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ORIGINAL ARTICLE

An easy trick to a periodic solution of relativistic harmonic oscillator

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Abstract In this paper, the relativistic harmonic oscillator equation which is a nonlinear ordinary differential equation is investigated by Homotopy perturbation method. Selection of a linear operator, which is a part of the main operator, is one of the main steps in HPM. If the aim is to obtain a periodic solution, this choice does not work here. To overcome this lack, a linear operator is imposed, and Fourier series of sines will be used in solving the linear equations arise in the HPM. Comparison of the results, with those of resulted by Differential Transformation and Harmonic Balance Method, shows an excellent agreement.

MATHEMATICS SUBJECT CLASSIFICATION: 34L30; 34C15; 74G10

1. Introduction

Mathematical model of Physical and mechanical oscillatory systems are often leads to a nonlinear differential equations of the second order. Many researchers are interested to study these equations. To solve nonlinear differential equations, there are several semianalytical methods known, such as Harmonic Balance [1–3], Differential Transformation [4–6], Adomian decomposition [7,8], and Homotopy perturbation [9–16]. But it is important to find the periodic solution to some of these equations. The relativistic harmonic oscillator introduced by Penfield and Zatzkis [17] in 1956. Mickens [1] has shown that all solutions to the relativistic oscillator are periodic and he has introduced a method for calculating an analytic approximation to the solution. This paper applies Homotopy perturbation method to find a periodic solution for relativistic oscillator, but in prior, a special linear operator should be imposed in the homotopy. HPM uses the parameter p to transfer a nonlinear problem into an infinite number of linear sub-problems, and then approximate it by the sum of solutions of the first several sub-problems. Fourier series of sines is used to solve these equations.

2. Definition of the problem

Consider the relativistic motion of a particle of rest mass m in a one dimensional harmonic oscillator force, $F = -k\bar{x}$. Where k is

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the elastic constant and $\bar{x}$ is the displacement (dimensional variable). Newton's equation of motion can be written in the form

$$F = \frac{dp}{d\bar{t}}, \tag{1}$$

where $\bar{t}$ is the time coordinate (dimensional variable) and p is the relativistic momentum which can be written as follows,

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}, \tag{2}$$

where $v = \frac{d\bar{x}}{d\bar{t}}$ is the speed of the particle and c is the speed of light. Substituting Eq. (2) into Eq. (1) leads to

$$F = \frac{d}{d\bar{t}}\left(\frac{mv}{\sqrt{1 - v^2/c^2}}\right) = \frac{m}{(1 - v^2/c^2)^{3/2}}\frac{dv}{d\bar{t}} = \frac{m}{[1 - (1/c^2)(d\bar{x}/d\bar{t})^2]^{3/2}}\frac{d^2\bar{x}}{d\bar{t}^2}. \tag{3}$$

Substituting Eq. (3) into Newton's equation of motion in the form

$$\frac{dv}{d\bar{t}} + k\bar{x} = 0, \tag{4}$$

results in

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \frac{k}{m}\left[1 - \frac{1}{c^2}\left(\frac{d\bar{x}}{d\bar{t}}\right)^2\right]^{3/2}\bar{x} = 0. \tag{5}$$

From Eq. (5), one can write the non-dimensional nonlinear differential equation of motion for the relativistic oscillator as follows

$$\frac{d^2x}{dt^2} + \left[1 - \left(\frac{dx}{dt}\right)^2\right]^{3/2} x = 0, \tag{6}$$

where x and t are dimensionless variables defined as follows:

$$x = \frac{\omega_0\bar{x}}{c}, t = \omega_0\bar{t},$$

where $\omega_0 = \sqrt{k/m}$ is the angular frequency for the non-relativistic oscillator (linear oscillator). Let us consider the following initial conditions on Eq. (6).

$$x(0) = 0, x'(0) = \beta. \tag{7}$$

Mickens [1] has shown that all the motions corresponding to Eq. (6) are periodic and the period depends on the values ofβ. In addition, he has shown that the period is

$$2p = \frac{2\pi}{\omega}, \tag{8}$$

where

$$\omega = \sqrt[4]{\frac{2 - 2\beta^2}{2 - \beta^2}}. \tag{9}$$

3. Mathematical formulation of the method

3.1. Homotopy perturbation method (HPM)

HPM is a known method for solving the following nonlinear functional equations

$$\begin{aligned} &\mathcal{A}(u(r)) = 0, \quad r \in \Omega, \\ &\mathcal{B}\left(u, \tfrac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \end{aligned} \tag{10}$$

where $\mathcal{A}$ is a general differential operator, $\mathcal{B}$ is a boundary operator, and Γ is the boundary of the domain Ω. This method is well addressed in [9–14] and has been used by many researchers. There are some papers regarding convergence of the method [9,12]. The major advantage of Homotopy perturbation method is that the homotopy can be freely constructed in many forms by selecting different linear operators or initial approximations, according to initial conditions. This is a useful property to find a periodic solution.

3.2. Periodic solution

To find a periodic solution of Eq. (10), with period $2p$, let us consider the solution as the following series,

$$u(t) = \sum_{k=1}^{\infty} a_k \sin(k\omega t), \tag{11}$$

where $\omega = \frac{\pi}{p}$. Homotopy can be constructed, with linear and nonlinear operators, as follows

$$(1 - p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[\mathcal{N}(v) + \mathcal{L}(v)] = 0, \quad p \in [0, 1] \tag{12}$$

where

$$\mathcal{L}(v) = \tfrac{\partial^2}{\partial t^2} v + \omega^2 v,$$
$$\mathcal{N}(v) = \mathcal{A}(v) - \mathcal{L}(v).$$

Assume the solution of (12) have the form

$$v(t, p) = v_0(t) + v_1(t)p + v_2(t)p^2 + \cdots. \tag{13}$$

Substituting Eq. (13) into Eq. (12) and equating the coefficients of the terms with identical powers of p, results in

$$\begin{aligned} &p^0 : \mathcal{L}(v_0) = \mathcal{L}(u_0), \\ &p^1 : \mathcal{L}(v_1) = R_1, \\ &p^2 : \mathcal{L}(v_2) - \mathcal{L}(v_1) = R_2, \\ &\vdots \end{aligned} \tag{14}$$

where R_k is the coefficient of p^k in $-p[\mathcal{A}(v)]$. To solve this linear equations, rewrite R_k as

$$R_k = \sum_{n=1}^{\mu_k} \beta_n \sin(n\omega t), \tag{15}$$

where

$$\beta_n = \frac{1}{p}\int_{-p}^{p} R_k \sin(n\omega t)dt.$$

By determining R_k in the form (15), one can easily solve the Eq. (14). This approach is used to find a periodic solution to the nonlinear relativistic harmonic oscillator with a predetermined period.

4. Periodic solution to nonlinear relativistic harmonic oscillator

As previously mentioned, Mickens [1] has shown that all the motions corresponding to Eq. (6) are periodic with the period $2p$, in the forms (8) and (9). According to the Eq. (6), initial conditions (7), and the fact that the solutions are periodic; the solution can be expressed by a linear combination of the following base function

$$\{\sin(2n + 1)\omega t | n = 0, 1, 2, \ldots\}.$$

Table 1 periodic approximate solution by HPM.

β	Approximate solution by HPM
0.01	$1.0000 \times 10^{-2}\sin(t) - 4.687 \times 10^{-8}\sin(3t) + 5.305 \times 10^{-13}\sin(5t)$
0.1	$1.0010 \times 10^{-1}\sin(0.999t) - 4.689 \times 10^{-5}\sin(2.997t) + 5.062 \times 10^{-8}\sin(4.995t)$
0.2	$2.01 \times 10^{-1}\sin(0.995t) - 3.768 \times 10^{-4}\sin(2.985t) + 1.652 \times 10^{-6}\sin(4.974t)$

Table 2 Periodic approximate solution by Harmonic balance method [1].

β	Approximate solution using Harmonic balance method [1]
0.01	$1.00003 \times 10^{-2}\sin(t) - 4.167 \times 10^{-8}\sin(3t) + 4.687 \times 10^{-13}\sin(5t)$
0.1	$1.0025 \times 10^{-1}\sin(0.998t) - 4.173 \times 10^{-5}\sin(2.996t) + 4.369 \times 10^{-8}\sin(4.994t)$
0.2	$2.02 \times 10^{-1}\sin(0.995t) - 3.354 \times 10^{-4}\sin(2.985t) + 1.508 \times 10^{-6}\sin(4.974t)$

Table 3 Periodic approximate solution by DTM [4].

β	Approximate solution using DTM [4]
0.01	$1.00003 \times 10^{-2}\sin(t) - 4.688 \times 10^{-8}\sin(3t) + 6.121 \times 10^{-13}\sin(5t)$
0.1	$1.0033 \times 10^{-1}\sin(0.998t) - 4.7097 \times 10^{-5}\sin(2.997t) + 8.254 \times 10^{-8}\sin(4.841t)$
0.2	$2.03 \times 10^{-1}\sin(0.992t) - 3.695 \times 10^{-4}\sin(3.051t) + 9.257 \times 10^{-6}\sin(4.29t)$

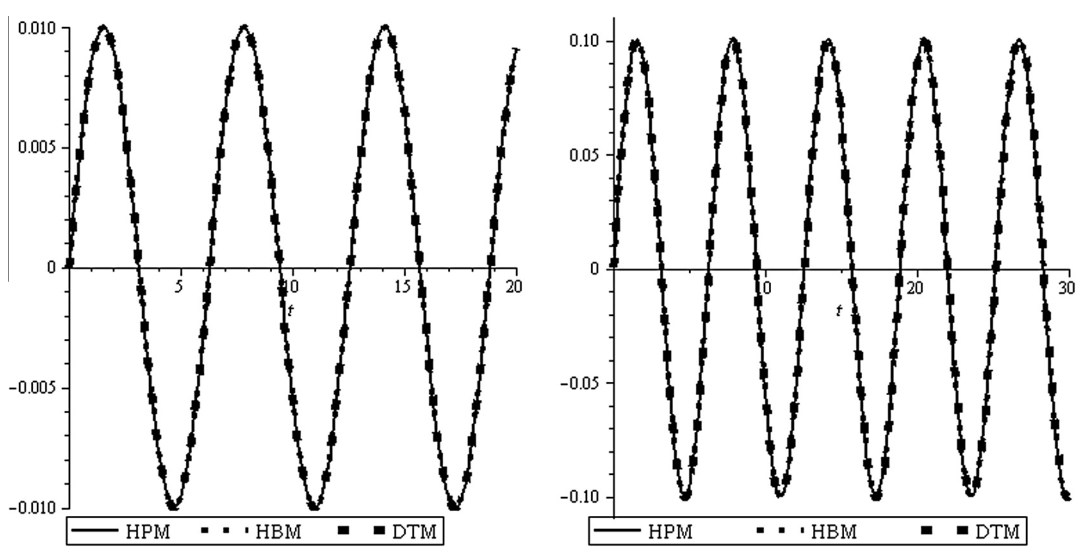

Figure 1 Plots of periodic approximate solution by HAM, HBM, and DTM at $\beta = 0.01$(left) and $\beta = 0.1$(right).

So

$$x(t) = \sum_{k=1}^{\infty} a_k[\sin(2k+1)\omega t]. \tag{16}$$

Consider a homotopy as the following

$$(1-p)[\mathcal{L}(v(t,p)) - \mathcal{L}(x_0(t))] + p[A(v(t,p))] = 0, \tag{17}$$

where

$$\mathcal{L}(v(t,p)) = \frac{\partial^2}{\partial t^2}v(t,p) + \omega^2 v(t,p),$$
$$\mathcal{A}(v(t,p)) = \frac{\partial^2 v(t,p)}{\partial t^2} + \left[1 - \left(\frac{\partial v(t,p)}{\partial t}\right)^2\right]^{3/2} v(t,p), \tag{18}$$

and $x_0(t)$ is an initial approximation of the solution. Now suppose that the solution of Eq. (17) is as the series (13). As $p \to 1$, the solution of Eq. (6) will be obtained.

$$x = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots.$$

Substituting (13) into (17) results in linear differential equations of the form (14), where

$$R_k = \frac{-1}{(k-1)!} \frac{\partial^{k-1}\mathcal{A}[v(t;p)]}{\partial p^{k-1}}\bigg|_{p=0}.$$

The corresponding linear equations are as follows:

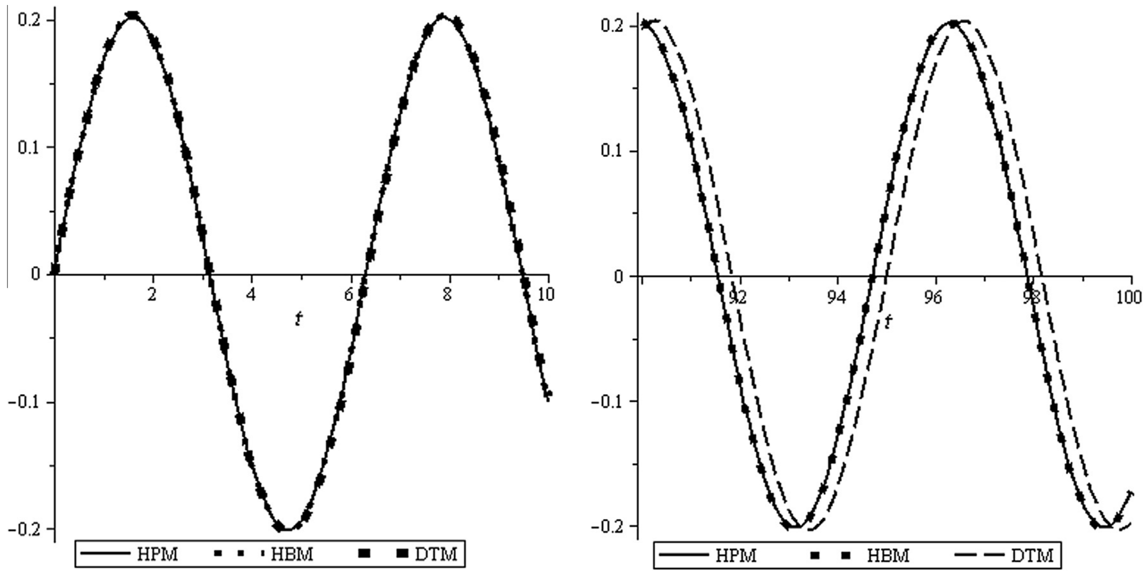

Figure 2 Plots of periodic approximate solution using HAM, HBM, and DTM at $\beta = 0.2$ with respect to t at the interval [0, 10] (left) and [90, 100] (right).

$$v_0''(t) + \omega^2 v_0(t) = u_0''(t) + \omega^2 u_0(t), v_0(0) = 0, v_0'(0) = \beta. \quad (19)$$

$$v_1''(t) + \omega^2 v_1(t) = R_1 = v_0''(t) + \left[1 - (v_0'(t))^{3/2}\right] v_0(t),$$
$$v_1(0) = 0, v_1'(0) = 0. \quad (20)$$

$$v_2''(t) + \omega^2 v_2(t) = R_2$$
$$= v_1''(t) - 3\sqrt{1 - (v_0'(t))^2}\, v_0(t) v_0'(t) v_1'(t)$$
$$+ \left[1 - \left(v_0'(t)\right)^{3/2}\right] v_1(t), v_2(0) = 0, v_2'(0) = 0. \quad (21)$$

$\vdots$

Considering initial conditions (7) and expression (16), the initial approximation to the solution can be selected as $u_0(t) = \frac{\beta}{\omega}\sin(\omega t)$. Thus (19), yields

$$v_0(t) = u_0(t) = \frac{\beta}{\omega}\sin(\omega t).$$

and from (20),

$$v_1''(t) + \omega^2 v_1(t) = R_1 = \sum_{n=1}^{\mu_1} \beta_{1n} \sin[(2n+1)\omega t], \quad (22)$$

where

$$\beta_{1n} = \frac{1}{p}\int_{-p}^{p} R_1 \sin[(2n+1)n\omega t]dt.$$

Note that the term $\sin(\omega t)$ in the right hand side of the Eq. (22) has been eliminated, to avoid the secular term $t\sin(\omega t)$ in $v_1(t)$. Thus, the solution of (22) is

$$v_1(t) = \sum_{n=1}^{\mu_1} a_n \sin[(2n+1)\omega t] + c_1 \sin(\omega t) + c_2 \cos(\omega t). \quad (23)$$

From initial conditions in (20), $c_1 = 0$ and $c_2 = 0$. Substituting (22) into Eq. (20) leads to

$$a_n = \frac{\beta_{1n}}{(-4n^2 - 4n)\omega^2}.$$

Similarly from (21),

$$v_2(t) = v_1(t) + \sum_{n=1}^{\mu_2} \frac{\beta_{2n}}{(-4n^2 - 4n)\omega^2} \sin[(2n+1)\omega t],$$

where

$$\beta_{2n} = \frac{1}{p}\int_{-p}^{p} R_2 \sin[(2n+1)n\omega t]dt,$$

and so on.

The nonlinear relativistic harmonic oscillator (6), with initial conditions (7), has been solved for different values of β. Tables 1–3 show the corresponding periodic approximate solutions obtained by HPM, DTM [4], and Harmonic balance method [1].The results of three methods are shown in Figs. 1 and 2. It can be seen that the results of HPM and Harmonic balance method are in more agreement.

5. Conclusions

In this study, Homotopy perturbation method has been applied to obtain the periodic solution of relativistic harmonic oscillator. To find a periodic solution, a reliable periodic base functions and linear operator is proposed. The equation for different values of β has been solved. Comparing with DTM and Harmonic balance method shows the solution obtained by Homotopy perturbation method is in a good agreement with those of two other methods. Thus, the Homotopy perturbation method is an effective method to find the periodic solution of the equations such as relativistic harmonic oscillator.

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