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Original Article

Homomorphisms and subalgebras of decomposable *MS*-algebras

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ABSTRACT

Decomposable *MS*-algebras were introduced and characterized by Badawy et al. [1] in terms of decomposable *MS*-triple (M, D, φ), where M is a de Morgan algebra, D is a distributive lattice with unit and φ is a bounded lattice homomorphism of M into the lattice of filters of D. In this paper we study homomorphisms, subalgebras and solve the “Fill-in” problem for such decomposable *MS*-algebras.

1. Introduction

Blyth and Varlet [2] have studied a new variety of the so-called Morgan Stone algebras (briefly *MS*-algebras) as a common abstraction of the classes of de Morgan and Stone algebras. Such algebras are bounded distributive lattices with a unary operation satisfying certain identities. Blyth and Varlet [3] described the lattice of subvarieties of the variety **MS** of all *MS*-algebras. The class **MS** contains the well-known classes such as Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras. In 2012 Badawy et al. [1] presented a simple triple construction of principal *MS*-algebras and they showed that there exists a one-to-one correspondence between the principal *MS*-algebras and the principal *MS*-triples. They also introduced the class of decomposable *MS*-algebras which contains the class of principal *MS*-algebras and they presented a triple construction of decomposable *MS*-algebras generalizing the construction of principal *MS*-algebras. Moreover, they investigated that there exists a one-to-one correspondence between the decomposable *MS*-algebras and the decomposable *MS*-triples. Luo [4] considered special kind of Principal congruences on *MS*-algebras. Also, Luo [5] investigated the relationship between principal congruence and Kernel ideals of Symmetric de-Morgan algebras. Also, Badawy [6] introduced the notion of d_L-filters of principal *MS*-algebras. S. El-Assar and A. Badawy [7] studied homomorphisms and subalgebras of *MS*-algebras for the class K_2. Recently, Badawy [8] characterized a subclass of the class of modular generalized *MS*-algebras which contains the class of K_2-algebras by means of quadruples. Moreover, Badawy [9] constructed principal generalized K_2-algebras in terms of triples.

According to the characterization [1] of decomposable *MS*-algebras by means of the decomposable *MS*-triple (M, D, φ), we study some properties of this triple. In Section 3, we define the homomorphism between two decomposable *MS*-triples. We show that homomorphisms of decomposable *MS*-algebras are the same as the homomorphisms of their associated decomposable *MS*-triples. In Section 4, using decomposable *MS*-triples, we characterize subalgebras of decomposable *MS*-algebras. Also, we solve the following fill in problem:

“Let L be a decomposable *MS*-algebra, M_1 a subalgebra of $L^{\circ\circ}$, and D_1 a sublattice of $D(L)$ containing 1. We can fill in $(M_1, D_1, ?)$ such that it will become the decomposable *MS*-triple associated with a subalgebra of L.”

Moreover, we solve the above fill in problem to obtain K_2-subalgebras and Stone subalgebras of a decomposable *MS*-algebra L.

Finally, a subalgebra of a decomposable *MS*-algebra $L = \{(a, a^{\circ}\varphi \vee [x)) : a \in M, x \in D\}$ associated with the decomposable *MS*-triple (M, D, φ) is characterized. Also, the greatest Stone subalgebra of L is determined.

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2. Preliminaries

In this section, we present certain definitions and results. We refer the reader to Refs. [1–3,9–12] as a guide references.

A de Morgan algebra is an algebra $(L; \vee, \wedge, ^{-}, 0, 1)$ of type (2,2,1,0,0) where $(L;\ \vee,\ \wedge,\ 0,\ 1)$ is a bounded distributive lattice and $^{-}$ the unary operation of involution satisfies:

$$\bar{\bar{x}} = x,\ \overline{(x \vee y)} = \bar{x} \wedge \bar{y},\ \overline{(x \wedge y)} = \bar{x} \vee \bar{y}.$$

An *MS*-algebra is an algebra $(L;\ \vee,\ \wedge,\ ^{\circ},\ 0,\ 1)$ of type (2,2,1,0,0) where $(L;\ \vee,\ \wedge,\ 0,\ 1)$ is a bounded distributive lattice and the unary operation $^{\circ}$ satisfies:

$$x \leq x^{\circ\circ},\ (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ},\ 1^{\circ} = 0.$$

The class **MS** of all *MS*-algebras is equational. A de Morgan algebra is an *MS*-algebra satisfying the identity, $x = x^{\circ\circ}$. A K_2-algebra is an *MS*-algebra satisfying the additional two identities,

$$x \wedge x^{\circ} = x^{\circ} \wedge x^{\circ\circ},\ (x \wedge x^{\circ}) \vee (y \vee y^{\circ}) = y \vee y^{\circ}.$$

The class **S** of Stone algebras is a subclass of **MS** and is characterized by the identity $x \wedge x^{\circ} = 0$. A Boolean algebra is an *MS*-algebra satisfying the identity $x \vee x^{\circ} = 1$.

We recall some of the basic properties of *MS*-algebras which were proved in [2] or [11].

Theorem 2.1. *[2] For any two elements a, b of an MS-algebra L, we have*

(1) $0^{\circ} = 1$,
(2) $a \leq b \Rightarrow b^{\circ} \leq a^{\circ}$,
(3) $a^{\circ\circ\circ} = a^{\circ}$,
(4) $(a \vee b)^{\circ} = a^{\circ} \wedge b^{\circ}$,
(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Definition 2.2. [11] A bounded sublattice L_1 of an *MS*-algebra L is called a subalgebra of L if $x^{\circ} \in L_1$ for every $x \in L_1$.

Definition 2.3. A subalgebra L_1 of an *MS*-algebra L is called a de Morgan (Boolean) subalgebra of L if $x^{\circ\circ} = x$ $(x \vee x^{\circ} = 1)$ for every $x \in L_1$.

Definition 2.4. A de Morgan subalgebra L_1 of an *MS*-algebra L is called a Kleene subalgebra of L if $x \wedge x^{\circ} \leq y \vee y^{\circ}$ for every $x, y \in L_1$.

Theorem 2.5. *[1] Let L be an MS-algebra. Then*

(1) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ *is a de Morgan subalgebra of L,*
(2) $D(L) = \{x \in L : x^{\circ} = 0\}$ *is a filter (filter of dense elements) of L.*

For any lattice L, let $F(L)$ denote to the set of all filters of L. It is known that $(F(L); \wedge, \vee)$ is a distributive lattice if and only if L is a distributive lattice, where

$$F \wedge G = F \cap G \text{ and } F \vee G = \{x \in L : x \geq f \wedge g, f \in F, g \in G\} \quad \text{for every } F, G \in F(L).$$

If L is a distributive lattice, then $F \vee G = \{x \in L : x = f \wedge g, f \in F, g \in G\}$. Also, $[a) = \{x \in L : x \geq a\}$ is a principal filter of L generated by a.

Definition 2.6. [12] Let $L = (L; \vee, \wedge, 0_L, 1_L)$ and $L_1 = (L_1; \vee, \wedge, 0_{L_1}, 1_{L_1})$ be bounded lattices. The map h: $L \to L_1$ is called (0,1) lattice homomorphism if

(1) $0_L h = 0_{L_1}$ and $1_L h = 1_{L_1}$,
(2) h is a $\vee$-homomorphism, that is, $(x \vee y)h = xh \vee yh$ for every $x, y \in L$,
(3) h is a $\wedge$-homomorphism, that is, $(x \wedge y)h = xh \wedge yh$ for every $x, y \in L$.

Now we recall some important definitions and results from [1] which needed throughout this paper.

Definition 2.7. [1] An *MS*-algebra $(L;\ \vee,\ \wedge,\ ^{\circ},\ 0,\ 1)$ is called decomposable *MS*-algebra if for every $x \in L$ there exists $d \in D(L)$ such that $x = x^{\circ\circ} \wedge d$.

The class of decomposable *MS*-algebras contains the class **M** of all de Morgan algebras and the class **S** of all Stone algebras.

Definition 2.8. [1] A decomposable *MS*-triple is (M, D, φ), where

(i) M is a de Morgan algebra,
(ii) D is a distributive lattice with 1,
(iii) φ is a (0, 1)-homomorphism from M into $F(D)$

such that for every element $a \in M$ and for every $y \in D$ there exists an element $t \in D$ with $a\varphi \cap [y) = [t)$.

Let L be a decomposable *MS*-algebra. Define $\varphi(L)$: $L^{\circ\circ} \to F(D(L))$ by

$$a\varphi(L) = [a^{\circ}) \cap D(L) \text{ for all } a \in L^{\circ\circ}.$$

It is known that $\varphi(L)$ is a (0,1) lattice homomorphism and $a\varphi(L) \cap [y)$ is a principal filter of $D(L)$ (see [1]). The triple $(L^{\circ\circ}, D(L), \varphi(L))$ is called the decomposable *MS*-triple associated with L.

The following Theorem presents a triple construction for decomposable *MS*-algebras which was proved in [1].

Theorem 2.9. *[1] Let* (M, D, φ)*be a decomposable MS-triple. Then*

$$L = \{(a, a^{\circ}\varphi \vee [x)) : a \in M, x \in D\}$$

is a decomposable MS-algebra, if we define

$$(a, a^{\circ}\varphi \vee [x)) \vee (b, b^{\circ}\varphi \vee [y)) = (a \vee b, (a^{\circ}\varphi \vee [x)) \cap (b^{\circ}\varphi \vee [y))),$$
$$(a, a^{\circ}\varphi \vee [x)) \wedge (b, b^{\circ}\varphi \vee [y)) = (a \wedge b, (a^{\circ}\varphi \vee [x)) \vee (b^{\circ}\varphi \vee [y))),$$
$$(a, a^{\circ}\varphi \vee [x))^{\circ} = (a^{\circ}, a\varphi),$$
$$1_L = (1, [1)),$$
$$0_L = (0, D).$$

Conversely, every decomposable MS-algebra L can be associated with the decomposable MS-triple $(L^{\circ\circ}, D(L), \varphi(L))$, *where* $a\varphi(L) = [a^{\circ}) \cap D(L)$.

The decomposable *MS*-algebra L constructed by Theorem 2.9 is called the decomposable *MS*-algebra associated with the decomposable *MS*-triple$(M,\ D,\ \varphi)$, the construction of L described in Theorem 2.9 is called a decomposable *MS*-construction and Theorem 2.9 is called the construction Theorem.

Lemma 2.10. *[1] Let L be a decomposable MS-algebra associated with the decomposable MS-triple* (M, D, φ). *Then*

(1) $L^{\circ\circ} = \{(a, a^{\circ}\varphi) : a \in M\}$,
(2) $D(L) = \{(1, [x)) : x \in D\}$,
(3) $D \cong D(L)$ *and* $M \cong L^{\circ\circ}$.

3. Homomorphisms of decomposable *MS*-algebras

In this section, we define a homomorphisms between two decomposable *MS*-triples. A one-to-one correspondence between homomorphisms of decomposable *MS*-algebras and homomorphisms of decomposable *MS*-triples is obtained.

Definition 3.1. A (0,1) lattice homomorphism h: $L \to L_1$ of an *MS*-algebra L into an *MS*-algebra L_1 is called a homomorphism if $x^{\circ}h = (xh)^{\circ}$ for all $x \in L$.

Let h: $L \to L_1$ be a homomorphism of an *MS*-algebra into an *MS*-algebra L_1. Then, we use $h_{L^{\circ\circ}}, h_{D(L)}$ to denote the restrictions of a homomorphism h to $L^{\circ\circ}$ and $D(L)$, respectively.

Lemma 3.2. *A homomorphism h: $L \to L_1$ of a decomposable MS-algebra into a decomposable MS-algebra L_1 is onto (one-to-one) if and only if $h_{L^{\circ\circ}}$ and $h_{D(L)}$ are onto (one-to-one).*

Proof. Combine homomorphism's properties with Definition 2.7. □

Now, we define a homomorphism of decomposable *MS*-triples.

Definition 3.3. Let (M, D, φ) and (M_1, D_1, φ_1) be decomposable *MS*-triples. A homomorphism of the triple (M, D, φ) into (M_1, D_1, φ_1) is a pair (f, g), where f is a homomorphism of M into M_1, g is a homomorphism of D into D_1 preserving 1 such that for every $a \in M$,

$$a\varphi g \subseteq af\varphi_1 \quad (1)$$

Lemma 3.4. *Let (f, g) be a homomorphism of a decomposable MS-triple (M, D, φ) into a decomposable MS-triple (M_1, D_1, φ_1). Let a, b $\in M$ and x, y, t $\in D$. Then*

(i) $a\varphi \cap [y) = [t)$ *implies* $af\varphi_1 \cap [yg) = [tg)$,
(ii) $(a^\circ f\varphi_1 \vee [xg)) \cap (b^\circ f\varphi_1 \vee [yg)) = (a \vee b)^\circ f\varphi_1 \vee [tg)$.

Proof.

(i) Let $a\varphi \cap [y) = [t)$. Then $t = x_1 \vee y, x_1 \in a\varphi$, so $x_1 g \in a\varphi g \subseteq af\varphi_1$. If $t_1 \in af\varphi_1 \cap [yg)$, then $t_1 \geq x_1 g \vee yg = (x_1 \vee y)g = tg$,. Hence $af\varphi_1 \cap [yg) \subseteq [tg)$. Conversely, $af\varphi_1 \cap [yg) \supseteq a\varphi g \cap [yg) = (a\varphi \cap [y))g = [t)g = [tg)$.

(ii) Let $a^\circ\varphi \cap [y) = [t_1)$ and $b^\circ\varphi \cap [x) = [t_2)$ for some $t_1, t_2 \in D$. Then by (1), $a^\circ f\varphi_1 \cap [yg) = [t_1 g)$ and $b^\circ f\varphi_1 \cap [xg) = [t_2 g)$. So by distributivity of $F(D_1)$, we get

$$\begin{aligned}
&(a^\circ f\varphi_1) \vee [xg)) \cap (b^\circ f\varphi_1 \vee [yg)) \\
&= (a^\circ f\varphi_1 \cap (b^\circ f\varphi_1 \vee [yg))) \vee ([xg) \cap (b^\circ f\varphi_1 \vee [yg))) \\
&= (a^\circ f\varphi_1) \cap b^\circ f\varphi_1) \vee (a^\circ f\varphi_1 \cap [yg)) \vee ([xg) \cap b^\circ f\varphi_1) \\
&\quad \vee ([xg) \cap [yg)) \\
&= (a \vee b)^\circ f\varphi_1 \vee [t_1 g) \vee [t_2 g) \vee [(x \vee y)g) \\
&= (a \vee b)^\circ f\varphi_1 \vee [(t_1 \wedge t_2 \wedge (x \vee y))g = (a \vee b)^\circ f\varphi_1 \vee [tg),
\end{aligned}$$

where, $t = t_1 \wedge t_2 \wedge (x \vee y) \in D$. □

The following Theorem shows that homomorphisms of decomposable *MS*-algebras are the same as homomorphisms of decomposable *MS*-triples.

Theorem 3.5. *Let L and L_1 be decomposable MS-algebras, (M, D, φ) and (M_1, D_1, φ_1) be the associated decomposable MS-triples, respectively. Let h be a homomorphism of L into L_1 and h_M, h_D the restrictions of h to M and D, respectively. Then (h_M, h_D) is a homomorphism of the decomposable MS-triples. Conversely, every homomorphism (f, g) of the decomposable MS-triples uniquely determines a homomorphism h of L into L_1 with $h_M = f, h_D = g$ by the following rule:*

$$xh = x^{\circ\circ} f \wedge dg, \quad \text{for all } x \in L,$$

where $x = x^{\circ\circ} \wedge d$ for some $d \in D(L)$.

Proof. To prove the first statement, we have to verify that $a\varphi g \subseteq af\varphi_1$, $\forall a \in M$ with $g = h_D$ and $f = h_M$. Evidently,

$$\begin{aligned}
a\varphi h &= \{xh : x \in a\varphi\} \\
&= \{xh : x \in [a^\circ) \cap D\} \\
&\subseteq \{y : y \in [(ah)^\circ) \cap D_1\} \\
&= ah\varphi_1.
\end{aligned}$$

Then (h_M, h_D) is a homomorphism of the decomposable *MS*-triples (M, D, φ) and (M_1, D_1, φ_1).

Conversely, let $a\varphi g \subseteq af\varphi_1$, $\forall a \in M$ holds. We represent the elements of L and L_1 as in the construction Theorem (Theorem 2.9), that is,

$$L = \{(a, a^\circ\varphi \vee [x)) : a \in M, x \in D\},$$

and

$$L_1 = \{(b, b^\circ\varphi_1 \vee [y)) : b \in M_1, y \in D_1\}.$$

Define h: $L \to L_1$ by

$$(a, a^\circ\varphi \vee [x))h = (af, (a^\circ f)\varphi_1 \vee [xg))$$

We will show that h is well defined. Let $(a, a^\circ\varphi \vee [x)) = (b, b^\circ\varphi \vee [y))$. Then $a = b$ and $a^\circ\varphi \vee [x) = b^\circ\varphi \vee [y)$. Hence, $x \geq x_1 \wedge y$ and $y \geq y_1 \wedge x$ for some $x_1, y_1 \in a^\circ\varphi$. Since g is a homomorphism and $a\varphi g \subseteq af\varphi_1$, $\forall a \in M$, we have $xg \geq x_1 g \wedge yg$ and $yg \geq y_1 g \wedge xg$ for some $x_1 g, y_1 g \in a^\circ\varphi g \subseteq a^\circ f\varphi_1$. So we obtain $(a^\circ f)\varphi_1 \vee [xg) = (b^\circ f)\varphi_1 \vee [yg)$. Thus $(a, a^\circ\varphi \vee [x))h = (b, b^\circ\varphi_1 \vee [y))h$. Therefore h is a map of L into L_1. Obviously, $h_M = f$ and $h_D = g$.

To prove that h is a homomorphism, let $(a, a^\circ\varphi \vee [x)), (b, b^\circ\varphi \vee [y)) \in L$. We get

$$\begin{aligned}
&((a, a^\circ\varphi \vee [x)) \wedge (b, b^\circ\varphi \vee [y)))h \\
&= (a \wedge b, (a^\circ\varphi \vee [x)) \vee (b^\circ\varphi \vee [y)))h \\
&= (a \wedge b, (a \wedge b)^\circ\varphi \vee [x \wedge y))h \\
&= ((a \wedge b)f, (a \wedge b)^\circ f\varphi \vee [(x \wedge y)g)) \\
&= (af, (a^\circ f)\varphi_1 \vee [xg)) \wedge (bf, (b^\circ f)\varphi_1 \vee [yg)) \\
&= (a, a^\circ\varphi \vee [x))h \wedge (b, b^\circ\varphi \vee [y))h,
\end{aligned}$$

Let $a^\circ\varphi \cap [y) = [t_1)$ and $b^\circ\varphi \cap [x) = [t_2)$ for $t_1, t_2 \in D$. Thus by Lemma 4.2 (ii), we have

$$\begin{aligned}
&((a, a^\circ\varphi \vee [x)) \vee (b, b^\circ\varphi \vee [y)))h \\
&= (a \vee b, (a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)))h \\
&= (a \vee b, (a \vee b)^\circ\varphi \vee [t))h \\
&= ((a \vee b)f, (a \vee b)^\circ f\varphi_1 \vee [tg)) \\
&= ((a \vee b)f, (a^\circ f\varphi_1 \vee [xg)) \cap (b^\circ f\varphi_1 \vee [yg)) \\
&= (af, a^\circ f\varphi_1 \vee [xg)) \vee (bf, b^\circ f\varphi_1 \vee [yg)) \\
&= (a, a^\circ\varphi \vee [x))h \vee (b, b^\circ\varphi \vee [y))h,
\end{aligned}$$

where

$$\begin{aligned}
&(a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)) \\
&= ((a^\circ\varphi \vee [x)) \cap b^\circ\varphi) \vee ((a^\circ\varphi \vee [x)) \cap [y)) \\
&= (a^\circ\varphi \cap b^\circ\varphi) \vee ([x) \cap b^\circ\varphi) \vee (a^\circ\varphi \cap [y)) \vee ([x) \cap [y)) \\
&= (a \vee b)^\circ\varphi \vee [t_1) \vee [t_2) \vee [x \vee y) \\
&= (a \vee b)^\circ\varphi \vee [t_1 \wedge t_2 \wedge (x \vee y)) \\
&= (a \vee b)^\circ\varphi \vee [t) \text{ where } t = t_1 \wedge t_2 \wedge (x \vee y).
\end{aligned}$$

Also

$$\begin{aligned}
(a, a^\circ\varphi \vee [x))^\circ h &= (a^\circ, a\varphi)h \\
&= (a^\circ f, af\varphi_1) \\
&= (af, (af)^\circ\varphi_1 \vee [xg))^\circ \\
&= ((a, a^\circ\varphi \vee [x))h)^\circ.
\end{aligned}$$

Therefore h s a homomorphism of L into L_1. It is easy to see the uniqueness of h with $h_M = f$ and $h_D = g$. □

4. Subalgebras of decomposable *MS*-algebras

According to the characterization of a decomposable *MS*-algebra by means of the decomposable *MS*-triple (M, D, φ), we characterize the subalgebras of decomposable *MS*-algebra and solve the fill-in problem for their associated decomposable *MS*-triples.

Definition 4.1. A bounded sublattice L_1 of a decomposable *MS*-algebra L is called a subalgebra of L if

(1) $x^\circ \in L_1$ for every $x \in L_1$,

(2) for every $x \in L_1$, there exists $d \in D(L_1)$ such that $x = x^{\circ\circ} \wedge d$.

The decomposable *MS*-triple associated with a subalgebra of a decomposable *MS*-algebras is determined in the following:

Theorem 4.2. *Let L_1 be a subalgebra of the decomposable MS-algebra L. Then $L_1^{\circ\circ} = L_1 \cap L^{\circ\circ}$ is a subalgebra of $L^{\circ\circ}$ and $D(L_1) = L_1 \cap D(L)$ is a sublattice of $D(L)$ containing 1. The decomposable MS-triple associated with L_1 is $(L_1^{\circ\circ}, D(L_1), \varphi_1)$, where φ_1 is given by $a\varphi_1 = a\varphi \cap D(L_1)$ for every $a \in L_1^{\circ\circ}$.*

Proof. Obviously $0, 1 \in L_1 \cap L^{\circ\circ} = L_1^{\circ\circ}$. Let $x, y \in L_1^{\circ\circ}$. Then $x, y \in L_1 \cap L^{\circ\circ}$, so $x\wedge y, x\vee y \in L_1, L^{\circ\circ}$. Thus $x \wedge y, x \vee y \in L_1^{\circ\circ}$. Then $L_1^{\circ\circ}$ is a bounded sublattice of $L^{\circ\circ}$. Now, let $x \in L_1^{\circ\circ}$. Then $x \in L_1$ and $x \in L^{\circ\circ}$. Thus $x^\circ \in L_1$ and $x^\circ \in L^{\circ\circ}$ as L_1 is a subalgebra of a decomposable *MS*-algebra L and $L_1^{\circ\circ}$ is a subalgebra of a de Morgan algebra $L^{\circ\circ}$, respectively. Therefore $L_1^{\circ\circ}$ is a subalgebra of $L^{\circ\circ}$. Since each of L_1 and $D(L)$ is a sublattice of L containing 1, then $D(L_1) = L_1 \cap D(L)$ is a sublattice of $D(L)$ containing 1. Recall that the map $\varphi: L^{\circ\circ} \to F(D(L))$ defined by $a\varphi = [a^\circ) \cap D(L)$ is a (0,1) lattice homomorphism. Now, define $\varphi_1 : L_1^{\circ\circ} \to F(D(L_1))$ by

$$a\varphi_1 = a\varphi \cap D(L_1) \text{ for all } a \in L_1^{\circ\circ}.$$

We have to show that φ_1 is a (0,1) lattice homomorphism and $a\varphi_1 \cap [x)$ is a principal filter of $D(L_1)$ for every $a \in L_1^{\circ\circ}$ and for every $x \in D(L_1)$. It is clear that $0\varphi_1 = [1)$ and $1\varphi_1 = D(L_1)$. Let $a, b \in L_1^{\circ\circ}$. Then we get

$$\begin{aligned}(a \wedge b)\varphi_1 &= (a \wedge b)\varphi \cap D(L_1)\\ &= (a\varphi \cap b\varphi) \cap D(L_1) \text{ as } \varphi \text{ is a } \wedge-\text{homomorphism}\\ &= (a\varphi \cap D(L_1)) \cap (b\varphi \cap D(L_1)) \text{ by distributivity of } F(D(L_1))\\ &= a\varphi_1 \cap b\varphi_1,\end{aligned}$$

and

$$\begin{aligned}(a \vee b)\varphi_1 &= (a \vee b)\varphi_1 \cap D(L_1)\\ &= (a\varphi \vee b\varphi) \cap D(L_1) \text{ as } \varphi \text{ is a } \vee-\text{homomorphism}\\ &= (a\varphi \cap D(L_1)) \vee (b\varphi \cap D(L_1)) \text{ by distributivity of } F(D(L_1))\\ &= a\varphi_1 \vee b\varphi_1.\end{aligned}$$

For every $a \in L_1^{\circ\circ}$ and every $x \in D(L_1)$, we have

$$\begin{aligned}a\varphi_1 \cap [x) &= a\varphi \cap D(L_1) \cap [x)\\ &= a\varphi \cap [x) \text{ as } x \in D(L_1)\\ &= [a^\circ) \cap D(L) \cap [x) \text{ where } a\varphi = [a^\circ) \cap D(L)\\ &= [a^\circ) \cap [x) \text{ as } x \in D(L)\\ &= [a^\circ \vee x).\end{aligned}$$

Since $D(L_1)$ is a filter of L_1 and $x \in D(L_1)$, then $a^\circ\vee x \in D(L_1)$. Therefore $a\varphi_1 \cap [x)$ is a principal filter of $D(L_1)$ and $(L_1^{\circ\circ}, D(L_1), \varphi_1)$ is the decomposable *MS*-triple associated with L_1. □

A characterization of subalgebras of a decomposable *MS*-algebra is given by solving the following fill-in problem.

Theorem 4.3. *Let L be a decomposable MS-algebra, M_1 a subalgebra of $L^{\circ\circ}$, and D_1 a sublattice of $D(L)$ containing 1. We can fill-in (M_1, D_1, ?) such that it will become the decomposable MS-triple associated with a subalgebra of L iff*

$$a \vee d \in D_1 \text{ for every } a \in M_1 \text{ and for every } d \in D_1. \qquad (2)$$

Proof. If (M_1, D_1, φ_1) is the decomposable *MS*-triple associated with a subalgebra L_1 of L, then $M_1 = L_1^{\circ\circ}$ and $D_1 = D(L_1)$. Thus for every $a \in M_1$ and for every $d \in D_1$, we have $a\vee d \in L_1$. Then $(a \vee d)^\circ = a^\circ \wedge d^\circ = a^\circ \wedge 0 = 0$ implies that $a\vee d \in D(L)$. Hence $a\vee d \in L_1 \cap D(L)$. Thus by Theorem 4.2, $a\vee d \in D_1$.

Conversely, assume that $a\vee d \in D_1$, $\forall a \in M_1, \forall d \in D_1$. Let $M = L^{\circ\circ}, D = D(L)$ and $\varphi = \varphi(L)$. Represent the elements of L as in the construction Theorem (Theorem 2.9), that is,

$$L = \{(a, a^\circ\varphi \vee [x)) : a \in M, x \in D\}.$$

Let

$$L_1 = \{(a, a^\circ\varphi \vee [x)) : a \in M_1, x \in D_1\}.$$

We will show that L_1 is a subalgebra of L. It is clear that $0_L = (0, D)$ and $1_L = (1, [1))$ belong to L_1. Now, let $(a, a^\circ\varphi \vee [x))$, $(b, b^\circ\varphi \vee [y)) \in L_1$. Then we get

$$\begin{aligned}(a, a^\circ\varphi \vee [x)) \wedge (b, b^\circ\varphi \vee [y)) &= (a \wedge b, (a \wedge b)^\circ\varphi \vee [x \wedge y)) \in L_1,\\ (a, a^\circ\varphi \vee [x)) \vee (b, b^\circ\varphi \vee [y)) &= (a \vee b, (a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)))\\ &= (a \vee b, (a \vee b)^\circ\varphi \vee [t) \in L_1,\end{aligned}$$

where

$$\begin{aligned}&(a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y))\\ &= ((a^\circ\varphi \vee [x)) \cap b^\circ\varphi) \vee ((a^\circ\varphi \vee [x)) \cap [y))\\ &= (a^\circ\varphi \cap b^\circ\varphi) \vee ([x) \cap b^\circ\varphi) \vee (a^\circ\varphi \cap [y)) \vee ([x) \cap [y))\\ &= (a \vee b)^\circ\varphi \vee ([b) \cap D \cap [x)) \vee ([a) \cap D \cap [y)) \vee [x \vee y)\\ &= (a \vee b)^\circ\varphi \vee [b \vee x) \vee [a \vee y) \vee [x \vee y) \text{ where } b \vee x, a \vee y \in D_1 \text{ by (2)}\\ &= (a \vee b)^\circ\varphi \vee [t_1) \vee [t_2) \vee [x \vee y) \text{ where } t_1 = b \vee x, t_2 = a \vee y\\ &= (a \vee b)^\circ\varphi \vee [t), \text{ where } t = t_1 \wedge t_2 \wedge (x \vee y) \in D_1.\end{aligned}$$

Then L_1 is a bounded sublattice of L. Let $(a, a^\circ\varphi\vee[x)) \in L_1$. Then $(a, a^\circ\varphi \vee [x))^\circ = (a^\circ, a\varphi) \in L_1$ as $a^\circ \in M_1$. Then L_1 is a subalgebra of an *MS*-algebra L. Now, we show that L_1 is decomposable. It is observed that

$$L_1^{\circ\circ} = \{(a, a^\circ\varphi) : a \in M_1\} \text{ and } D(L_1) = \{(1, [x)) : x \in D_1\}.$$

For any $(a, a^\circ\varphi \vee [x)) \in L_1$, we have

$$(a, a^\circ\varphi \vee [x)) = (a, a^\circ\varphi) \wedge (1, [x)) = (a, a^\circ\varphi \vee [x))^{\circ\circ} \wedge (1, [x)), \text{ where } (1, [x)) \in D(L_1).$$

Thus L_1 is a decomposable *MS*-algebra. Therefore L_1 is a subalgebra of L.

Now we show that $L_1^{\circ\circ} \cong M_1$ and $D(L_1) \cong D_1$. Define

$$\psi : M_1 \longrightarrow L_1^{\circ\circ} \text{ by } a\psi = (a, a^\circ\varphi), \text{ for all } a \in M_1,$$

and

$$\chi : D_1 \longrightarrow D(L_1) \text{ by } x\chi = (1, [x)), \text{ for all } a \in D_1.$$

By easy computation we can prove that ψ and χ are isomorphisms. Hence we can fill-in (M_1, D_1, ?) by $a\varphi_1 = a\varphi(L_1) = a\varphi(L) \cap D_1$ such that it will become the decomposable *MS*-triple associated with a subalgebra of L. □

For Stone subalgebras of a decomposable *MS*-algebra, we consider the following fill in problem.

Corollary 4.4. *Let L be a decomposable MS-algebra, B a Boolean subalgebra of $L^{\circ\circ}$, and D_1 a sublattice of $D(L)$ containing 1. We can fill in (B, D_1, ?) such that it will become the decomposable MS-triple associated with a Stone subalgebra S of L iff*

$$a \vee d \in D_1 \text{ for every } a \in B \text{ and for every } d \in D_1.$$

Proof. By the above Theorem 4.3, it is observed that

$$S = \{(a, a^\circ\varphi \vee [x)) : a \in B, x \in D_1\}$$

is a subalgebra of a decomposable *MS*-algebra

$$L = \{(a, a^\circ\varphi \vee [x)) : a \in L^{\circ\circ}, x \in D(L)\}.$$

It reminds only to prove that S satisfies the Stone identity, $z^\circ \vee z^{\circ\circ} = 1$ for every $z \in S$. Since B is a Boolean algebra, then for any a

$\in B$, we have $a \wedge a^\circ = 0$ and $a \vee a^\circ = 1$. Let $(a, a^\circ \vee [x)) \in S$. Then we have

$$\begin{aligned}
&(a, a^\circ \vee [x))^\circ \vee (a, a^\circ \vee [x))^{\circ\circ}\\
&= (a^\circ, a\varphi) \vee (a, a^\circ\varphi)\\
&= (a^\circ \vee a, a\varphi \cap a^\circ\varphi)\\
&= (1, (a \wedge a^\circ)\varphi) \text{ as } \varphi \text{ is a } \vee-homomorphism\\
&= (1, 0\varphi) \text{ where } 0\varphi = [1)\\
&= (1, [1)).
\end{aligned}$$

Therefore S is a Stone subalgebra of L. □

Definition 4.5. A subalgebra L_1 of a decomposable *MS*-algebra L is called a K_2-subalgebra of L if for every $x, y \in L_1$, the following two condition are hold.

(1) $x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ$,
(2) $x \wedge x^\circ \le y \vee y^\circ$.

For K_2-subalgebras of a decomposable *MS*-algebra, we solve the following fill-in problem.

Theorem 4.6. *Let L be a decomposable MS-algebra, K a Kleene subalgebra of $L^{\circ\circ}$, and D_1 a sublattice of $D(L)$containing 1. We can fill-in (K, D_1, ?) such that it will become the decomposable MS-triple associated with a K_2-subalgebra L_1 if the following conditions are hold:*

(1) *$a \vee d \in D_1$ for every $a \in K$ and for every $d \in D_1$,*
(2) *$a\varphi \vee a^\circ\varphi = D$ for every $a \in K$.*

Proof. By Theorem 4.3,

$$L_1 = \{(a, a^\circ\varphi \vee [x)) : a \in K, x \in D_1\}.$$

is a subalgebra of $L = \{(a, a^\circ\varphi \vee [x)) : a \in L^{\circ\circ}, x \in D(L)\}$ such that $L_1^{\circ\circ} \cong K$ nd $D(L_1) \cong D_1$. Therefore we can fill-in (K, D_1, ?) by $a\varphi_1 = a\varphi(L_1) = a\varphi(L) \cap D_1$ such that it will become the decomposable *MS*-triple associated with a subalgebra L_1 of L.

Now, we have to prove that L_1 satisfies conditions (1) and (2) of Definition 4.5. Let $(a, a^\circ\varphi \vee [x)), (b, b^\circ\varphi \vee [y)) \in L_1$. Then we get

$$\begin{aligned}
&(a, a^\circ\varphi \vee [x)) \wedge (a, a^\circ\varphi \vee [x))^\circ\\
&= (a, a^\circ\varphi \vee [x)) \wedge (a^\circ, a\varphi)\\
&= (a \wedge a^\circ, a^\circ\varphi \vee a\varphi \vee [x))\\
&= (a \wedge a^\circ, a^\circ\varphi \vee a\varphi) \text{ as } x \in D = a\varphi \vee a^\circ\varphi\\
&= (a, a^\circ\varphi) \wedge (a^\circ, a\varphi)\\
&= (a, a^\circ\varphi \vee [x))^{\circ\circ} \wedge (a, a^\circ\varphi \vee [x))^\circ.
\end{aligned}$$

Since K is a Kleene subalgebra of $L^{\circ\circ}$, then $a \wedge a^\circ \le b \vee b^\circ$ for any $a \in K$ and

$$\begin{aligned}
&((a, a^\circ\varphi \vee [x)) \wedge (a, a^\circ\varphi \vee [x))^\circ) \vee ((b, b^\circ\varphi \vee [y)) \vee (b, b^\circ\varphi \vee [y))^\circ)\\
&= ((a, a^\circ\varphi \vee [x)) \wedge (a^\circ, a\varphi)) \vee ((b, b^\circ\varphi \vee [y)) \vee (b^\circ, b\varphi))\\
&= (a \wedge a^\circ, a^\circ\varphi \vee a\varphi \vee [x)) \vee (b \vee b^\circ, (b^\circ\varphi \vee [y)) \cap b\varphi)\\
&= (a \wedge a^\circ, D) \vee (b \vee b^\circ, (b^\circ\varphi \vee [y)) \cap b\varphi), \text{ as } a^\circ\varphi \vee a\varphi = D \supseteq [x)\\
&= ((a \wedge a^\circ) \vee (b \vee b^\circ), D \cap (b^\circ\varphi \vee [y)) \cap b\varphi)\\
&= (b \vee b^\circ, (b^\circ\varphi \vee [y)) \cap b\varphi) \text{ as } a \wedge a^\circ \le b \vee b^\circ\\
&\quad \text{and } (b^\circ\varphi \vee [y)) \cap b\varphi) \subseteq D\\
&= (b, b^\circ\varphi \vee [y)) \vee (b^\circ, b\varphi)\\
&= (b, b^\circ\varphi \vee [y)) \vee (b, b^\circ\varphi \vee [y))^\circ.
\end{aligned}$$

Consequently L_1 is a K_2-subalgebra of L. □

In closing this paper, we introduce important results concerning subalgebras of a decomposable *MS*-algebra constructing from the decomposable *MS*-triple (M, D, φ).

Theorem 4.7. *Let L be a decomposable MS-algebra associated with the decomposable MS-triple (M, D, φ), M_1 a subalgebra of M, and D_1 a sublattice of D containing 1. Then $L_1 = \{(a, a^\circ\varphi \vee [x)) \in L : a \in M_1, x \in D_1\}$ is a subalgebra of L iff*

$$a\varphi \cap [x) \text{ is a principal filter of } D_1 \text{ for every } a \in M_1 \text{ and for every } x \in D_1. \quad (3)$$

Proof. Let $L_1 = \{(a, a^\circ\varphi \vee [x)) \in L : a \in M_1, x \in D_1\}$ is a subalgebra of a decomposable *MS*-algebra $L = \{(a, a^\circ\varphi \vee [x)) : a \in M, x \in D\}$. Then $L_1^{\circ\circ} = \{(a, a^\circ\varphi) : a \in M_1\} \cong M_1$ and $D(L_1) = \{(1, [x)) : x \in D_1\} \cong D_1$. Also, a map $\varphi(L_1) : L_1^{\circ\circ} \to F(D(L_1))$ defined by $(a, a^\circ\varphi)\varphi(L_1) = [(a, a^\circ\varphi)^\circ) \cap D(L_1)$ is (0,1) lattice homomorphism and $(a, a^\circ\varphi)\varphi(L_1) \cap [(1, [x)))$ is a principal filter of $D(L_1)$ for every $(a, a^\circ\varphi) \in L_1^{\circ\circ}$ and for every $(1, [x)) \in D(L_1)$. Hence

$$(a, a^\circ\varphi)\varphi(L_1) \cap [(1, [x))) = [(1, [z))) \text{ for some } z \in D.$$

Then

$$\begin{aligned}
&(a, a^\circ\varphi)\varphi(L_1) \cap [(1, [x)) = [(1, [z))) \text{ for some } z \in D_1\\
&\Rightarrow [(a, a^\circ\varphi)^\circ) \cap D(L_1) \cap [(1, [x)) = [(1, [z)))\\
&\Rightarrow [(a^\circ, a\varphi)) \cap [(1, [x)) = [(1, [z))) \text{ as } (1, [x)) \in D(L_1)\\
&\Rightarrow [(a^\circ, a\varphi) \vee (1, [x))) = [(1, [z)))\\
&\Rightarrow [(a^\circ \vee 1, a\varphi \cap [x)) = [(1, [z)))\\
&\Rightarrow [(1, a\varphi \cap [x))) = [(1, [z)))\\
&\Rightarrow (1, a\varphi \cap [x)) = (1, [z))\\
&\Rightarrow a\varphi \cap [x) = [z) \text{ where } z \in D_1.
\end{aligned}$$

Then $a\varphi \cap [x)$ is a principal filter of D_1 for every $a \in M_1$ and for every $x \in D_1$. Then condition (3) is hold. Conversely, let

$$L = \{(a, a^\circ\varphi \vee [x)) : a \in M, x \in D\}.$$

be the decomposable *MS*-algebra constructing from (M, D, φ) (see Theorem 2.9). Then by Lemma 2.10 (1) and (2), respectively, we have

$$L^{\circ\circ} = \{(a, a^\circ\varphi) : a \in M\} \text{ and } D(L) = \{(1, [x)) : x \in D_1\}.$$

It is clear that $(0, D), (1, [1)) \in L_1$. Let $(a, a^\circ\varphi \vee [x)), (b, b^\circ\varphi \vee [y)) \in L_1$. Then we get

$$\begin{aligned}
&(a, a^\circ\varphi \vee [x)) \wedge (b, b^\circ\varphi \vee [y))\\
&= (a \wedge b, a^\circ\varphi \vee [x) \vee b^\circ\varphi \vee [y))\\
&= (a \wedge b, (a \wedge b)^\circ\varphi \vee [x \wedge y)) \in L_1 \text{ as } a \vee b \in M_1 \text{ and } x \vee y \in D_1,
\end{aligned}$$

and

$$\begin{aligned}
&(a, a^\circ\varphi \vee [x)) \vee (b, b^\circ\varphi \vee [y))\\
&= (a \vee b, (a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y)))\\
&= (a \vee b, (a \vee b)^\circ\varphi \vee [t)) \in L_1 \text{ for some } t \in D_1
\end{aligned}$$

where

$$\begin{aligned}
&(a^\circ\varphi \vee [x)) \cap (b^\circ\varphi \vee [y))\\
&= ((a^\circ\varphi \vee [x)) \cap b^\circ\varphi) \vee ((a^\circ\varphi \vee [x)) \cap [y)) \text{ by distributivity}\\
&= (a^\circ\varphi \cap b^\circ\varphi) \vee ((b^\circ\varphi \cap [x)) \vee ((a^\circ\varphi \cap [y)) \vee ([x) \cap [y))\\
&\quad \text{by distributivity}\\
&= (a \vee b)^\circ\varphi \vee [t_1) \vee [t_2) \vee [x \vee y) \text{ as } b^\circ\varphi \cap [x) = [t_1), t_1 \in D_1\\
&\quad \text{and } a^\circ\varphi \cap [y) = [t_2), t_2 \in D_1 \text{ by (3)}\\
&= (a \vee b)^\circ\varphi \vee [t) \text{ where } t = t_1 \wedge t_2 \wedge (x \vee y) \in D_1.
\end{aligned}$$

Hence L_1 is a bounded sublattice of L. Also, let $(a, a^\circ\varphi \vee [x)) \in L_1$. Then $(a, a^\circ\varphi \vee [x))^\circ = (a^\circ, a\varphi) \in L_1$, because $a^\circ \in M_1$. Now,

$$\begin{aligned}
D(L_1) &= \{(a, a^\circ\varphi \vee [x)) \in L_1 : (a, a^\circ\varphi \vee [x))^\circ = (0, D)\}\\
&= \{(a, a^\circ\varphi \vee [x)) \in L_1 : (a^\circ, a\varphi) = (0, D)\}\\
&= \{(a, a^\circ\varphi \vee [x)) \in L_1 : a = 1\}\\
&= \{(1, [x)) : x \in D_1\}
\end{aligned}$$

$\cong D_1$.

If $(a, a^\circ\varphi \vee [x)) \in L_1$, then $(1, [x)) \in D(L_1)$ and

$$\begin{aligned}(a, a^\circ\varphi \vee [x))^{\circ\circ} \wedge (1, [x)) &= (a, a^\circ\varphi) \wedge (1, [x)) \\ &= (a \wedge 1, a^\circ\varphi \vee [x)) \\ &= (a, a^\circ\varphi \vee [x)).\end{aligned}$$

Therefore L_1 is a subalgebra of L. Consequently, $L_1^{\circ\circ}$ is a subalgebra of $L^{\circ\circ}$ and

$$\begin{aligned}L_1^{\circ\circ} &= \{(a, a^\circ\varphi \vee [x))^{\circ\circ} : (a, a^\circ\varphi \vee [x)) \in L_1\} \\ &= \{(a, a^\circ\varphi) : a \in M_1\} \\ &\cong M_1,\end{aligned}$$

completing the proof of Theorem 4.7. □

Let $(L; \vee, \wedge, ^\circ, 0, 1)$ be a de Morgan algebra. Consider the set

$$Z(L) = \{x \in L : x \vee x^\circ = 1\}$$

The set $Z(L)$ is called the center of L. Then we verify the following:

Lemma 4.8. *Let $(L; \vee, \wedge, ^\circ, 0, 1)$ be a de Morgan algebra. Then the center $Z(L)$ is the greatest Boolean subalgebra of L.*

Proof. Clearly, $0, 1 \in Z(L)$. Let $x, y \in Z(L)$. Then $x \vee x^\circ = 1$ and $y \vee y^\circ = 1$. Now

$$\begin{aligned}(x \vee y) \vee (x \vee y)^\circ &= (x \vee y) \vee (x^\circ \wedge y^\circ) \\ &= (x \vee y \vee x^\circ) \wedge (x \vee y \vee y^\circ) \\ &= 1,\end{aligned}$$

$$\begin{aligned}(x \wedge y) \vee (x \wedge y)^\circ &= (x \wedge y) \vee (x^\circ \vee y^\circ) \\ &= (x \vee x^\circ \vee y^\circ) \wedge (y \vee x^\circ \vee y^\circ) \\ &= 1.\end{aligned}$$

Thus $Z(L)$ is a bounded sublattice of L. Let $x \in Z(L)$. Thus

$$x^\circ \vee x^{\circ\circ} = x^\circ \vee x = 1 \text{ as } x = x^{\circ\circ} \text{ for all } x \in L.$$

Then $x^\circ \vee x^{\circ\circ} = 1$ implies $x^\circ \in Z(L)$. Therefore $Z(L)$ is a subalgebra of L. Now, suppose that B is any Boolean subalgebra of L. Let $x \in B$. Thus $x \vee x^\circ = 1$. Then $x \in Z(L)$. This deduce that $B \subseteq Z(L)$ and $Z(L)$ is the largest Boolean subalgebra of L. □

In the following Theorem we obtain the greatest Stone subalgebra of a decomposable MS-algebra constructing from the decomposable MS-triple (M, D, φ).

Theorem 4.9. *Let L be a decomposable MS-algebra associated with the decomposable MS-triple (M, D, φ), $Z(M)$ the center of M and $D_1 = D$. Then we have*

(1) $Z(L^{\circ\circ}) = \{(a, a^\circ\varphi) : a \in Z(M)\}$,
(2) $L_1 = \{(a, a^\circ\varphi \vee [x)) \in L : a \in Z(M)\}$ *is the greatest Stone subalgebra of L such that $D(L_1) = D(L)$ and $L_1^{\circ\circ} = Z(L^{\circ\circ})$.*

Proof. Since L is constructed from the decomposable MS-triple (M, D, φ), then by Theorem 2.9, we have

$$L = \{(a, a^\circ\varphi \vee [x)) : a \in M, x \in D\}.$$

Consequently by Corollary 2.10 (1) and (2), respectively, we have

$$L^{\circ\circ} = \{(a, a^\circ\varphi) : a \in M\}.$$

and

$$D(L) = \{(1, [x)) : x \in D\}.$$

(1) By Lemma 4.8, the center $Z(L^{\circ\circ})$ of $L^{\circ\circ}$ is the greatest Boolean subalgebra of the de Morgan algebra $L^{\circ\circ}$. Now we get

$$\begin{aligned}Z(L^{\circ\circ}) &= \{(a, a^\circ\varphi) \in L^{\circ\circ} : (a, a^\circ\varphi)^\circ \vee (a, a^\circ\varphi)^{\circ\circ} = (1, [1))\} \\ &= \{(a, a^\circ\varphi) \in L^{\circ\circ} : (a^\circ, a\varphi) \vee (a, a^\circ\varphi) = (1, [1))\} \\ &= \{(a, a^\circ\varphi) \in L^{\circ\circ} : (a \vee a^\circ, (a \vee a^\circ)\varphi) = (1, [1))\} \\ &= \{(a, a^\circ\varphi) \in L^{\circ\circ} : a \vee a^\circ = 1\} \\ &= \{(a, a^\circ\varphi) \in L^{\circ\circ} : a \in Z(M)\}.\end{aligned}$$

(2) Since $D_1 = D$, then $a\varphi \cap [x)$ is a principal filter of D for every $a \in Z(M)$ and for every $x \in D$. Hence the sufficient condition (3) of Theorem 4.7 holds. Thus by Theorem 4.7, $L_1 = \{(a, a^\circ\varphi \vee [x)) \in L : a \in Z(M), x \in D\}$ is a subalgebra of a decomposable MS-algebra L such that $L_1^{\circ\circ} \cong Z(M)$ and $D(L_1) \cong D_1$.

It is observed that

$$\begin{aligned}D(L_1) &= \{(a, a^\circ\varphi \vee [x)) \in L_1 : (a, a^\circ\varphi \vee [x))^\circ = (0, D)\} \\ &= \{(a, a^\circ\varphi \vee [x)) \in L_1 : (a^\circ, a\varphi)^\circ = (0, D)\} \\ &= \{(a, a^\circ\varphi \vee [x)) \in L_1 : a = 1, x \in D\} \\ &= \{(1, [x)) : x \in D\} \\ &= D(L),\end{aligned}$$

and

$$\begin{aligned}L_1^{\circ\circ} &= \{(a, a^\circ\varphi \vee [x))^{\circ\circ} : (a, a^\circ\varphi \vee [x)) \in L_1\} \\ &= \{(a, a^\circ\varphi) : a \in Z(M)\} \\ &= Z(L^{\circ\circ}).\end{aligned}$$

Now we will verify that the Stone identity, $z^\circ \vee z^{\circ\circ} = 1$ holds for every $z = (a, a^\circ\varphi \vee [x)) \in L_1$.

$$\begin{aligned}(a, a^\circ\varphi \vee [x))^\circ \vee (a, a^\circ\varphi \vee [x))^{\circ\circ} &= (a^\circ, a\varphi) \vee (a, a^\circ\varphi) \\ &= (a \vee a^\circ, (a \vee a^\circ)\varphi) \\ &= (1, [1)) \text{ as } a \in Z(M).\end{aligned}$$

Therefore L_1 is a Stone subalgebra of a decomposable MS-algebra L. To prove that L_1 is the greatest Stone subalgebra of L, let S be any Stone subalgebra of L. Let $(a, a^\circ\varphi \vee [x)) \in S$. Hence $(a, a^\circ\varphi \vee [x))^\circ \vee (a, a^\circ\varphi \vee [x))^{\circ\circ} = (1, [1))$. Then $a \vee a^\circ = 1$ implies $a \in Z(M)$. This deduce that $(a, a^\circ\varphi \vee [x)) \in L_1$. Therefore $S \subseteq L_1$. □

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