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Extremal A-statistical limit points via ideals☆

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Abstract In this paper, following the line of recent work of Savaş et al. [20] we apply the notion of ideals to A-statistical limit superior and inferior for a sequence of real numbers.

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1. Introduction and background

In [8] Fridy and Orhan introduced the concepts of statistical limit superior and inferior. In [1] Connor and Kline extended the concept of a statistical limit (cluster) point of a number sequence to a A-statistical limit (cluster) point where A is a nonnegative regular summability matrix. In [3] Demirci extended the concepts of statistical limit superior and inferior to A-statistical limit superior and inferior and given some A-statistical analogue of properties of statistical limit superior and inferior for a sequence of real numbers. More works on matrix summability can be seen from [4] where many references can be found.

On the other hand, the notion of ideal convergence was introduced first by Kostyrko et al. [12] as an interesting generalization of statistical convergence [5,22]. More recent applications of ideals can be seen from [2,9–11,13,15–19,23] where more references can be found.

Naturally the purpose of this paper is to unify the above approaches and present the idea of A-summability with respect to ideal concept and make certain observations.

First we introduce some notation. Let $A = (a_{nk})$ denote a summability matrix which transforms a number sequence $x = (x_k)$ into the sequence Ax whose nth term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$.

The notion of a statistically convergent sequence can be defined using the asymptotic density of subsets of the set of positive integers $\mathbb{N} = \{1, 2, \ldots\}$. For any $K \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we denote

$$K(n) := card K \cap \{1, 2, \ldots, n\}$$

and we define lower and upper asymptotic density of the set K by the formulas

$$\underline{\delta}(K) := \liminf_{n\to\infty} \frac{K(n)}{n}; \quad \bar{\delta}(K) := \limsup_{n\to\infty} \frac{K(n)}{n}.$$

If $\underline{\delta}(K) = \bar{\delta}(K) =: \delta(K)$, then the common value $\delta(K)$ is called the asymptotic density of the set K and

$$\delta(K) = \lim_{n\to\infty} \frac{K(n)}{n}.$$

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Obviously all three densities $\underline{\delta}(K)$, $\bar{\delta}(K)$ and $\delta(K)$ (if they exist) lie in the unit interval [0,1].

$$\delta(K) = \lim_n \frac{1}{n}|K_n| = \lim_n (C_1\chi_K)_n = \lim_n \frac{1}{n}\sum_{k=1}^{n}\chi_K(k),$$

if it exists, where C_1 is the Cesaro mean of order one and χ_K is the characteristic function of the set K [6].

The notion of statistical convergence was originally defined for sequences of numbers in the paper [5] and also in [21]. We say that a number sequence $x = (x_k)_{k\in\mathbb{N}}$ statistically converges to a point L if for each $\varepsilon > 0$ we have

$$\delta(K(\varepsilon)) = 0,$$

where

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$$

and in such situation we will write $L = st\text{-}\lim x_k$.

Statistical convergence can be generalized by using a regular nonnegative summability matrix A in place of C_1. Following Freedman and Sember [6], we say that a set $K \subseteq \mathbb{N}$ has A-density if

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk} = \lim_n \sum_{k=1} a_{nk}\chi_K(k) = \lim_n (A\chi_K)_n$$

exists where A is a nonnegative regular summability matrix.

The number sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be A-statistically convergent to L if for every $\varepsilon > 0$, $\delta_A(\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}) = 0$. In this case it is denoted as $st_A\text{-}\lim x_k = L$ [1,14].

The notion of statistical convergence was further generalized in the paper [12,13] using the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subseteq A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal.

Recall the generalization of statistical convergence from [12,13].

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)_{k\in\mathbb{N}}$ be a sequence of points in a metric space (X, ρ). We say that the sequence x is $\mathcal{I}$-convergent (or $\mathcal{I}$-converges) to a point $\xi \in X$, and we denote it by $\mathcal{I} - \lim x = \xi$, if for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, \xi) \geqslant \varepsilon\} \in \mathcal{I}.$$

This generalizes the notion of usual convergence, which can be obtained when we take for $\mathcal{I}$ the ideal $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. A sequence is statistically convergent if and only if it is $\mathcal{I}_\delta$-convergent, where $\mathcal{I}_\delta := \{K \subset \mathbb{N} : \delta(K) = 0\}$ is the admissible ideal of the sets of zero asymptotic density.

The concept of $A^{\mathcal{I}}$-statistically convergent was studied in [20] and the following definition was given:

Definition 1. Let $A = (a_{nk})$ be a non-negative regular matrix. A sequence $(x_k)_{k\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geqslant \delta\right\} \in \mathcal{I}$$

where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$. In this case we write $L = \mathcal{I}\text{-}st_A\text{-}\lim x_k$. We denote the class of all $A^{\mathcal{I}}$-statistically convergent sequences by $S_A(\mathcal{I})$.

We say that a set $K \subseteq \mathbb{N}$ has $A^{\mathcal{I}}$-density if

$$\delta_{A^{\mathcal{I}}}(K) := \mathcal{I}\text{-}\lim_n \sum_{k\in K} a_{nk} = \mathcal{I}\text{-}\lim_n \sum_{k=1} a_{nk}\chi_K(k) = \mathcal{I}\text{-}\lim_n (A\chi_K)_n,$$

exists where A is a nonnegative regular summability matrix. Then a sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for each $\varepsilon > 0$ the set $K(\varepsilon)$ has $A^{\mathcal{I}}$-density zero, where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$.

Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $A^{\mathcal{I}}$-statistically convergent is the A-statistical convergence introduced by [1,14]. Also $A^{\mathcal{I}}$-density coincides with usual A-density in [6].

2. Main results

In this section we study the concepts of extremal $A^{\mathcal{I}}$- statistical limit points ($A^{\mathcal{I}}$-statistical $\liminf x$, $A^{\mathcal{I}}$-statistical $\limsup x$). The result are analogues to those given by Fridy [7], Fridy and Orhan [8] and Kostyrko et al. [13]. These notions generalize the notions of A-statistical limit point and A-statistical cluster point.

Following the line of Savaş et al. [20] we now introduce the following definition using ideals.

Definition 2. Let $\mathcal{I}$ be an ideal of $\mathcal{P}(\mathbb{N})$. A number ζ is said to be an $A^{\mathcal{I}}$-statistical cluster point of the number sequence $x = (x_k)$ if for each $\varepsilon > 0$, $\delta_{A^{\mathcal{I}}}(K_\varepsilon) \neq 0$ where $K_\varepsilon = \{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}$. We denote the set of all $A^{\mathcal{I}}$-statistically cluster points of x by $\Gamma_{A^{\mathcal{I}}}(x)$.

Note that the statement $\delta_{A^{\mathcal{I}}}(K_\varepsilon) \neq 0$ means that either $\delta_{A^{\mathcal{I}}}(K_\varepsilon) > 0$ or K_ε fails to have $A^{\mathcal{I}}$-density.

Throughout the paper $A = (a_{nk})$ will be a nonnegative regular matrix summability method. For a number sequence $x = (x_k)$, we write

$$M_t = \{k : x_k > t\} \quad \text{and} \quad M^t = \{k : x_k < t\}, \quad \text{for} \quad t \in \mathbb{R}.$$

Definition 3. Let $A = (a_{nk})$ be a nonnegative regular matrix summability method and x be a number sequence. Then if there is a $t \in \mathbb{R}$ such that $\delta_{A^{\mathcal{I}}}(M_t) \neq 0$, we define

$$\mathcal{I}\text{-}st_A\text{-}\limsup x = \sup\{t \in \mathbb{R} : \delta_{A^{\mathcal{I}}}(M_t) \neq 0\}.$$

If $\delta_{A^{\mathcal{I}}}(M_t) = 0$ holds for each $t \in \mathbb{R}$, then we define $\mathcal{I}\text{-}st_A\text{-}\limsup x = -\infty$.

Also, if there is a $t \in \mathbb{R}$ such that $\delta_{A^{\mathcal{I}}}(M^t) \neq 0$, we define

$$\mathcal{I}\text{-}st_A\text{-}\liminf x = \inf\{t \in \mathbb{R} : \delta_{A^{\mathcal{I}}}(M^t) \neq 0\}.$$

If $\delta_{A^{\mathcal{I}}}(M^t) = 0$ holds for each $t \in \mathbb{R}$ then we define $\mathcal{I}\text{-}st_A\text{-}\liminf x = +\infty$.

Remark 1. If $\mathcal{I} = \mathcal{I}_f$, then the above Definition 3 yields the usual definition of $st\text{-}\lim\sup_{k\to\infty} x_k$ and $st\text{-}\lim\inf_{k\to\infty} x_k$ introduced by [8].

Definition 4. The real number sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-statistically bounded if there is a number K such that $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k| > K\}) = 0$.

Note that if we take $A = C_1$ (the Cesaro matrix of order 1) and $\mathcal{I} = \mathcal{I}_f$ in Definitions 1 and 2, then we get Definitions 1 and 2 of [8].

The next statement is an analogue of Theorems 1 and 2 of [3].

Theorem 1. $\beta = \mathcal{I}\text{-}st_A\text{-}\limsup x_k$ *if and only if for each* $\varepsilon > 0$,

$$\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0 \quad \text{and} \quad \delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0. \tag{2.1}$$

Proof. We prove the necessity first. Let $\varepsilon > 0$ be given. Since $\beta + \varepsilon > \beta$, we have $(\beta + \varepsilon) \notin \{t : \delta_{A^{\mathcal{I}}}(M_t) \neq 0\}$ and $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0$. Similarly, since $\beta - \varepsilon < \beta$, there exists some t' such that $\beta - \varepsilon < t' < \beta$ and $t' \in \{t : \delta_{A^{\mathcal{I}}}(M_t) \neq 0\}$. Thus $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > t'\}) \neq 0$ and $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0$.

Now let us prove the sufficiency. If $\varepsilon > 0$ then $(\beta + \varepsilon) \notin \{t : \delta_{A^{\mathcal{I}}}(M_t) \neq 0\}$ and $\mathcal{I}\text{-}st_A\text{-}\limsup x \leqslant \beta + \varepsilon$. On the other hand, we already have $\mathcal{I}\text{-}st_A\text{-}\limsup x \geqslant \beta - \varepsilon$, and this means that $\mathcal{I}\text{-}st_A$-lim sup $x = \beta$, as desired. □

The dual statement for $\mathcal{I}\text{-}st_A\text{-}\liminf x$ is as follows.

Theorem 2. $\alpha = \mathcal{I}\text{-}st_A\text{-}\liminf x$ *if and only if for each* $\varepsilon > 0$,

$$\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}) \neq 0 \quad \text{and} \quad \delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}) = 0. \tag{2.2}$$

Proof. Similarly as in Theorem 1. □

By Definition 2 we see that Theorem 1 can be interpreted by saying that $\mathcal{I}\text{-}st_A\text{-}\limsup x$ and $\mathcal{I}\text{-}st_A\text{-}\liminf x$ are the greatest and the least $A^{\mathcal{I}}$-statistically cluster points of (x_k). The next theorem reinforces this observation.

Theorem 3. *For every real sequence* x,

$$\mathcal{I}\text{-}st_A\text{-}\liminf x \leqslant \mathcal{I}\text{-}st_A\text{-}\limsup x.$$

Proof. If x_k is any real number sequence then we have three possibilities:

(1) $\mathcal{I}\text{-}st_A\text{-}\limsup x_k = +\infty$. In this case there is nothing to prove.

(2) $\mathcal{I}\text{-}st_A\text{-}\limsup x_k = -\infty$. If this is the case, then we have

$$t \in \mathbb{R} \Rightarrow \delta_{A^{\mathcal{I}}}(M_t) = 0$$

and

$$t \in \mathbb{R} \Rightarrow \delta_{A^{\mathcal{I}}}(M^t) \neq 0.$$

Thus, $\mathcal{I}\text{-}st_A\text{-}\liminf x_k = \inf\{t : \delta_{A^{\mathcal{I}}}(M^t) \neq 0\} = \inf \mathbb{R} = -\infty$ and $\mathcal{I}\text{-}st_A\text{-}\liminf x_k \leqslant \mathcal{I}\text{-}st_A\text{-}\limsup x_k$.

(3) $-\infty < \mathcal{I}\text{-}st_A\text{-}\limsup x_k < +\infty$. For this case there exists a $\beta \in \mathbb{R}$ such that $\beta = \mathcal{I}\text{-}st_A\text{-}\limsup x_k = \sup\{t : \delta_{A^{\mathcal{I}}}(M_t) \neq 0\}$. For any $t \in \mathbb{R}$,

$$\beta < t \Rightarrow \delta_{A^{\mathcal{I}}}(M_t) = 0 \quad \text{and} \quad \delta_{A^{\mathcal{I}}}(M^t) \neq 0.$$

But this means that $\mathcal{I}\text{-}st_A\text{-}\liminf x_k = \inf\{t : \delta_{A^{\mathcal{I}}}(M^t) \neq 0\} \leqslant \beta$. □

Remark 2. If $\mathcal{I}\text{-}st_A\text{-}\lim x_k$ exists, then a sequence x_k is $A^{\mathcal{I}}$-statistically bounded.

Remark 3. Note that ideal boundedness of real number sequences implies that $\mathcal{I}\text{-}st_A\text{-}\limsup$ and $\mathcal{I}\text{-}st_A\text{-}\liminf$ are finite.

Theorem 4. *A real number sequence x_k is $\mathcal{I}\text{-}st_A$-convergent if and only if* $\mathcal{I}\text{-}st_A\text{-}\liminf x = \mathcal{I}\text{-}st_A\text{-}\limsup x$.

Proof. We prove the necessity first. Let $L = \mathcal{I}\text{-}st_A\text{-}\lim x_k$. Then

$$\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > L + \varepsilon\}) = 0 \quad \text{and} \quad \delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k < L - \varepsilon\}) = 0.$$

Then for any $t \geqslant L + \varepsilon$ and $t' < L - \varepsilon$, the sets $\delta_{A^{\mathcal{I}}}(M_t) = 0$ and $\delta_{A^{\mathcal{I}}}(M^{t'}) = 0$. We conclude $\sup\{t : \delta_{A^{\mathcal{I}}}(M_t) \neq 0\} \leqslant L + \varepsilon$ and $\inf\{t' : \delta_{A^{\mathcal{I}}}(M^{t'}) \neq 0\} \geqslant L - \varepsilon$. Combining with Theorem 3, we conclude that $L = \mathcal{I}\text{-}st_A\text{-}\liminf x_k = \mathcal{I}\text{-}st_A\text{-}\limsup x_k$.

To prove sufficiency, let $\varepsilon > 0$ and $L = \mathcal{I}\text{-}st_A\text{-}\liminf x_k = \mathcal{I}\text{-}st_A\text{-}\limsup x_k$. Since

$$\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\} \subseteq \{k \in \mathbb{N} : x_k > L + \varepsilon\} \cup \{k \in \mathbb{N} : x_k < L - \varepsilon\}$$

and the property of additivity of the ideal $\mathcal{I}$, the union of these sets on the righ-hand side also belongs to $\mathcal{I}$. We conclude that $L = \mathcal{I}\text{-}st_A\text{-}\lim x_k$. □

We have for bounded sequences the following result.

Theorem 5. *Suppose that $x = (x_k)$ is a bounded real sequence. Then*

$$\mathcal{I}\text{-}st_A\text{-}\limsup x_k = \max \Gamma_{A^{\mathcal{I}}}(x)$$

and

$$\mathcal{I}\text{-}st_A\text{-}\liminf x_k = \min \Gamma_{A^{\mathcal{I}}}(x).$$

Proof. Let $\mathcal{I}\text{-}st_A\text{-}\limsup x = L = \sup\{t : \delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > t\}) \neq 0\}$. If $L' > L$, then there exists some $\varepsilon > 0$ such that $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > L' - \varepsilon\}) = 0$. This means that there exists some $\varepsilon > 0$ such that $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - L'| < \varepsilon\}) = 0$, that is, $L' \notin \Gamma_{A^{\mathcal{I}}}(x)$.

Now, we show that L is in fact an $A^{\mathcal{I}}$-statistically cluster point of x. Clearly, for each $\varepsilon > 0$ there exists some $t \in (L - \varepsilon, L + \varepsilon)$ such that $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : x_k > t\}) \neq 0$, and this means $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} : |x_k - L| < \varepsilon\}) \neq 0$. □

Let $\mathcal{I} = \mathcal{I}_f$. Then all these results imply the similar theorems for A-statistical of a sequence and extremal $\mathcal{I}$-limit points which are investigated in [3,13].

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