Journal of the Egyptian Mathematical Society 25 (2017) 141–150

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original Article

Numerical approaches to system of fractional partial differential equations

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ARTICLE INFO

Article history:
Received 26 May 2016
Revised 7 December 2016
Accepted 14 December 2016
Available online 28 January 2017

MSC:
45A05
45B05

Keywords:
Adomian decomposition method
Variational iteration method
Lagrange multiplier
Caputo fractional derivative

ABSTRACT

In this paper, by introducing the fractional derivative in sense of Caputo, the Laplace- variational iteration method (LVIM) and the Laplace-Adomian decomposition method (LADM) are directly extended to study the linear and nonlinear systems of fractional partial differential equations, as a result the approximated numerical solutions are acquired in the form of rapidly convergent series with easily computable components. Numerical results show that the two approaches are easy to implement and accurate when are applied. Compassions are made between the two methods and exact solutions. Figures are used to show the efficiency as well as the accuracy of the achieved approximated results.

1. Introduction

Fractional order partial differential equations are generalizations of classical partial differential equations. These have been of considerable interest in the recent literatures. These topics have received a great deal of attention especially in the fields of viscoelasticity materials, electrochemical processes, dielectric polarization, colored noise, anomalous diffusion, signal processing, control theory and others. Increasingly, these models are used in applications such as fluid flow, finance and others. Most nonlinear fractional differential equations do not have analytic solutions, so approximation and numerical techniques must be used. The Laplace-Adomian decomposition method (LADM) [1–7], which is the combination of the Laplace transform and the Adomian decomposition method, was presented by [1,2]. This technique solves many types of nonlinear equations such as ordinary and partial differential equations of integer and fractional order. Jafari et al. [3] applied this method to obtain the solution of linear/nonlinear fractional diffusion and wave equations. Daftardar-Gejji and Bhalekar used LADM for solving fractional boundary value problem and evolution equations [4,5].The modified variational iteration method [8], which is a novel modification of the variational iteration method proposed by means of the Laplace transform, is relatively new approach to provide analytical approximation to linear and nonlinear problems. The structure of this paper is as follow. In Section 2 we begin by introducing some basic definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In Section 3 the basic idea of the Laplace-Adomian decomposition method is illustrated. While in Section 4 the modified variational iteration method is discussed. Numerical examples are presented in Section 5 which show the efficiency and accuracy of numerical techniques.

2. Basic Definitions

We give some basic definitions and properties of the fractional calculus theory [9–13] and Laplace transform which are used further in this paper.

Definition 1. The fractional derivative of $f(x)$ in the *Caputo* sense [5, 6] is defined as

$$D_*^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (1)$$

for $m-1 < \alpha < m$, $m \in \aleph$, $x > 0$, $f \in C_{-1}^m$.

Definition 2. The Laplace transform of $f(t)$, $t > 0$ is defined by

$$F(s) = \pounds[f(t)] = \int_0^\infty e^{-st} f(t)dt, \quad (2)$$

where s can be either real or complex. It has following property:

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http://dx.doi.org/10.1016/j.joems.2016.12.004
1110-256X/

The Laplace transform £[$f(t)$] of the Caputo derivative is defined as [9–13]

$$£[D_*^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(0), m-1<\alpha<m \tag{3}$$

Definition 4. The Mittag –Leffler function $E_\alpha(z)$ with $\alpha. > 0$ is defined by the following series representation, valid in the whole complex plane

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \alpha > 0, z \in C \tag{4}$$

3. Analysis of LADM

In this section, we present LADM [1–7] for solving system of fractional differential equations written in the form

$$D_{*t}^\alpha u(x,t) + R_1(u,v) + N_1(u,v) = g_1(x,t)$$
$$D_{*t}^\alpha v(x,t) + R_2(u,v) + N_2(u,v) = g_2(x,t), \text{ where } m-1<\alpha\le m, \tag{5}$$

with initial data

$$u(x,0) = f_1(x), v(x,0) = f_2(x). \tag{6}$$

Where is $D_{*t}^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ the Caputo fractional derivative of order α, R_1, R_2 and N_1, N_2 are linear and nonlinear operators, respectively, and g_1 and g_2 are source terms. The procedure of the method follows the following steps:

Step 1 applying the Laplace transform to both sides of (5) to obtain

$$\begin{aligned} £_t[D_{*t}^\alpha u(x,t)] + £_t[R_1(u,v)] + £_t[N_1(u,v)] &= £_t[g_1(x,t)] \\ £_t[D_{*t}^\alpha v(x,t)] + £_t[R_2(u,v)] + £_t[N_2(u,v)] &= £_t[g_2(x,t)] \end{aligned} \tag{7}$$

Using the differentiation property of Laplace transform, we get

$$\begin{aligned} £_t[u(x,t)] &= \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k} \frac{\partial^k u(x,t)}{\partial^k t}\bigg|_{t=0} + \frac{1}{s^\alpha}£_t[g_1(x,t)] \\ &\quad - \frac{1}{s^\alpha}£_t[R_1(u,v)] - \frac{1}{s^\alpha}£_t[N_1(u,v)] \\ £_t[v(x,t)] &= \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k} \frac{\partial^k v(x,t)}{\partial^k t}\bigg|_{t=0} + \frac{1}{s^\alpha}£_t[g(x,t)] \\ &\quad - \frac{1}{s^\alpha}£_t[R_2(u,v)] - \frac{1}{s^\alpha}£_t[N_2(u,v)] \end{aligned} \tag{8}$$

Step 2

The LADM defines the solutions $u(x, t)$ and $v(x, t)$ by the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) \tag{9}$$

The nonlinear terms N_1 *and* N_2 are usually represented by an infinite series of the so-called Adomian polynomials

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n, N_2(u,v) = \sum_{n=0}^{\infty} B_n \tag{10}$$

The Adomian polynomials can be generated for all forms of nonlinearity. They are determined by the following relations:

$$\begin{aligned} A_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N_1\left(\sum_{k=0}^{\infty}\lambda^k u_k, \sum_{k=0}^{\infty}\lambda^k v_k\right)\right]\right]_{\lambda=0} \\ B_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N_2\left(\sum_{k=0}^{\infty}\lambda^k u_k, \sum_{k=0}^{\infty}\lambda^k v_k\right)\right]\right]_{\lambda=0} \end{aligned} \tag{11}$$

Substituting (9) and (10) into (8), gives

$$\begin{aligned} £_t\left[\sum_{n=0}^{\infty} u_n(x,t))\right] &= \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k}\frac{\partial^k u(x,t)}{\partial^k t}\bigg|_{t=0} + \frac{1}{s^\alpha}£_t[g_1(x,t)] \\ &\quad - \frac{1}{s^\alpha}£_t\left[R_1\left(\sum_{n=0}^{\infty}u_n(x,t), \sum_{n=0}^{\infty}v_n(x,t)\right)\right] \\ &\quad - \frac{1}{s^\alpha}£_t\left[\sum_{n=0}^{\infty}A_n\right] \\ £_t\left[\sum_{n=0}^{\infty} v_n(x,t)\right] &= \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k}\frac{\partial^k v(x,t)}{\partial^k t}\bigg|_{t=0} + \frac{1}{s^\alpha}£_t[g_2(x,t)] \\ &\quad - \frac{1}{s^\alpha}£_t\left[R_2\left(\sum_{n=0}^{\infty}u_n(x,t), \sum_{n=0}^{\infty}v_n(x,t)\right)\right] \\ &\quad - \frac{1}{s^\alpha}£_t\left[\sum_{n=0}^{\infty}B_n\right] \end{aligned} \tag{12}$$

Applying the linearity of the Laplace transform, we define the following recursions

$$\begin{aligned} £_t[u_0(x,t)] &= \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k}\frac{\partial^k u(x,t)}{\partial^k t}\bigg|_{t=0} + \frac{1}{s^\alpha}£_t[g_1(x,t)], \\ £_t[v_0(x,t)] &= \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k}\frac{\partial^k v(x,t)}{\partial^k t}\bigg|_{t=0} + \frac{1}{s^\alpha}£_t[g_2(x,t)] \end{aligned} \tag{13}$$

$$\begin{aligned} £_t[u_1(x,t)] &= -\frac{1}{s^\alpha}£_t[R_1(u_0(x,t), v_0(x,t))] - \frac{1}{s^\alpha}£_t[A_0] \\ £_t[v_1(x,t)] &= -\frac{1}{s^\alpha}£_t[R_2(u_0(x,t), v_0(x,t))] - \frac{1}{s^\alpha}£_t[B_0] \end{aligned} \tag{14}$$

In general, for $k \ge 1$, the recursive relations are given by

$$\begin{aligned} £_t[u_{k+1}(x,t)] &= -\frac{1}{s^\alpha}£_t[R_1(u_k(x,t), v_k(x,t))] - \frac{1}{s^\alpha}£_t[A_k] \\ £_t[v_{k+1}(x,t)] &= -\frac{1}{s^\alpha}£_t[R_2(u_k(x,t), v_k(x,t))] - \frac{1}{s^\alpha}£_t[B_k] \end{aligned} \tag{15}$$

Step 3

Applying the inverse Laplace transform, we can evaluate u_k and v_k $(k \ge 0)$. In some cases the exact solution in the closed form may also be obtain

The convergence of this method was shown in [14–20].

4. Analysis of LVIM

In this section, we present LVIM[8] for solving the system of fractional partial differential equations (5). The method consists of the following steps:

Step 1

Applying the Laplace transform to both sides of (5), using initial conditions (6) and using the differentiation property of Laplace transform, we get

$$\begin{aligned} & £_t[u(x,t)] - \sum_{k=0}^{m-1} s^{\alpha-1-k}\frac{\partial^k u(x,t)}{\partial^k t}\bigg|_{t=0} \\ &\quad = £_t[g_1(x,t)] - £_t[R_1(u,v)] - £_t[N_1(u,v)], \\ & £_t[v(x,t)] - \sum_{k=0}^{m-1} s^{\alpha-1-k}\frac{\partial^k v(x,t)}{\partial^k t}\bigg|_{t=0} \\ &\quad = £_t[g_2(x,t)] - £_t[R_2(u,v)] - £_t[N_2(u,v)] \end{aligned} \tag{16}$$

Step 2

The iteration formula of Eq. (16) can be used to suggest the main iterative scheme involving the Lagrange multiplier as

$$\begin{aligned}\pounds_t[u_{n+1}(x,t)] &= \pounds[u_n(x,t)] + \lambda(s)[s^{\alpha}\pounds[u_n(x,t)] \\ &\quad - \sum_{k=0}^{m-1} s^{\alpha-1-k} \left.\frac{\partial^k u(x,t)}{\partial^k t}\right|_{t=0} + \pounds[g_1(x,t)] \\ &\quad - \pounds[R_1(u_n,v_n)] - \pounds[N_1(u_n,v_n)]] \\ \pounds_t[v_{n+1}(x,t)] &= \pounds[v_n(x,t)] + \lambda(s)[s^{\alpha}\pounds[v_n(x,t)] \\ &\quad - \sum_{k=0}^{m-1} s^{\alpha-1-k} \left.\frac{\partial^k v(x,t)}{\partial^k t}\right|_{t=0} + \pounds[g_2((x,t))] \\ &\quad - \pounds[R_2(u_n,v_n)] - \pounds[N_2(u_n,v_n)]] \end{aligned} \tag{17}$$

Considering $\pounds_t[[R_1(u_n,v_n)] + [[N_1(u_n,v_n)]]$ and $\pounds_t[[R_2(u_n,v_n)] + [N_2(u_n,v_n)]]$ as restricted terms, one can derive a Lagrange multiplier as

$$\lambda(s) = -\frac{1}{s^{\alpha}} \tag{18}$$

With Eq. (18) and the inverse Laplace transform $\pounds_t^{-1}$, the iteration formula (17) can be given as

$$\begin{aligned} u_{n+1}(x,t) &= \pounds_t^{-1}\left[\frac{1}{s^{\alpha}}\left\{\sum_{k=0}^{m-1} s^{\alpha-1-k}\left.\frac{\partial^k u(x,t)}{\partial^k t}\right|_{t=0} - \pounds[g_1(x,t)]\right.\right. \\ &\quad \left.\left. + \pounds[R_1(u_n,v_n)] + \pounds[N_1(u_n,v_n)]\right\}\right] \\ v_{n+1}(x,t) &= \pounds_t^{-1}\left[\frac{1}{s^{\alpha}}\left\{\sum_{k=0}^{m-1} s^{\alpha-1-k}\left.\frac{\partial^k v(x,t)}{\partial^k t}\right|_{t=0} - \pounds[g_2(x,t)]\right.\right. \\ &\quad \left.\left. = \pounds[R_2(u_n,v_n)] + \pounds[N_2(u_n,v_n)]\right\}\right]. \end{aligned} \tag{19}$$

Where the initial iterations $u_0(x, t)$ and $v_0(x, t)$ can be determined by

$$\begin{aligned} u_0(x,t) &= \pounds_t^{-1}\left[\frac{1}{s^{\alpha}}\left\{\sum_{k=0}^{m-1} s^{\alpha-1-k}\left.\frac{\partial^k u(x,t)}{\partial^k t}\right|_{t=0}\right\}\right], \\ v_0(x,t) &= \pounds_t^{-1}\left[\frac{1}{s^{\alpha}}\left\{\sum_{k=0}^{m-1} s^{\alpha-1-k}\left.\frac{\partial^k v(x,t)}{\partial^k t}\right|_{t=0}\right\}\right] \end{aligned} \tag{20}$$

The convergence of this method was shown in [21-22].

5. Applications

In this section, we use the LADM and LVIM to solve the inhomogeneous linear and nonlinear fractional partial differential equations

5.1. Example 1 (The inhomogeneous linear system [7,23,24])

Consider the inhomogeneous linear system of FPDEs:

$$\begin{aligned} D_{*t}^{\alpha}u(x,t) - v_x(x,t) - u(x,t) + v(x,t) = -2, \\ D_{*t}^{\alpha}v(x,t) + u_x(x,t) - u(x,t) + v(x,t) = -2 \end{aligned}, \quad 0 < \alpha \le 1 \tag{21}$$

With the initial conditions

$$u(x,0) = 1 + e^x, v(x,0) = -1 + e^x \tag{22}$$

Firstly, we will solve this system by using the **LADM.** According to Eqs. (13) and (15) for $k \ge 1$, the recursive relations are given by

$$\begin{aligned} \pounds_t[u_{k+1}(x,t)] &= \frac{1}{s^{\alpha}}\pounds_t[(v_k(x,t))_x + u_k(x,t) - v_k(x,t)], \\ \pounds_t[v_{k+1}(x,t)] &= \frac{1}{s^{\alpha}}\pounds_t[-(u_k(x,t))_x + u_k(x,t) - v_k(x,t)], \end{aligned} \tag{23}$$

with

$$\pounds_t[u_0(x,t)] = \frac{1}{s}(1+e^x) - \frac{2}{s^{\alpha+1}}, \pounds_t[v_0(x,t)] = \frac{1}{s}(-1+e^x) - \frac{2}{s^{\alpha+1}}$$

$$\pounds_t[u_1(x,t)] = \frac{1}{s^{\alpha}}\pounds_t[(v_0(x,t))_x + u_0(x,t) - v_0(x,t)],$$

$$\pounds_t[v_1(x,t)] = \frac{1}{s^{\alpha}}\pounds_t[-(u_0(x,t))_x + u_0(x,t) - v_0(x,t)]$$

Applying the inverse Laplace transform, we can evaluate u_k and v_k $(k \ge 0)$ so the first few terms of the decomposition series are given

$$\begin{aligned} u_0(x,t) &= 1 + e^x - \frac{2t^{\alpha}}{\Gamma(\alpha+1)}, v_0(x,t) = -1 + e^x - \frac{2t^{\alpha}}{\Gamma(\alpha+1)} \\ u_1(x,t) &= (2+e^x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, v_1(x,t) = (2-e^x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2(x,t) &= e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, v_2(x,t) = e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_3(x,t) &= e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, v_3(x,t) = -e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{aligned} \tag{24}$$

Hence, The solution in series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \\ v(x,t) &= v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots \\ u(x,t) &= 1 + e^x[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots]. \\ v(x,t) &= -1 + e^x[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots]. \end{aligned} \tag{25}$$

The approximate solution by **LVIM**.

According to the Eqs. (19) and (20) the iteration formulas for system (21) are

$$\begin{aligned} u_{n+1}(x,t) &= \pounds^{-1}\frac{1}{s}(1+e^x) \\ &\quad + \frac{1}{s^{\alpha}}\pounds[(v_n(x,t))_x + u_n(x,t) - v_n(x,t) - 2]] \\ v_{n+1}(x,t) &= \pounds^{-1}\left[\frac{1}{s}(-1+e^x)\right. \\ &\quad \left. + \frac{1}{s^{\alpha}}\pounds[(u_n(x,t))_x - u_n(x,t) + v_n(x,t) + 2]\right] \end{aligned} \tag{26}$$

Where

$$u_0(x,t) = \pounds^{-1}\left[\frac{1}{s}u(x,0)\right] = 1 + e^x, v_0(x,t)$$

$$= \pounds^{-1}\left[\frac{1}{s}v(x,0)\right] = -1 + e^x,$$

$$u_1(x,t) = 1 + e^x + e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)}, v_1(x,t) = -1 + e^x - e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = 1 + e^x + e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$v_2(x,t) = -1 + e^x - e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = 1 + e^x + e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$v_3(x,t) = -1 + e^x - e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$u_n(x,t) = 1 + e^x + e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + e^x \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$
$$v_n(x,t) = -1 + e^x - e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + (-1)^n e^x \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \tag{27}$$

$$u_n(x,t) = 1 + e^x \sum_{k=0}^{n} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, v_n(x,t) = -1 + e^x \sum_{k=0}^{n} \frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{28}$$

For $n \to \infty$, $u_n(x, t)$ *and* $v_n(x, t)$ tend to the exact solution

$$u_n(x,t) = 1 + e^x \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = 1 + e^x E_{\alpha}(\mathrm{t}^{\alpha}),$$
$$v_n(x,t) = -1 + e^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+1)} = -1 + e^x E_{\alpha}(-\mathrm{t}^{\alpha}) \tag{29}$$

It's noted that the approximated solutions obtained by two techniques are coincided,

Substituting $\alpha = 1$, in Eq. (25) or Eq. (27), we obtain

$$u(x,t) = 1 + e^x \left[1 + \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} + \frac{t^3}{\Gamma(4)} + \ldots \right] = 1 + e^{x+t}$$
$$v(x,t) = -1 + e^x \left[1 - \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} - \frac{t^3}{\Gamma(4)} + \ldots \right] = -1 + e^{x-t}$$

Which represent the exact solutions of (21) when $\alpha = 1$. Also we note that the results obtained here are similar to the solutions obtained by VIM and HAM [14,15]. Fig. 1 and 2 illustrate the exact and the approximate solutions obtained by LADM and LVIM when $\alpha = 1, 0 \le t \le 1, -4 \le x \le 2$.

Tables 1 and 3 represent the approximate solutions $u(x, t)$ and $v(x, t)$ at $t = 0.05, 0.1, 1, -4 \le x \le 2$, $\alpha = 0.25, 0.5, 0.75, 0.95$. While Tables 2 and 4 discuss the error analysis at:$t = 0.05, 0.1, 1, -4 \le x \le 2$ and $\alpha = 1$ between the two approximated methods and the exact solution Tables 5–10.

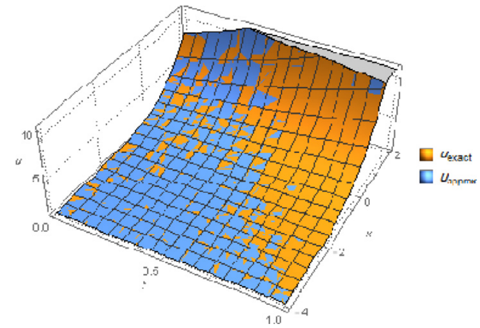

Fig. 1. The space-time graph of approximate solutions $u(x, t)$ and exact solutions at $0 \le t \le 1, -4 \le x \le 2$.

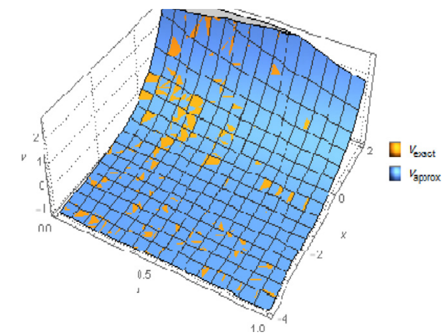

Fig. 2. The space-time graph of approximate solutions $v(x, t)$ and exact solutions at $0 \le t \le 1, -4 \le x \le 2$.

5.2. Example 2 (The inhomogeneous and nonlinear system [7,23,24])

Consider the inhomogeneous nonlinear system of FPDEs:

$$\begin{matrix} D_{*t}^{\alpha} u(x,t) + v(x,t) u_x(x,t) + u(x,t) = 1 \\ D_{*t}^{\alpha} v(x,t) - u(x,t) v_x(x,t) - v(x,t) = 1 \end{matrix}, \quad 0 < \alpha \le 1 \tag{30}$$

with initial conditions

$$u(x,0) = e^x, v(x,0) = e^{-x} \tag{31}$$

At first, we solve this system by using the **LADM** and according to Eqs. (13) and (15) the recursive formulas for the system (30) are

Table 1
The approximate solution $u(x, t)$ of the system (21) using LADM and LVIM at: $t = 0.05, 0.1, 1, -4 \le x \le 2$, $\alpha = 0.25, 0.5, 0.75, 0.95$.

t	x	$\alpha = 0.25$ $u_{LADM} = u_{LVIM}$	$\alpha = 0.5$ $u_{LADM} = u_{LVIM}$	$\alpha = 0.75$ $u_{LADM} = u_{LVIM}$	$\alpha = 0.95$ $u_{LADM} = u_{LVIM}$
0.05	−4	1.036051428687405	1.024033098837850	1.0205857657256676	1.019435781599878
	−3	1.097997943490960	1.065328735852483	1.0559579128969974	1.052831931944848
	−2	1.266386029017834	1.177581915544005	1.1521093777864020	1.143612080568064
	−1	1.724112302034542	1.482717694086218	1.4134761575749883	1.390378108955365
	0	2.968341312384143	2.312162736110218	2.1239447256371595	2.061157719801573
	1	6.350506421659045	4.566828121549507	4.0551985238918780	3.884525746865651
	2	15.54418437904921	10.69564406804474	9.3048906298301900	8.840953921427154
0.1	−4	1.042934504973291	1.027230434426905	1.022338863485111	1.020544421115751
	−3	1.116708084682781	1.074019995083702	1.0607233266800047	1.055845526595156
	−2	1.317245465827464	1.201207207578656	1.1650631154778390	1.151803880144340
	−1	1.862362584919820	1.546937896116047	1.4486880673522466	1.412645728885934
	0	3.344144544130518	2.486731344307873	2.2196606201300200	2.121687386421872
	1	7.372045517591398	5.041354797032579	4.3153813005865230	4.049062439722293
	2	18.32101554058261	11.98554130712945	10.012140743797262	9.288211023734114
1	−4	1.127377392190869	1.088797875387869	1.063713729379336	1.052007549816477
	−3	1.346247650548940	1.241377651072615	1.1731918727952064	1.141371177608807
	−2	1.941198696633821	1.656132482706818	1.4707843206559998	1.384286703161876
	−1	3.558443314029053	2.783553004803663	2.2797244639626406	2.044599562123363
	0	7.954569969667713	5.848199723051324	4.4786517558241385	3.839516007736213
	1	19.90448117329472	14.17877320791059	10.455955855393906	8.718604765447921
	2	52.38770764981316	36.82361973244628	26.703952972428162	21.98134307497448

Table 2
The compassion between the approximate solution $u(x, t)$ of the system (21) using LADM and LVIM at: $t = 0.05, 0.1, 1, -4 \le x \le 2$ and $\alpha = 1$.

t	x	u_{Exact}	$u_{LADM} = u_{LVIM}$	Error $\lvert u_{Exact} - u_{LADM} \rvert$
0.05	−4	1.019254701775387	1.019254701775384	2.8865798640e−015
	−3	1.052339705948432	1.052339705948425	7.7715611723e−015
	−2	1.142274071586514	1.142274071586493	2.1094237468e−014
	−1	1.386741023454501	1.386741023454444	5.7287508071e−014
	0	2.051271096376024	2.051271096375860	1.5631940187e−013
	1	3.857651118063164	3.857651118062740	4.2410519541e−013
	2	8.767901106306770	8.767901106305619	1.1510792319e−013
0.1	−4	1.020241911445804	1.020241911445436	3.6792791036e−012
	−3	1.055023220056407	1.055023220055407	1.0003109452e−013
	−2	1.149568619222635	1.149568619219916	2.7191582319e−012
	−1	1.406569659740599	1.406569659733208	7.3914208087e−012
	0	2.105170918075648	2.105170918055556	2.0091928121e−012
	1	4.004166023946434	4.004166023891818	5.4615867384e−011
	2	9.166169912567652	9.166169912419190	1.4846079921e−011
1	−4	1.049787068367864	1.049782924035073	4.1443327905e−010
	−3	1.135335283236613	1.135324017772097	1.1265464516e−006
	−2	1.367879441171442	1.367848818463960	3.0622707483e−005
	−1	2.000000000000000	1.999916758850712	8.3241149288e−005
	0	3.718281828459045	3.718055555555555	2.2627290349e−004
	1	8.389056098930650	8.388441025408820	6.1507352183e−014
	2	21.08553692318767	21.08386498001012	1.6719431776e−013

Table 3
The approximate solutions $v(x, t)$ of the system (21) using LADM and LVIM at: $t = 0.05, 0.1, 1, -4 \le x \le 2, \alpha = 0.25, 0.5$.

t	x	$\alpha = 0.25$ $v_{LADM} = v_{LVIM}$	$\alpha = 0.5$ $v_{LADM} = v_{LVIM}$	$\alpha = 0.75$ $v_{LADM} = v_{LVIM}$	$\alpha = 0.95$ $v_{LADM} = v_{LVIM}$
0.05	−4	−0.988037944965103	−0.9855237049226642	−0.9836456385909189	−0.982736860925000
	−3	−0.967483763167613	−0.9606493501478669	−0.9555442365656431	−0.953073922750266
	−2	−0.911611704288652	−0.8930338435688919	−0.8791567060861135	−0.872441696931182
	−1	−0.759735701919378	−0.7092358407132113	−0.6715138700627470	−0.653260582698960
	0	−0.346893924499977	−0.2096210694435512	−0.1070821220907282	−0.057464542740106
	1	0.7753263770879149	1.1484726845284885	1.4272024418269855	1.5620770061479066
	2	3.8258374304221094	4.8401542572944130	5.5978202916097190	5.9644473689246090
0.1	−4	−0.988697753585911	−0.9867467672698400	−0.9848300617494617	−0.983660452692930
	−3	−0.969277308951815	−0.9639739783012671	−0.9587638325147162	−0.955584505469944
	−2	−0.916487067202359	−0.9020711198646629	−0.8879084752494593	−0.879266168316926
	−1	−0.772988312334850	−0.7338017046467693	−0.6953036451463379	−0.671811419255666
	0	−0.382918254572002	−0.2763980109745393	−0.1717494354035815	−0.107890944654913
	1	0.6774020952707183	0.9669541378047313	1.2514184591533892	1.4250038341483138
	2	3.5596516345935223	4.3467356902069300	5.1199898857739210	5.5918438563088735
1	−4	−0.985224885455625	−0.9911144679812868	−0.9927214237421447	−0.993188030547617
	−3	−0.959837074620625	−0.9758466197773410	−0.9802147784212185	−0.981483147221568
	−2	−0.890825849763489	−0.9343443054448839	−0.9462181917103625	−0.949665975572137
	−1	−0.703233891274634	−0.8215293185559702	−0.8538058878246102	−0.863177936044527
	0	−0.193306079349337	−0.5148663897179910	−0.6026032014459347	−0.628079069817576
	1	1.1928214256330807	0.3187298772043172	0.0802364961973154	0.0109859061384672
	2	4.9607066343540620	2.5846794618505236	1.9363872380514313	1.7481446174843973

$$
\begin{aligned}
&\pounds_t[u_0(x,t)] = \frac{1}{s}(e^x) + \frac{1}{s^{\alpha+1}}, \pounds_t[v_0(x,t)] = \frac{1}{s}(e^{-x}) + \frac{1}{s^{\alpha+1}} \\
&\pounds_t[u_n(x,t)] = -\frac{1}{s^\alpha}\pounds_t[A_n + \mathrm{u_n}], \\
&\pounds_t[v_{k+1}(x,t)] = \frac{1}{s^\alpha}\pounds_t[B_n + \mathrm{v_n}], \quad n \ge 1.
\end{aligned}
\tag{32}
$$

Where A_n and B_n are the Adomian polynomials that represent the nonlinear terms $v(x, t)u_x(x, t)$ and $u(x, t)v_x(x, t)$, respectively. By using Eq. (11), the first few terms of the Adomian polynomials for v(x, t)u$_x$(x, t) and u(x, t)v$_x$(x, t) are

$$
\begin{aligned}
&A_0 = v_0(u_0)_x \\
&A_1 = v_0(u_1)_x + v_1(u_0)_x \\
&A_2 = v_0(u_2)_x + v_1(u_1)_x + v_2(u_0)_x \\
&A_3 = v_0(u_3)_x + v_1(u_2)_x + v_2(u_1)_x + v_0(u_3)_x \\
&\vdots
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
&B_0 = u_0(v_0)_x \\
&B_1 = u_0(v_1)_x + u_1(v_0)_x \\
&B_2 = u_0(v_2)_x + u_1(v_1)_x + u_2(v_0)_x \\
&B_3 = u_0(v_3)_x + u_1(v_2)_x + u_2(v_1)_x + u_0(v_3)_x \\
&\vdots
\end{aligned}
\tag{34}
$$

By applying the inverse Laplace transform, we can evaluate u_n and $v_n (n \ge 0)$. The first few terms of the decomposition series are given:

$$
\begin{aligned}
&u_0(x,t) = e^x + \frac{t^\alpha}{\Gamma(\alpha+1)}, v_0(x,t) = e^{-x} + \frac{t^\alpha}{\Gamma(\alpha+1)} \\
&u_1(x,t) = -e^x\left\{\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right\} \\
&\quad -\left\{\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right\}
\end{aligned}
$$

Table 4
The compassion between the approximate solutions $v(x, t)$ of the system (21) using LADM and LVIM at $t = 0.05, 0.1, 1$, $-4 \le x \le 2$ and $\alpha = 1$.

t	x	v_{Exact}	$v_{LADM} = v_{LVIM}$	Error $\lvert v_{Exact} - v_{LADM}\rvert$
0.05	−4	−0.9825776253605065	−0.9825776253605036	2.8865798640e−015
	−3	−0.9526410756088590	−0.9526410756088514	7.6605388699e−015
	−2	−0.8712650964121957	−0.8712650964121749	2.0872192862e−014
	−1	−0.6500622508888447	−0.6500622508887880	5.6732396558e−014
	0	−0.0487705754992860	−0.0487705754991320	1.5398793352e−013
	1	1.58570965931584600	1.5857096593162647	4.1877612489e−013
	2	6.02868758058929300	6.0286875805904310	1.1386447341e−012
0.1	−4	−0.9834273245982388	−0.9834273245978798	3.5893510386e−013
	−3	−0.9549507976064422	−0.9549507976054665	9.7566399405e−013
	−2	−0.8775435717470181	−0.8775435717443660	2.6521007612e−012
	−1	−0.6671289163019205	−0.6671289162947114	7.2091221881e−012
	0	−0.0951625819640405	−0.0951625819444444	1.9596102518e−011
	1	1.45960311115694990	1.4596031112102170	5.3267168454e−011
	2	5.68589444227926850	5.6858944424240660	1.4479706322e−010
1	−4	−0.9932620530009145	−0.9932588273534521	3.2256474624e−06
	−3	−0.9816843611112658	−0.9816755928923834	8.7682188824e−06
	−2	−0.9502129316321360	−0.9501890971420800	2.3834490061e−05
	−1	−0.8646647167633873	−0.8645999279021774	6.4788861210e−05
	0	−0.6321205588285577	−0.6319444444444444	1.7611438411e−04
	1	0.00000000000000000	0.0004787285300654	4.7872853007e−04
	2	1.71828182845904500	1.7195831475230867	1.3013190640e−03

Table 5
The approximate solutions $u(x, t)$ of the system (30) using LADM and LVIM at: $t = 0.01, 0.05, 0.1$, $-2 \le x \le 2$, $\alpha = 0.25, 0.5$.

t	x	$\alpha = 0.25$		$\alpha = 0.5$	
		u_{LADM}	u_{LVIM}	u_{LADM}	u_{LVIM}
0.01	−2	0.04218709537885074	0.06252656894315434	0.1208033556111244	0.120806856144133
	−1.5	0.10481072407351527	0.12758934647798814	0.1995098720671460	0.199513990733647
	−1	0.20805963275083572	0.23485973173049910	0.3292749798908985	0.329280117687238
	−0.5	0.37828830466370844	0.41171869761251123	0.5432214733546151	0.543228291411996
	0	0.65894793692949560	0.70330983657621300	0.8959596079199504	0.895969196259451
	0.5	1.12167744247297430	1.18406234973334470	1.4775264733649032	1.477540629127460
	1	1.88458942084306070	1.97668924411805120	2.4363681347583954	2.436389820928498
	1.5	3.14241862725373900	3.28351006461910100	4.0172307771312970	4.017264878844496
	2	5.21622839477088450	5.43809334837297700	6.6236326416667090	6.623687213149899
0.05	−2	−0.4037990217070789	−0.1192375319380223	0.1006788039808965	0.101237640049464
	−1.5	−0.3782715066271705	−0.0622759207765685	0.1700411131975995	0.170683467044438
	−1	−0.3361837495268067	0.03163789915867726	0.2844002277880571	0.285180279172412
	−0.5	−0.2667927691593774	0.18647561169861426	0.4729465325117781	0.473953608755160
	0	−0.1523863838328614	0.44175984186976036	0.7838068356216850	0.785188212579122
	0.5	0.03623785715888020	0.86265238222723630	1.2963288295752775	1.298327324081615
	1	0.34722665545163120	1.55658686619361900	2.1413347427082080	2.144350692043941
	1.5	0.85996050214636370	2.70069141038131100	3.5345139655578550	3.539207414312787
	2	1.70531570139986400	4.58700090828823500	5.8314781841675440	5.838937361896498
0.1	−2	−1.3176691443433968	−0.4159422285932652	0.0754120441332074	0.080632628849818
	−1.5	−1.3637628139138667	−0.3652521747133969	0.1383914967914239	0.144324748557672
	−1	−1.4397584273793242	−0.2816784046687222	0.2422270600060774	0.249335301095990
	−0.5	−1.5650540117797294	−0.1438885523234663	0.4134229617332041	0.422468432713888
	0	−1.7716315069054810	0.08328850812478751	0.6956772863674070	0.707916709475242
	0.5	−2.1122202171672580	0.45784015990095240	1.1610359951389166	1.178541355056383
	1	−2.6737560683361723	1.07537143516018240	1.9282827967960499	1.954470218741718
	1.5	−3.5995721704190666	2.09350838400264960	3.1932589185648093	3.233760640849911
	2	−5.1259848706798060	3.77213242804495770	5.2788519574527175	5.342953971182636

$$v_1(x,t) = e^{-x}\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right\} - \left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right\}$$

$$u_2(x,t) = e^{x}\left\{2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + C_1\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + C_2\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\right\} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$v_2(x,t) = e^{-x}\left\{2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - C_1\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + C_2\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\right\} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$u_3(x,t) = -e^{x}\left\{(C_1+1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - (C_1+2C_2+3)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + (C_2+C_1C_4-C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - C_2C_5\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}\right\} - (5-C_1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - (2C_2+C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)},$$

$$v_3(x,t) = e^{-x}\left\{(C_1+1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + (C_1+3)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\right.$$

Table 6
The approximate solutions $u(x,\ t)$ of the system (30) using LADM and LVIM at: $t = 0.01, 0.05, 0.1,\ -2 \le x \le 2, \alpha = 0.75, 0.95$.

t	x	$\alpha = 0.75$		$\alpha = 0.95$	
		u_{LADM}	u_{LVIM}	u_{LADM}	u_{LVIM}
0.01	−2	0.1307736545984738	0.130773655013575	0.1336081756574512	0.1336081756576903
	−1.5	0.2156126918872653	0.215612692403025	0.2202826661980245	0.2202826661983340
	−1	0.3554886172510174	0.355488617932736	0.3631847423793647	0.3631847423797902
	−0.5	0.5861050306570988	0.586105031612436	0.5987904350067503	0.5987904350073671
	0	0.9663272168122795	0.966327218218738	0.9872385519395576	0.9872385519404898
	0.5	1.5932076227184300	1.593207624868661	1.6276812248900876	1.6276812248915398
	1	2.6267586821210300	2.626758685497535	2.6835926824476717	2.6835926824499810
	1.5	4.3307962981127490	4.330796303511039	4.4244963624988370	4.4244963625025580
	2	7.1402793616714320	7.140279370403082	7.2947612900393235	7.2947612900453755
0.05	−2	0.1206534766232478	0.120654165347700	0.1006788039808965	0.1275545641060124
	−1.5	0.1990476376292254	0.199048471922175	0.1275545615338730	0.2103043536621810
	−1	0.3282977583784716	0.328298832673298	0.2103043503995918	0.3467356918493954
	−0.5	0.5413951816983135	0.541396651689338	0.3467356874484471	0.5716729411087387
	0	0.8927334362571264	0.892735558640891	0.5716729348309533	0.9425317685353943
	0.5	1.4719922897588975	1.471995487756448	0.9425317591632277	1.553974605740630
	1	2.4270286827686367	2.427033654153516	1.5539745912666911	2.5620734172581376
	1.5	4.0016174982165220	4.001625393422811	2.5620733943727982	4.2241473707745705
	2	6.5976755708519680	6.597688286624806	4.2241473340211770	6.9644440514137710
0.1	−2	0.1111908692197648	0.111208880068764	0.1207070938664902	0.1207072441853156
	−1.5	0.1839085697153036	0.1839299941281838	0.1990297273472040	0.1990299143478740
	−1	0.3037997892786997	0.3038268417069596	0.3281619191441071	0.3281621666229143
	−0.5	0.5014669931430502	0.5015033245999785	0.5410649104917897	0.5410652576821884
	0	0.8273651166740236	0.8274167466628888	0.8920826009223993	0.8920831125094196
	0.5	1.3646802850207973	1.3647571380246775	1.4708129335273783	1.4708137161586057
	1	2.2505632321439430	2.2506816708691897	2.4249779428925670	2.4249791724001435
	1.5	3.7111372904163895	3.7113242924050110	3.9981300895907403	3.9981320558728592
	2	6.1192168077229825	6.1195168514223175	6.5918194958995890	6.5918226769175690

Table 7
The compassion between the approximate solution $u(x,\ t)$ of the system (30) using LADM and LVIM with the exact solution at $t = 0.01, 0.05, 0.1\ , -2 \le x \le 2$ and $\alpha = 1$.

t	x	u_{Exact}	u_{LADM}	u_{LVIM}	$Error(1)\ \lvert u_{Exact} - u_{LADM}\rvert$	$Error(2)\ \lvert u_{Exact} - u_{LVIM}\rvert$
0.01	−2	0.133988674668805	0.1339886746687563	0.1339886746687921	4.8738790781e − 014	1.2906342661e − 014
	−1.5	0.220909977959378	0.2209099779593174	0.2209099779593642	6.0840221749e − 014	1.3961054535e − 014
	−1	0.364218979571523	0.3642189795714426	0.3642189795715075	8.0768725041e − 014	1.5820678101e − 014
	−0.5	0.600495578812266	0.6004955788121523	0.6004955788122471	1.1357581542e − 013	1.8762769116e − 014
	0	0.990049833749168	0.9900498337490001	0.9900498337491443	1.6797674362e − 013	2.3758772727e − 014
	0.5	1.632316219955379	1.6323162199551216	1.6323162199553471	2.5734969711e − 013	3.1752378504e − 014
	1	2.691234472349262	2.6912344723488570	2.6912344723492170	4.0500935938e − 013	4.4853010195e − 013
	1.5	4.437095519003664	4.4370955190030160	4.4370955190035980	6.4837024638e − 013	6.6613381478e − 013
	2	7.315533762309567	7.3155337623085170	7.3155337623094650	1.0498268921e − 012	1.0214051827e − 013
0.05	−2	0.128734903587804	0.1287349027665015	0.1287349033837376	8.2130269252e − 010	2.0406665247e − 010
	−1.5	0.212247973826743	0.2122479728142209	0.2122479736061528	1.0125221217e − 09	2.2059021276e − 010
	−1	0.349937749111155	0.3499377477833657	0.3499377488633224	1.3277896560e − 09	2.4783297636e − 010
	−0.5	0.576949810380487	0.5769498085329086	0.5769498100877378	1.8475780905e − 09	2.9274882518e − 010
	0	0.951229424500714	0.9512294217961498	0.9512294241339114	2.7045642392e − 09	3.6680258830e − 010
	0.5	1.568312185490169	1.5683121813726733	1.5683121850012725	4.1174956778e − 09	4.8889647886e − 09
	1	2.585709659315846	2.5857096528688204	2.5857096586256505	6.4470255801e − 09	6.9019545634e − 09
	1.5	4.263114515168817	4.2631145048810460	4.2631145141467370	1.0287770635e − 08	1.0220801983e − 09
	2	7.028687580589293	7.0286875639692035	7.0286875700200280	1.6620089305e − 08	1.5692647182e − 09
0.1	−2	0.122456428252982	0.1224563708620622	0.1224564149289104	5.7390919753e − 08	1.3324071471e − 08
	−1.5	0.201896517994655	0.2018964481857558	0.2018965036510477	6.9808899633e − 08	1.4343607663e − 08
	−1	0.332871083698080	0.3328709934153923	0.3328710676735410	9.0282687215e − 08	1.6024538585e − 08
	−0.5	0.548811636094027	0.5488115120557701	0.5488116172981012	1.2403825644e − 07	1.8795925261e − 08
	0	0.904837418035960	0.9048372383441787	0.9048373946707904	1.7969178079e − 07	2.3365169133e − 08
	0.5	1.491824697641270	1.4918244261923397	1.4918246667426909	2.7144893067e − 07	3.0898579473e − 08
	1	2.459603111156950	2.4596026884260540	2.4596030678378770	4.2273089607e − 07	4.3319072773e − 08
	1.5	4.055199966844675	4.0551992946919850	4.0551999030476710	6.7215268995e − 07	6.3797003236e − 07
	2	6.685894442279269	6.6858933588995640	6.6858943447198650	1.0833797042e − 06	9.7559403223e − 07

$$+ (C_2 + C_1 C_4 - C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + C_2 C_5 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} \Bigg\}$$

$$-(5 - C_1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - (2C_2 + C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}, \qquad (35)$$

Hence, the solution in series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots$$

$$u(x,t) = e^x\Bigg[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (C_1 - 1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - (2C_1 + C_2 + 3)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - (C_2 + C_1 C_4 - C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - C_2 C_5 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} \cdots\Bigg]$$

Table 8
The approximate solutions $v(x, t)$ of the system (30) using LADM and LVIM at: $t = 0.01, 0.05, 0.1,\ -2 \le x \le 2, \alpha = 0.25, 0.5$.

t	x	$\alpha = 0.25$		$\alpha = 0.5$	
		v_{LADM}	v_{LVIM}	v_{LADM}	v_{LVIM}
0.01	−2	11.019632067904507	11.16214673773801	8.3023169334219720	8.302362814681851
	−1.5	6.6640460800750950	6.750826801968885	5.0353778048675680	5.035405189886335
	−1	4.0222496374418190	4.075226011123347	3.0538790599844490	3.053895226465894
	−0.5	2.4199190982649714	2.452392098324162	1.8520393190307363	1.852048681125338
	0	1.4480564992603373	1.468093574590036	1.1230866680812202	1.123091903106580
	0.5	0.8585920359360224	0.871086341635406	0.6809545358015372	0.680957267632868
	1	0.5010637661187722	0.508983150778221	0.4127878419297877	0.412789055497192
	1.5	0.2842119087605991	0.289356463543562	0.2501365201827998	0.250136812876584
	2	0.1526846086572363	0.156146144044625	0.1514835067004673	0.151483240856166
0.05	−2	12.741331742901039	14.28387501109960	9.6900046753926910	9.695583771667678
	−1.5	7.620713021220675	8.578791758443340	5.8743826621614295	5.877754053413207
	−1	4.514900769823024	5.118483849494242	3.5600909252622270	3.562123275729825
	−0.5	2.6311304160392264	3.019701010670502	2.1564020313132570	2.157622212490645
	0	1.4885659406116394	1.746724870845184	1.3050216804350916	1.305749256037090
	0.5	0.7955655555663252	0.974625812958494	0.7886333945505843	0.789062189767965
	1	0.3752395748436826	0.506324062014976	0.4754280668451784	0.475675642598766
	1.5	0.1202989804616185	0.222284692070623	0.2854594328065070	0.285597093399168
	2	−0.034330306436466	0.050006105633914	0.1702376318783237	0.170308625555916
0.1	−2	11.746064849649155	15.91961714640930	10.945743230783265	10.98983220943679
	−1.5	6.8608297538241130	9.485753886324932	6.6297184046013790	6.656617463534122
	−1	3.8977848883020440	5.583418558685089	4.0119170194411750	4.028389865025268
	−0.5	2.1006073312588107	3.216532537991779	2.4241402183033123‘	2.434289245826742
	0	1.0105640419646411	1.780943598396056	1.4611049076327490	1.467418345615942
	0.5	0.3494193665937173	0.910214891786902	0.8769944653251712	0.880981500654139
	1	−0.051585149524503	0.382091234936523	0.5227135734073182	0.525289574199660
	1.5	−0.294806683233433	0.061768044937215	0.3078313503088025	0.309551515392768
	2	−0.442328000530228	−0.13251779081432	0.1774986937723425	0.178699768259501

Table 9
The approximate solutions$v(x, t)$of the system (30) using LADM and LVIM at: $t = 0.01, 0.05, 0.1,\ -2 \le x \le 2, \alpha = 0.75, 0.95$.

t	x	$\alpha = 0.75$		$\alpha = 0.95$	
		v_{LADM}	v_{LVIM}	v_{LADM}	v_{LVIM}
0.01	−2	7.648942626862513	7.648942634909978	7.484632581547689	7.484632581553438
	−1.5	4.639316108297522	4.639316113100426	4.5396591220907440	4.539659122094185
	−1	2.813885350503663	2.813885353338637	2.7534424268901270	2.753442426892169
	−0.5	1.706705628719220	1.706705630360587	1.6700472363603770	1.670047236361570
	0	1.035167181644853	1.035167182562259	1.0129348367188737	1.012934836719551
	0.5	0.627858524318440	0.627858524796742	0.6143760194589611	0.614376019459326
	1	0.380813335683583	0.380813335895555	0.3726378770920195	0.372637877092195
	1.5	0.230972854442051	0.230972854492486	0.2260162821244919	0.226016282124553
	2	0.140090008502967	0.140090008455424	0.1370857894007188	0.137085789400710
0.05	−2	8.304742823778044	8.304753842196892	7.8409504703710220	7.840950525128538
	−1.5	5.036999525995561	5.037006139208576	4.7557753396036790	4.755775372471040
	−1	3.055013027820015	3.055016969140631	2.8845220322103520	2.884522051800663
	−0.5	1.852877449540069	1.852879770276026	1.7495495291876306	1.749549540725003
	0	1.123745364181905	1.123746701983579	1.0611539081735606	1.061153914826579
	0.5	0.681504399431970	0.681505141053865	0.6436208580166085	0.643620861707116
	1	0.413271695330240	0.413272075350820	0.3903742616530845	0.390374263546739
	1.5	0.250580336354915	0.250580497053211	0.2367724364907372	0.236772437294545
	2	0.151903039066066	0.151903066738672	0.1436082201419541	0.143608220284736
0.1	−2	9.011452617977437	9.011697844201986	8.2881865183590280	8.288189326527712
	−1.5	5.465319897317157	5.465467920846330	5.0270280997583540	5.027029791072664
	−1	3.314481678826521	3.314570745940746	3.0490355326970790	3.049036546604968
	−0.5	2.009932355230251	2.009985663471222	1.8493223960907188	1.849322999130844
	0	1.218683193361735	1.218714812749721	1.1216595958789501	1.121659949715179
	0.5	0.738766317216556	0.738784781650233	0.6803097976181642	0.680310000304590
	1	0.447682017720994	0.447692503271555	0.4126176133150120	0.412617724324448
	1.5	0.271130465515962	0.271136111629283	0.2502540961697051	0.250254151574236
	2	0.164046536083754	0.164049246930013	0.1517756450022987	0.151775666680750

$$-(3-C_1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - (2C_2+C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots$$

$$v(x,t) = e^x\Bigg[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$-(C_1+3)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + (C_2+C_1C_4-C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$-C_2C_5\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}\Bigg].$$

$$-(3-C_1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - (2C_2+C_3)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots \tag{36}$$

Table 10
The compassion between the $v(x, t)$ approximate solutions of the system (30) using LADM and LVIM with the exact solution at $t = 0.01, 0.05, 0.1$, $-2 \le x \le 2$ and $\alpha = 1$.

t	x	v_{Exact}	v_{LADM}	v_{LVIM}	Error(1) $\lvert v_{Exact} - v_{LADM}\rvert$	Error(2) $\lvert v_{Exact} - v_{LVIM}\rvert$
0.01	−2	7.463317347319193	7.4633173473182090	7.4633173473191120	9.84101689028e−013	8.0824236193e−014
	−1.5	4.526730794314252	4.5267307943136650	4.5267307943142060	5.87085935422e−013	4.6185277824e−014
	−1	2.745601015016916	2.7456010150165717	2.7456010150168932	3.44613226844e−013	2.3092638912e−014
	−0.5	1.665291194945886	1.6652911949456886	1.6652911949458766	1.97841742988e−013	9.7699626167e−015
	0	1.010050167084168	1.0100501670840591	1.0100501670841666	1.08801856413e−013	1.3322676296e−015
	0.5	0.612626394184416	0.6126263941843613	0.6126263941844193	5.48450174165e−014	3.2196467714e−015
	1	0.371576691022046	0.3715766910220235	0.3715766910220521	2.22044604925e−014	6.3837823916e−015
	1.5	0.225372655539439	0.2253726555394364	0.2253726555394470	2.33146835171e−015	8.2434059578e−015
	2	0.136695425445524	0.1366954254455336	0.1366954254455332	9.74220704109e−015	9.3258734069e−015
0.05	−2	7.767901106306771	7.7679010910175550	7.7679011049770590	1.52892152272e−08	1.3297114521e−09
	−1.5	4.711470182590742	4.7114701734689080	4.7114701818505990	9.12183395485e−09	7.4014305795e−10
	−1	2.857651118063164	2.8576511126820376	2.8576511176806116	5.38112621129e−09	3.8255221213e−10
	−0.5	1.733253017867395	1.7332530147551228	1.7332530177017336	3.11227243977e−09	1.6566170658e−10
	0	1.051271096376024	1.0512710946398809	1.0512710963419130	1.73614322918e−09	3.4111158342e−11
	0.5	0.637628151621773	0.6376281507202947	0.6376281516674515	9.01478669491e−10	4.5678127947e−11
	1	0.386741023454501	0.3867410230592723	0.3867410235485741	3.95228960670e−10	9.4072860613e−11
	1.5	0.234570288093798	0.2345702880056246	0.2345702882172234	8.81730521929e−11	1.2342576961e−10
	2	0.142274071586514	0.1422740716845793	0.1422740717277428	9.80657499650e−11	1.4122925052e−10
0.1	−2	8.166169912567652	8.1661689412265550	8.1661698234362030	9.71341096445e−07	8.9131448888e−08
	−1.5	4.953032424395115	4.9530318431730760	4.9530323744583710	5.81222038498e−07	4.9936743629e−08
	−1	3.004166023946433	3.0041656793435640	3.0041659977824793	3.44602869351e−07	2.6163954114e−08
	−0.5	1.822118800390509	1.8221185993044209	1.8221187886454813	2.01086088003e−07	1.1745027617e−08
	0	1.105170918075648	1.1051708040368870	1.1051709150761404	1.14038760657e−07	2.9995073092e−08
	0.5	0.670320046035639	0.6703199847937517	0.6703200483405587	6.12418876722e−08	2.3049193665e−09
	1	0.406569659740599	0.4065696305216338	0.4065696652628159	2.92189653028e−08	5.5222167394e−09
	1.5	0.246596963941607	0.2465969541455254	0.2465969714152126	9.79608114071e−09	7.4736060951e−09
	2	0.149568619222635	0.1495686212071287	0.1495686278798187	1.98449362499e−09	8.6571836644e−09

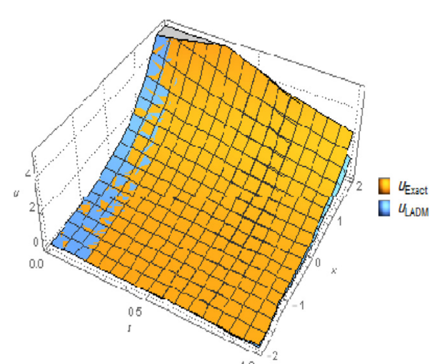

Fig. 3. The space-time graph of $u(x, t)$ using **LADM** and exact solutions at $0 \le t \le 1, -2 \le x \le 2$.

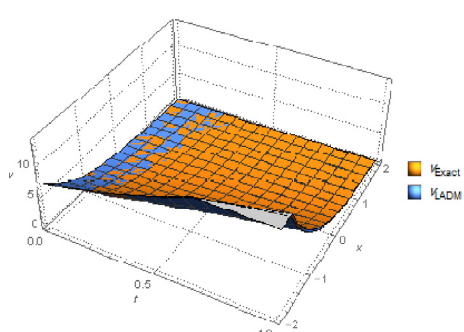

Fig. 4. The space-time graph of $v(x, t)$ using **LADM** and exact solutions at $0 \le t \le 1, -2 \le x \le 2$.

where

$$C_1 = \frac{\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^2}, C_2 = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}, C_3 = \frac{\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2},$$
$$C_4 = \frac{\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}, C_5 = \frac{\Gamma(5\alpha+1)}{\Gamma(\alpha+1)\Gamma(4\alpha+1)} \tag{37}$$

Figs. 3 and 4 illustrate the behavior of the approximate solutions using **LADM** and exact solutions when $\alpha = 1, 0 \le t \le 1, -2 \le x \le 2$

In order to solve system (30) using the **LVIM.** According to Eqs. (19) and (20), the iteration formulas will be

$$\begin{aligned} \pounds[u_n(x,t)] &= \frac{e^x}{s} - \pounds[v(x,t)u_x(x,t) + u(x,t)] + \frac{1}{s^{\alpha+1}} \\ \pounds[v_n(x,t)] &= \frac{e^{-x}}{s} + \pounds[u(x,t)v_x(x,t) + v(x,t)] + \frac{1}{s^{\alpha+1}} \end{aligned} \tag{38}$$

Applying the inverse Laplace transform, we can evaluate u_n and $v_n (n > 0)$.

$$\begin{aligned} u_n(x,t) &= \pounds^{-1}\left[\frac{e^x}{s} - \pounds[v_n(x,t)(u_n(x,t))_x + u_n(x,t)] + \frac{1}{s^{\alpha+1}}\right] \\ v_n(x,t) &= \pounds^{-1}\left[\frac{e^{-x}}{s} + \pounds[u_n(x,t)(v_n(x,t))_x + v_n(x,t)] + \frac{1}{s^{\alpha+1}}\right]. \end{aligned} \tag{39}$$

Where

$$u_0(x,t) = \pounds^{-1}[u_0(x,0)] = e^x, \quad v_0(x,t) = \pounds^{-1}[v_0(x,0)] = e^{-x}.$$

Then

$$u_1(x,t) = e^x\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)}\right], v_1(x,t) = e^{-x}\left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)}\right]$$

$$u_2(x,t) = e^x\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + C_1\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$v_2(x,t) = e^{-x}\left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + C_1\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$u_3(x,t) = e^x\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - C_1\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + C_1C_4\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - C_1C_6\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}\right]$$

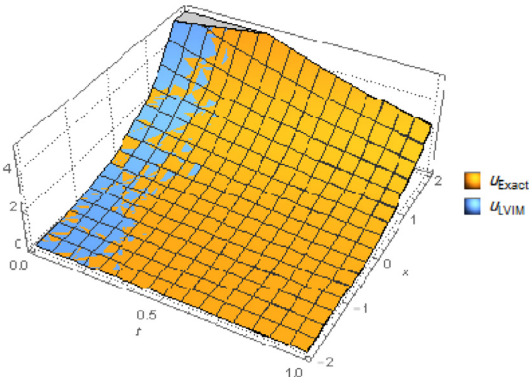

Fig. 5. The space-time graph of $u(x, t)$ using **LVIM** and exact solutions at $0 \le t \le 1, -2 \le x \le 2$.

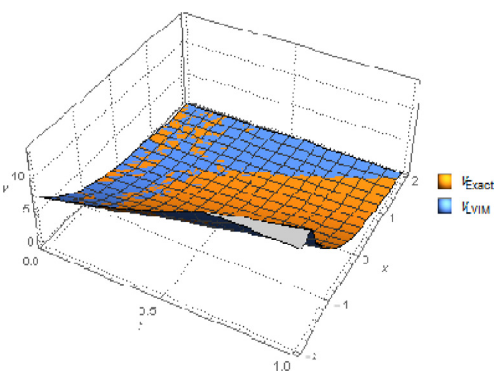

Fig. 6. The space-time graph of $v(x, t)$ using **LVIM** and exact solutions at $0 \le t \le 1, -2 \le x \le 2$.

$$+(C_1-2)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-C_1\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+C_3\frac{t^{5\alpha}}{\Gamma(5\alpha+1)},$$

$$v_3(x,t)=e^{-x}\Bigg[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$-C_1\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-C_1C_4\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-C_1C_6\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}\Bigg]$$

$$+(C_1-2)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+C_1\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-C_3\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}, \quad (40)$$

Where C_1, C_2, C_3, C_4 and C_5 are defined by Eq. (37), while $C_6 = \frac{\Gamma(5\alpha+1)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)}$

Figs. 5 and 6 illustrate the behavior of the approximate solutions using **LVIM** and exact solutions when $\alpha = 1, 0 \le t \le 1, -2 \le x \le 2$.

Tables [5–10] show the numerical results of the approximate solutions of system (30). Also a comparison between the approximate solutions using LADM and LVIM is given for different values of α ($\alpha = 0.25, 0.5, 0.75, 0.95, 0.1$).

The absolute errors between the two approximated methods with the exact solution when $\alpha = 1$ are tabulated.

6. Conclusions

LVIM and **LADM** have been applied to solve effectively, easily and accurately a large class of linear and nonlinear problems with the approximations which rapidly converge to exact solutions. In this work, the **LVIM** and **LADM** have been successfully employed to obtain the analytical approximate solution to the linear and nonlinear systems of FPDES. There are some important points are noted here

First, LVIM and **LADM** provide the solutions in terms of convergent series with easily computable components and provide the components of the exact solution.

Second, it seems that the approximate solution of the inhomogeneous and nonlinear system in second example using **LVIM** converges faster than the approximate solution using **LADM** to exact solution.

References

[1] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl 316 (2006) 753–763.
[2] H. Jafari, Iterative Methods for Solving System of Fractional Differential Equations Ph.D. Thesis, Pune University, 2006.
[3] H. Jafari, S. Seifi, A. Alipoor, M. Zabihi, An iterative method for solving linear and nonlinear fractional diffusion-wave equation, J. Nonlinear Fract. Phenom. Sci. Eng. (2007).
[4] S. Bhalekar, V. Daftardar-Gejji, Solving evolution equations using a new iterative method, Numer. Methods Partial Differential Equ. 26 (4) (2010) 906–916.
[5] V. Daftardar-Gejji, S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, Comput. Math. Appl. 59 (2010) 1801–1809.
[6] H. Jafari, M. Nazari, D. Baleanu, C.M. Khalique, A new approach for solving a system of fractional partial differential equations, Comput. Math. Appl. 66 (2013) 838–843.
[7] J. Fadaei, Application of Laplace–Adomian decomposition method on linear and nonlinear system of PDEs, J. Appl. Math. Sci. 5 (27) (2011) 1307–1315.
[8] G.-C. Wu, D. Baleanu, Variational iteration method for fractional calculus - a universal approach by Laplace transform, Adv. Difference Equ. 2013 (2013).
[9] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John wiley and Sons, Inc., Canada, 1993.
[10] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[11] AY. Luchko, R. Groreflo, The initial value problem for some fractional differential equations with the Caputo derivative, FachbreichMathematik und Informatik preprint series A08-98, FreicUniversitat, Berlin, 1998.
[12] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[13] C. Li, D. Qian, Y.Q. Chen, On Riemann Liouville and Caputo, Derivative Discrete Dyn. Nat. Soc. (2011) 1–15.
[14] Avaz Naghipour, Jalil Manafian, Application of the Laplace decomposition and implicit method for solving Burgers' equation, TWMS J. Pure Appl. Math. 6 (1) (2015) 68–77.
[15] M.M. Hosseini, H. Nasabzadeh, On the convergence of Adomian decomposition method, Appl. Math. Comput. 182 (2006) 536–543.
[16] K. Abbaoui, Y. Cherruault, New ideas for proving convergence of decomposition methods, Comput. Math. Appl. 29 (1995) 103–108.
[17] G. Adomian, R. Rach, Noise terms in decomposition series solution, Comput. Math. Appl. 24 (11) (1992) 61.
[18] B. Babolian, J. Biazar, On the order of convergence of Adomian method, Appl. Math. Comp. 130 (2002) 383–387.
[19] Y. Cherruault, G. Adomian, K. Abbaoui, R. Rach, Further remarks on convergence of decomposition method, Bio-Med. Comput. 38 (1995) 89–93.
[20] IbrahimL. El-Kalla, Error analysis of Adomian series solution to a class of nonlinear differential equations, Appl. Math. E-Notes 7 (2007) 214–221.
[21] Z.M. Odibat, A study on the convergence of variational iteration method, Math. Comput. Modell. 51 (2010) 1181–1192.
[22] H.A. Zedan, S.Sh. Tantawy, Y.M. Sayed, Convergence of the variational iteration method for initial and boundary value problem of fractional integro-differential equations, J. Fract. Calculus Appl 5 (26) (2014) 1–14.
[23] A.-M. Wazwaz, The variational iteration method for solving linear and nonlinear systems of PDEs, Comput. Math. Appl. 54 (2007) 895–902.
[24] M.S.H. Chowdhury, I. Hashim, A.F. Ismail, Analytical treatment of system of linear and nonlinear PDEs by homotopy -perturbation Method, in: Proceedings of the World Congress on Engineering, Vol. III, 2010.