say). Therefore, there exist fuzzy soft sets $f_A \in \tau$, $g_B \in \tau^c$ such that $f_A \sqcap x_\alpha^e = g_B \sqcap x_\alpha^e = x_\beta^e$. Since $g_B \in \tau^c$, then $g_B^c \in \tau$ and $\mu^e_{g_B^c}(x) = 1 - \beta$.

Conversely, let there exist a $0 \neq \beta < \alpha$ such that there exist f_A and $g_B \in \tau$ satisfying $\mu^e_{f_A}(x) = \beta$ and $\mu^e_{g_B}(x) = 1 - \beta$. Then, $g_B^c \in \tau^c$ and $\mu^e_{g_B^c}(x) = \beta$. Also, $f_A \sqcap x_\alpha^e = g_B^c \sqcap x_\alpha^e = x_\beta^e$ and so x_β^e is non-null proper fuzzy soft clopen set in x_α^e. Therefore, x_α^e is not FSC_5-connected. □

Remark 3.2. E_5 is an equivalence relation iff $\widetilde{1}_E$ is a FSC_5-connected set and then it is the only C_5-component of (X, τ, E).

Remark 3.3. The C_5-component of a fuzzy soft set if it exists, may not be fuzzy soft closed as shown by the following example:

Example 3.2. Consider the fuzzy soft topological space (X, τ, E) defined in Example 3.1. $f_E = \{(e_1, \{a_{\frac{1}{2}}\}), (e_2, \{b_{\frac{1}{2}}\})\}$ is a FSC_5-connected.

Solution. Let g_E be any fuzzy soft subset of X containing f_E. Then, g_E is of the form $f_E \widetilde{\sqsubseteq} g_E = \{(e_1, \{a_\alpha, b_\beta\}), (e_2, \{a_\gamma, b_\delta\})\}$ where $\alpha, \delta \geq \frac{1}{2}$ and $\gamma, \beta > 0$. Then, $\{(e_1, \{a_{\frac{1}{2}}, b_\beta\}), (e_2, \{a_\gamma, b_{\frac{1}{2}}\})\}$ or $\{(e_1, \{a\frac{1}{2}, b_{\frac{1}{2}}\}), (e_2, \{a\frac{1}{2}, b_{\frac{1}{2}}\})\}$ is a non-null proper fuzzy soft clopen set in g_E according as $\gamma, \beta < \frac{1}{2}$ or $\gamma, \beta \geq \frac{1}{2}$. So, g_E is not FSC_5-connected. Therefore, f_E is the C_5-component of f_E and it is not fuzzy soft closed.

4. Conclusion

In this paper, we define on the set of fuzzy soft points in X an equivalence relation. The union of equivalence classes turns out to be a maximal fuzzy soft connected set which is called a fuzzy soft connected component. According to Remark 2.3, we have many types of connected components in fuzzy soft setting. The universe fuzzy soft set $\widetilde{1}_E$ is the disjoint union of its C_i-components for $i = 1, 2, S, M, O, O_q$. Furthermore, we introduced some very interesting properties for fuzzy soft connected components in discrete fuzzy soft topological spaces which is a departure from the general topology such that in a discrete fuzzy soft topological space, the C_3-component (C_4-component) are only the fuzzy soft points with value 1. Also, we find that: for each fuzzy soft point x_α^e in X, the C_3-component (C_4-component) $C_3(x_\alpha^e)$ ($C_4(x_\alpha^e)$) is a fuzzy soft closed set in X. Moreover, we prove that the C_5-component of a fuzzy soft set may not exist and the C_5-component of a fuzzy soft set if it exists, may not be fuzzy soft closed set.

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