Journal of the Egyptian Mathematical Society (2014) **22**, 36–44

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Certain subclasses of *p*-valently analytic functions involving a generalized fractional differintegral operator

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Received 2 April 2013; revised 29 May 2013; accepted 8 June 2013
Available online 24 July 2013

KEYWORDS

Analytic functions;
p-Valently functions;
Hadmard product (or convolution);
Differential subordination;
Superordination;
Fractional differintegral operator

Abstract By making use of the principle of subordination between analytic functions and the generalized fractional differintegral operator, we introduce and investigate some new subclasses of *p*-valently analytic functions in the open unit disk. Such results as inclusion relationships, integral-preserving properties, convolution properties, subordination and superordination properties, and sandwich theorems for these classes are derived.

(2010) MATHEMATICS SUBJECT CLASSIFICATION: 30C45; 30C80; 30A20

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} (p \in N = \{1, 2, \ldots\}), \quad (1.1)$$

which are analytic and p-valent in the open unit disk

$U = \{z : z \in C \quad and \ |z| < 1\}$.

Let $H[a, n]$ be the class of analytic functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in C; z \in U).$$

For the functions $f \in A_p$, given by (1.1) and $g \in A_p$ of the form

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p} (p \in N),$$

the Hadmard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

Let P be the class of functions $\phi(z)$ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\Re\{\phi(z)\} > 0$ for $z \in U$.

For two functions f and g, analytic in U, we say that the function f is subordinate to g in U, if there exists a Schwarz function ω, which is analytic in U with

$\omega(0) = 0 \quad and \ |\omega(z)| < 1 (z \in U),$

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Peer review under responsibility of Egyptian Mathematical Society.

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1110-256X
http://dx.doi.org/10.1016/j.joems.2013.06.009

such that

$f(z) = g(\omega(z))(z \in U).$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see, for details, [1,2]; see also [3]):

$f(z) \prec g(z)(z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and}\ f(U) \subset g(U).$

Recently, Goyal and Prajapat [4] (see also [5]) introduced and investigated the generalized fractional differintegral operator $I_p^\lambda(\mu,\eta) : A_p \to A_p$ as follows:

$$I_p^\lambda(\mu,\eta)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n} a_{n+p} z^{n+p} \qquad (1.2)$$

$(z \in U; p, n \in N; \mu, \eta \in R; \mu < p+1; -\infty < \lambda < \eta + p + 1)$

where $(v)_n$ is the Pochhammer symbol defined, in terms of Gamma function, by

$$(v)_n = \frac{\Gamma(v+n)}{\Gamma(v)} = \begin{cases} 1 & (n = 0), \\ v(v+1)\cdots(v+n-1) & (n \in N). \end{cases}$$

In particular, we have

$$I_p^0(0,0)f(z) = f(z) \quad \text{and}\ I_p^1(1,1)f(z) = I_p^1(0,0)f(z) = \frac{zf'(z)}{p}.$$

It is easily verified from (1.2) that

$$z\left(I_p^\lambda(\mu,\eta)f(z)\right)' = (p+\eta-\lambda)I_p^{\lambda+1}(\mu,\eta)f(z) - (\eta - \lambda)I_p^\lambda(\mu,\eta)f(z). \qquad (1.3)$$

We note that the operator $I_p^\lambda(\mu,\eta)$ is a generalization of several previously familiar operators, and we will show some of the interesting special cases as below:

(1) $I_p^\lambda(\lambda,\eta) = I_p^\lambda(\mu,0) = \Omega_p^\lambda$, where Ω_p^λ is the fractional differintegral operator studied recently by Patel and Mishra [6] (see also [7]);
(2) $I_p^{-\alpha}(0,\beta-1) = Q_{\beta,p}^\alpha(\beta > -p; \alpha+\beta > -p)$, where $Q_{\beta,p}^\alpha$ is the Liu-Owa operator (see [8,9]);
(3) $I_1^{-\alpha}(0,\beta-1) = Q_\beta^\alpha(\beta > -1; \alpha+\beta > -1)$, where Q_β^α is the Jung-Kim-Srivastava operator (see [10]);
(4) $I_1^{-1}(0,\beta-1) = J_\beta(\beta > -1)$, where J_β is the Bernardi integral operator (see [11]).

By making use of the operator $I_p^\lambda(\mu,\eta)$ and the above-mentioned subordination principle between analytic functions, we now introduce the following subclasses of the class A_p of p-valently analytic functions.

Definition 1. A function $f \in A_p$ is said to be in the class $M_p^\lambda(\mu,\eta;\gamma;\phi)$ if it satisfies the subordination condition:

$$1 + \frac{1}{\gamma}\left(\frac{z\left(I_p^\lambda(\mu,\eta)f(z)\right)'}{pI_p^\lambda(\mu,\eta)f(z)} - 1\right) \prec \phi(z)(z \in U; \phi \in P), \qquad (1.4)$$

where (and throughout this paper unless otherwise mentioned) the parameters γ, p, λ, μ and η are constrained as follows:

$\gamma \in C^* = C \setminus \{0\}; \quad p \in N; \quad \mu,\eta \in R; \quad \mu < p+1 \quad \text{and}\ -\infty < \lambda < \eta + p + 1.$

Definition 2. A function $f \in A_p$ is said to be in the class $N_p^\lambda(\mu,\eta;\alpha,\beta;\phi)$ if it satisfies the subordination condition:

$$(1+\alpha)\left(\frac{z^p}{I_p^\lambda(\mu,\eta)f(z)}\right)^\beta - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^\lambda(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^\lambda(\mu,\eta)f(z)}\right)^\beta \prec \phi(z)(z \in U; \phi \in P), \qquad (1.5)$$

where $\alpha \in C$, $0 < \beta < 1$ and all powers are understood as principle values.

For the sake of convenience, we set

$$M_p^\lambda\left(\mu,\eta;\gamma;\frac{1+Az}{1+Bz}\right) = M_p^\lambda(\mu,\eta;\gamma;A;B)(-1 \leqslant B < A \leqslant 1),$$

and

$$N_p^\lambda\left(\mu,\eta;\alpha,\beta;\frac{1+Az}{1+Bz}\right) = N_p^\lambda(\mu,\eta;\alpha,\beta;A;B)(-1 \leqslant B < A \leqslant 1).$$

The main objective of this paper is to derive such results as inclusion relationships, integral-preserving properties, convolution properties, subordination and superordination properties, and sandwich theorems for the classes $M_p^\lambda(\mu,\eta;\gamma;\phi)$ and $N_p^\lambda(\mu,\eta;\alpha,\beta;\phi)$. For some recent inclusion and subordination results in analytic function theory, one can find in [12–24,29,30] and the references cited therein.

2. Preliminary results

In order to establish our main results, we shall require the following known definition and lemmas.

Definition 3. ([25]) Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \{\varepsilon \in \partial U : \lim_{z\to\varepsilon} f(z) = \infty\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U \setminus E(f)$.

Lemma 1 [26]. *Let κ, $\tau \in C$. Suppose that ϕ is convex and univalent in U with*

$$\phi(0) = 1 \quad \text{and}\ \Re(\kappa\phi + \tau) > 0 \quad (z \in U).$$

If the function ρ is analytic in U with $\rho(0) = 1$, then the subordination

$$\rho(z) + \frac{z\rho'(z)}{\kappa\rho(z) + \tau} \prec \phi(z) \quad (z \in U)$$

implies that

$$\rho(z) \prec \phi(z) \quad (z \in U).$$

Lemma 2. ([2]) *Let the function h be analytic and convex (univalent) in U with $h(0) = 1$. Suppose also that the function $k(z)$ given by*

$$k(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots \qquad (2.1)$$

is analytic in U. If

$$k(z)+\frac{zk'(z)}{\xi}\prec h(z)\quad (\Re(\xi)>0;\xi\neq 0;z\in U),\tag{2.2}$$

then

$$k(z)\prec q(z)=\frac{\xi}{n}z^{-\frac{\xi}{n}}\int_0^z t^{\frac{\xi}{n}-1}h(t)dt\prec h(z)\quad (z\in U),$$

and q is the best dominant of (2.2).

Lemma 3. ([26])*Let q be univalent in U, and let θ and ϕ be analytic in the domain D containing q(U) with $\phi(\omega)\neq 0$ when $\omega\in q(U)$. Setting $Q(z)=zq'(z)\phi(q(z))$, $S(z)=\theta(q(z))+Q(z)$ and suppose that*

(1) *$Q(z)$ is a starlike function in U,*
(2) $\Re\left(\frac{zS'(z)}{Q(z)}\right)>0\quad (z\in U)$.

If p is analytic in U with $p(0)=q(0)$, $p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z)),$$

then $p\prec q$, and q is the best dominant.

Lemma 4. ([27])*Let q be convex function in U and $\zeta\in C$, $\delta\in C^*$ with*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{\zeta}{\delta}\right)\right\}.$$

If p is analytic in U and

$$\zeta p(z)+\delta zp'(z)\prec\zeta q(z)+\delta zq'(z),$$

then $p\prec q$, and q is the best dominant.

Lemma 5. ([28])*Let q be univalent in U, and let ϑ and φ be analytic in the domain D containing q(U). Suppose that*

(1) $\Re\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right)>0\quad (z\in U)$.
(2) *$zq'(z)\varphi(q(z))$ is starlike function in U.*

If $p\in H[q(0),1]\cap Q$ with $p(U)\subseteq D$, $\vartheta(p(z))+zp'(z)\varphi(p(z))$ is univalent in U, and

$$\vartheta(q(z))+zq'(z)\varphi(q(z))\prec\vartheta(p(z))+zp'(z)\varphi(p(z)),$$

then $q\prec p$, and q is the best subordinant.

Lemma 6. ([25])*Let q be convex univalent in U and $\varrho\in C$. Further assume that $\Re(\varrho)>0$. If $p\in H[q(0),1]\cap Q$ and $p(z)+\varrho zp'(z)$ is univalent in U, then*

$$q(z)+\varrho zq'(z)\prec p(z)+\varrho zp'(z),$$

which implies that $q\prec p$, and q is the best subordinant.

3. Properties of the function class $M_p^\lambda(\mu,\eta;\gamma;\phi)$

We begin by proving the following inclusion relationship given by Theorem 1.

Theorem 1. *Let $\lambda<p+\eta$, $\gamma=\gamma_1+i\gamma_2\neq 0$, $\tan\sigma=\frac{\gamma_2}{\gamma_1}$ and $\phi\in P$ with $\Im(\phi)<(\Re(\phi)-1)\cot\sigma$. Then*

$$M_p^{\lambda+1}(\mu,\eta;\gamma;\phi)\subset M_p^{\lambda}(\mu,\eta;\gamma;\phi).\tag{3.1}$$

Proof. Let $f\in M_p^{\lambda+1}(\mu,\eta;\gamma;\phi)$ and suppose that

$$\psi(z)=1+\frac{1}{\gamma}\left(\frac{z\left(I_p^\lambda(\mu,\eta)f(z)\right)'}{pI_p^\lambda(\mu,\eta)f(z)}-1\right),\tag{3.2}$$

where ψ is analytic in U with $\psi(0)=1$. In view of (1.3) and (3.2), we obtain

$$(p+\eta-\lambda)\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^\lambda(\mu,\eta)f(z)}=\gamma p(\psi(z)-1)+p+\eta-\lambda.\tag{3.3}$$

Differentiating both sides of (3.3) with respect to z logarithmically and using (3.2), we get

$$1+\frac{1}{\gamma}\left(\frac{z\left(I_p^{\lambda+1}(\mu,\eta)f(z)\right)'}{pI_p^{\lambda+1}(\mu,\eta)f(z)}-1\right)$$
$$=\psi(z)+\frac{z\psi'(z)}{\gamma p(\psi(z)-1)+p+\eta-\lambda}\prec\phi(z)\quad (z\in U).\tag{3.4}$$

Since $\Re(\gamma p(\phi(z)-1)+p+\eta-\lambda)>0$ for $\Im(\phi)<(\Re(\phi)-1)\cot\sigma$, and where $\tan\sigma=\frac{\gamma_2}{\gamma_1}$, so by applying Lemma 1 to (3.4), it follows that $\psi(z)\prec\phi(z)$, that is, that $f\in M_p^\lambda(\mu,\eta;\gamma;\phi)$. Thus, the assertion (3.1) of Theorem 1 holds true. □

Taking $\phi(z)=\frac{1+Az}{1+Bz}$ in Theorem 1, we have the following corollary.

Corollary 1. *Let $\lambda<p+\eta$, $\gamma=\gamma_1+i\gamma_2\neq 0$, and $\tan\sigma=\frac{\gamma_2}{\gamma_1}$ with*

$$\Im\left(\frac{1+Az}{1+Bz}\right)<\left[\Re\left(\frac{1+Az}{1+Bz}\right)-1\right]\cot\sigma\;(-1\leqslant B<A\leqslant 1;z\in U).$$

Then

$$M_p^{\lambda+1}(\mu,\eta;\gamma;A,B)\subset M_p^\lambda(\mu,\eta;\gamma;A,B).$$

Next, we discuss some integral-preserving properties for the function class $M_p^\lambda(\mu,\eta;\gamma;\phi)$.

Theorem 2. *Let $\gamma=\gamma_1+i\gamma_2\neq 0$, $\tan\sigma=\frac{\gamma_2}{\gamma_1}$ and $\phi\in P$ with $\Im(\phi)<(\Re(\phi)-1)\cot\sigma$. If $f\in M_p^\lambda(\mu,\eta;\gamma;\phi)$, then $F\in M_p^\lambda(\mu,\eta;\gamma;\phi)$, where the integral operator F defined by*

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt\quad (z\in U;c>-p).\tag{3.5}$$

Proof. Let $f\in M_p^\lambda(\mu,\eta;\gamma;\phi)$. Then from (3.5), we find that

$$z\left(I_p^\lambda(\mu,\eta)F(z)\right)'=(c+p)I_p^\lambda(\mu,\eta)f(z)-cI_p^\lambda(\mu,\eta)F(z).\tag{3.6}$$

Upon setting

$$q(z)=1+\frac{1}{\gamma}\left(\frac{z\left(I_p^\lambda(\mu,\eta)F(z)\right)'}{pI_p^\lambda(\mu,\eta)F(z)}-1\right),\tag{3.7}$$

where q is analytic in U with $q(0) = 1$. Combining (3.6) and (3.7), we obtain

$$(c+p)\frac{I_p^\lambda(\mu,\eta)f(z)}{I_p^\lambda(\mu,\eta)F(z)} = \gamma p(q(z)-1)+p+c. \tag{3.8}$$

Taking the logarithmical differentiation on both sides of (3.8) and using (3.7), we get

$$q(z)+\frac{zq'(z)}{\gamma p(q(z)-1)+p+c} = 1+\frac{1}{\gamma}\left(\frac{z\left(I_p^\lambda(\mu,\eta)f(z)\right)'}{pI_p^\lambda(\mu,\eta)f(z)}-1\right) \prec \phi(z) \quad (z\in U). \tag{3.9}$$

Hence, by virtue of Lemma 1, we conclude that $q \prec \phi$, which implies that $F \in M_p^\lambda(\mu,\eta;\gamma;\phi)$. □

Theorem 3. *Let $f \in M_p^\lambda(\mu,\eta;\gamma;\phi)$ with $\phi \in P$ and*

$$\Re(\gamma pm(\phi(z)-1)+pm+c) > 0 \quad (z\in U; c\in C; m\in C^*).$$

Then the function $K \in A_p$ defined by

$$I_p^\lambda(\mu,\eta)K(z) = \left(\frac{c+pm}{z^c}\int_0^z t^{c-1}\left(I_p^\lambda(\mu,\eta)f(z)\right)^m dt\right)^{\frac{1}{m}} \quad (z\in U). \tag{3.10}$$

belongs to the class $M_p^\lambda(\mu,\eta;\gamma;\phi)$.

Proof. Let $f \in M_p^\lambda(\mu,\eta;\gamma;\phi)$. Then from (3.10), it follows that

$$mz\left(I_p^\lambda(\mu,\eta)K(z)\right)' = (p+c)I_p^\lambda(\mu,\eta)f(z)\left(\frac{I_p^\lambda(\mu,\eta)f(z)}{I_p^\lambda(\mu,\eta)K(z)}\right)^m - cI_p^\lambda(\mu,\eta)K(z). \tag{3.11}$$

Suppose that

$$\varpi(z) = 1+\frac{1}{\gamma}\left(\frac{z\left(I_p^\lambda(\mu,\eta)K(z)\right)'}{pI_p^\lambda(\mu,\eta)K(z)}-1\right) \quad (z\in U). \tag{3.12}$$

Then, by using (3.11) and (3.12), we have

$$\gamma pm(\varpi(z)-1)+pm+c = (c+pm)\left(\frac{I_p^\lambda(\mu,\eta)f(z)}{I_p^\lambda(\mu,\eta)K(z)}\right)^m. \tag{3.13}$$

In view of (3.12) and (3.13), we easily get

$$\varpi(z)+\frac{z\varpi'(z)}{\gamma pm(\varpi(z)-1)+pm+c} = 1+\frac{1}{\gamma}\left(\frac{z\left(I_p^\lambda(\mu,\eta)f(z)\right)'}{pI_p^\lambda(\mu,\eta)f(z)}-1\right) \prec \phi(z) \quad (z\in U). \tag{3.14}$$

Since

$$\Re(\gamma pm(\phi(z)-1)+pm+c) > 0 \quad (z\in U),$$

so an application of Lemma 1 to (3.14) yields $\varpi \prec \phi$, that is, that $K \in M_p^\lambda(\mu,\eta;\gamma;\phi)$. This completes the proof of Theorem 3. □

Now, we derive certain convolution properties for the function class $M_p^\lambda(\mu,\eta;\gamma;\phi)$.

Theorem 4. *Let $f \in M_p^\lambda(\mu,\eta;\gamma;\phi)$. Then*

$$f(z) = \left[z^p\cdot\exp\left(\gamma p\int_0^z\frac{\phi(\omega(\varsigma))-1}{\varsigma}d\varsigma\right)\right] * \left(z^p+\sum_{n=1}^\infty\frac{(1+p-\mu)_n(1+p+\eta-\lambda)_n}{(1+p)_n(1+p+\eta-\mu)_n}z^{n+p}\right), \tag{3.15}$$

where ω is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z\in U)$.

Proof. Let $f \in M_p^\lambda(\mu,\eta;\gamma;\phi)$. We know that subordination condition (1.4) can be written as follows:

$$\frac{z\left(I_p^\lambda(\mu,\eta)f(z)\right)'}{I_p^\lambda(\mu,\eta)f(z)} = \gamma p[\phi(\omega(z))-1]+p, \tag{3.16}$$

where ω is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z\in U)$. By virtue of (3.16), we easily find that

$$\frac{\left(I_p^\lambda(\mu,\eta)f(z)\right)'}{I_p^\lambda(\mu,\eta)f(z)}-\frac{p}{z} = \frac{\gamma p[\phi(\omega(z))-1]}{z}, \tag{3.17}$$

which, upon integration, yields

$$\log\left(\frac{I_p^\lambda(\mu,\eta)f(z)}{z^p}\right) = \gamma p\int_0^z\frac{\phi(\omega(\varsigma))-1}{\varsigma}d\varsigma,$$

implies that

$$I_p^\lambda(\mu,\eta)f(z) = z^p\cdot\exp\left(\gamma p\int_0^z\frac{\phi(\omega(\varsigma))-1}{\varsigma}d\varsigma\right). \tag{3.18}$$

Then, from (1.2) and (3.18), we readily deduce that the assertion (3.15) of Theorem 4 holds true.

Taking $\phi(z) = \frac{1+Az}{1+Bz}$ in Theorem 4, we obtain the following corollary. □

Corollary 2. *Let $f \in M_p^\lambda(\mu,\eta;\gamma;A,B)$ with $-1 \leqslant B < A \leqslant 1$. Then*

$$f(z) = \left[z^p\cdot\exp\left(\gamma p\int_0^z\frac{(A-B)(\omega(\varsigma))}{\varsigma(1+B(\omega(\varsigma)))}d\varsigma\right)\right] * \left(z^p+\sum_{n=1}^\infty\frac{(1+p-\mu)_n(1+p+\eta-\lambda)_n}{(1+p)_n(1+p+\eta-\mu)_n}z^{n+p}\right),$$

where ω is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z\in U)$.

Theorem 5. *Let $f \in A_p$ and $\phi \in P$. Then $f \in M_p^\lambda(\mu,\eta;\gamma;\phi)$ if and only if*

$$\frac{1}{z^p}\left\{f*\left\{pz^p+\sum_{n=1}^\infty\frac{(n+p)(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n}z^{n+p}-p[\gamma(\phi(e^{i\theta})-1)+1]\times\left(z^p+\sum_{n=1}^\infty\frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n}z^{n+p}\right)\right\}\right\}\neq 0 \quad (z\in U; 0<\theta<2\pi). \tag{3.19}$$

Proof. Suppose that $f \in M_p^\lambda(\mu,\eta;\gamma;\phi)$. We know that (1.4) holds true, which implies that

$$1+\frac{1}{\gamma}\left(\frac{z\left(I_p^{\lambda}(\mu,\eta)f(z)\right)'}{pI_p^{\lambda}(\mu,\eta)f(z)}-1\right)\neq\phi(e^{i\theta})\quad(z\in U;0<\theta<2\pi).\tag{3.20}$$

It is easy to see that the condition (3.20) is equivalent to

$$\frac{1}{z^p}\left\{z\left(I_p^{\lambda}(\mu,\eta)f(z)\right)'-p[\gamma(\phi(e^{i\theta})-1)+1]I_p^{\lambda}(\mu,\eta)f(z)\right\}\neq 0\quad(z\in U;0<\theta<2\pi).\tag{3.21}$$

On the other hand, we find from (1.2) that

$$z\left(I_p^{\lambda}(\mu,\eta)f(z)\right)'=pz^p+\sum_{n=1}^{\infty}\frac{(n+p)(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n}a_{n+p}z^{n+p}.\tag{3.22}$$

Combining (1.2), (3.21), together with (3.22), we easily get the convolution property (3.19) asserted by Theorem 5. □

4. Properties of the function class $N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)$

In this section, we first present the following subordination property given by Theorem 6.

Theorem 6. *Let $f\in N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)$ with $0<\beta<1$ and $\Re\left(\frac{\beta(p+\eta-\lambda)}{\alpha}\right)>0$. Then*

$$\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\prec\frac{\beta(p+\eta-\lambda)}{n\alpha}z^{-\frac{\beta(p+\eta-\lambda)}{n\alpha}}\times\int_0^z t^{\frac{\beta(p+\eta-\lambda)}{n\alpha}-1}\phi(t)dt\prec\phi(z)\quad(z\in U).\tag{4.1}$$

Proof. Let $f\in N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)$ and suppose that

$$h(z)=\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\quad(z\in U).\tag{4.2}$$

Then $h(z)$ is of the form (2.1) and analytic in U. Differentiating (4.2) with respect to z logarithmically and using (1.3), we get

$$h(z)+\frac{\alpha zh'(z)}{\beta(p+\eta-\lambda)}=(1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}-\alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\prec\phi(z).\tag{4.3}$$

Thus, by applying Lemma 2 to (4.3) with $\xi=\frac{\beta(p+\eta-\lambda)}{\alpha}$, we immediately derive the assertion (4.1) of Theorem 6. □

Taking $\phi(z)=\frac{1+Az}{1+Bz}$ in Theorem 6, we have the following corollary.

Corollary 3. *Let $f\in N_p^{\lambda}(\mu,\eta;\alpha,\beta;A,B)$ with $-1\leqslant B<A\leqslant 1$, $0<\beta<1$ and $\Re\left(\frac{\beta(p+\eta-\lambda)}{\alpha}\right)>0$. Then*

$$\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\prec\frac{\beta(p+\eta-\lambda)}{n\alpha}z^{-\frac{\beta(p+\eta-\lambda)}{n\alpha}}\int_0^z\frac{1+At}{1+Bt}t^{\frac{\beta(p+\eta-\lambda)}{n\alpha}-1}dt\prec\frac{1+Az}{1+Bz}\quad(z\in U).$$

From Theorem 6, we easily get the following inclusion relationship.

Corollary 4. *Let $0<\beta<1$ and $\Re\left(\frac{\beta(p+\eta-\lambda)}{\alpha}\right)>0$. Then*

$$N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)\subset N_p^{\lambda}(\mu,\eta;0,\beta;\phi).$$

Now, we give another inclusion relationship for the function class $N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)$.

Theorem 7. *Let $0<\beta<1$ and $\alpha_2>\alpha_1\geqslant 0$. Then*

$$N_p^{\lambda}(\mu,\eta;\alpha_2,\beta;\phi)\subset N_p^{\lambda}(\mu,\eta;\alpha_1,\beta;\phi).\tag{4.4}$$

Proof. Suppose that $f\in N_p^{\lambda}(\mu,\eta;\alpha_2,\beta;\phi)$. We know from (1.5) that

$$(1+\alpha_2)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}-\alpha_2\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\prec\phi(z)\quad(z\in U).\tag{4.5}$$

Since $0\leqslant\frac{\alpha_1}{\alpha_2}<1$, and the function ϕ is convex and univalent in U, we deduce from (4.1) and (4.5) that

$$\begin{aligned}&(1+\alpha_1)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}-\alpha_1\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\\&=\left(1-\frac{\alpha_1}{\alpha_2}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\\&\quad+\frac{\alpha_1}{\alpha_2}\left[(1+\alpha_2)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}-\alpha_2\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}\right]\\&\prec\phi(z)\quad(z\in U),\end{aligned}$$

that is $f\in N_p^{\lambda}(\mu,\eta;\alpha_1,\beta;\phi)$, which implies that the assertion (4.4) of Theorem 7 holds true. □

In view of (4.1), and by applying the similar methods of proof of Theorems 4 and 5, respectively, we get the following convolution results.

Theorem 8. *Let $f\in N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)$. Then*

$$f(z)=\left[z^p(\phi(\omega(z)))^{\beta}\right]*\left(z^p+\sum_{n=1}^{\infty}\frac{(1+p-\mu)_n(1+p+\eta-\lambda)_n}{(1+p)_n(1+p+\eta-\mu)_n}z^{n+p}\right),\tag{4.6}$$

where ω is analytic in U with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in U)$.

Theorem 9. *Let $f\in N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)$. Then*

$$\frac{1}{z^p}\left[\left(z^p+\sum_{n=1}^{\infty}\frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n}z^{n+p}\right)*f(z)-z^p\left(\phi(e^{i\theta})\right)^{\beta}\right]\neq 0\quad\times\quad(z\in U;0<\theta<2\pi).\tag{4.7}$$

Theorem 10. *Let q be univalent in U, $\alpha \in C^*$ and $0 < \beta < 1$. Suppose also that q satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\beta(p+\eta-\lambda)}{\alpha}\right)\right\}. \quad (4.8)$$

If $f \in A_p$ satisfies the subordination condition

$$(1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \prec q(z) + \frac{\alpha z q'(z)}{\beta(p+\eta-\lambda)}, \quad (4.9)$$

then

$$\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \prec q(z),$$

and q is the best dominant.

Proof. Let the function $h(z)$ be defined by (4.2). We know that (4.3) holds true. Combining (4.3) and (4.9), we find that

$$h(z) + \frac{\alpha z h'(z)}{\beta(p+\eta-\lambda)} \prec q(z) + \frac{\alpha z q'(z)}{\beta(p+\eta-\lambda)}. \quad (4.10)$$

By using Lemma 4 and (4.10), we easily obtain the assertion of Theorem 10. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 10, we get the following result.

Corollary 5. *Let $\alpha \in C^*$, $0 < \beta < 1$ and $-1 \leqslant B < A \leqslant 1$. Suppose also that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\Re\left(\frac{\beta(p+\eta-\lambda)}{\alpha}\right)\right\}.$$

If $f \in A_p$ satisfies the subordination condition

$$(1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz} + \frac{\alpha(A-B)z}{\beta(p+\eta-\lambda)(1+Bz)^2},$$

then

$$\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 11. *Let $\alpha_j \in C$ $(j = 0, 1, \ldots, n)$, $b \in C^*$, $u, v, k \in C$ such that k, $u + v \neq 0$, and $q_1 \neq 0$ be convex univalent in U. Further assume that*

$$\Re\left(1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)}\right) > 0 \quad \text{and } \Re\left(\bar{b}\sum_{j=1}^{n} j\alpha_j q_1^j(z)\right) > 0 \quad (z \in U). \quad (4.11)$$

If $f \in A_p$ satisfies

$$\Phi(z) \prec \sum_{j=0}^{n} \alpha_j q_1^j(z) + b\frac{zq_1'(z)}{q_1(z)}, \quad (4.12)$$

where

$$\Phi(z) = \sum_{j=0}^{n} \alpha_j \left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{kj} + bk\left(p - \frac{uz\left(I_p^{\lambda+1}(\mu,\eta)f(z)\right)' + vz\left(I_p^{\lambda}(\mu,\eta)f(z)\right)'}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right), \quad (4.13)$$

then

$$\left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{k} \prec q_1(z),$$

and q_1 is the best dominant.

Proof. Define the function $p_1(z)$ by

$$p_1(z) = \left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{k} \quad (k, u + v \neq 0). \quad (4.14)$$

Differentiating both sides of (4.14) logarithmically and multiplying by z, we get

$$\frac{zp_1'(z)}{p_1(z)} = k\left(p - \frac{uz\left(I_p^{\lambda+1}(\mu,\eta)f(z)\right)' + vz\left(I_p^{\lambda}(\mu,\eta)f(z)\right)'}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right). \quad (4.15)$$

Therefore, by making use of (4.12)–(4.15), we obtain

$$\sum_{j=0}^{n} \alpha_j p_1^j(z) + b\frac{zp_1'(z)}{p_1(z)} \prec \sum_{j=0}^{n} \alpha_j q_1^j(z) + b\frac{zq_1'(z)}{q_1(z)}. \quad (4.16)$$

By setting

$$\theta(w) = \sum_{j=0}^{n} \alpha_j w^j \quad \text{and } \phi(w) = \frac{b}{w},$$

we observe that $\theta(w)$ is analytic in C and that $\phi(w) \neq 0$ is analytic in C^*.

Also, we let

$$Q(z) = zq_1'(z)\phi(q_1(z)) = b\frac{zq_1'(z)}{q_1(z)},$$

and

$$S(z) = \theta(q_1(z)) + Q(z) = \sum_{j=0}^{n} \alpha_j q_1^j(z) + b\frac{zq_1'(z)}{q_1(z)}.$$

From (4.11), we see that $Q(z)$ is starlike univalent in U and

$$\Re\left(\frac{zS'(z)}{Q(z)}\right) = \Re\left\{1 + \frac{zq_1''(z)}{q_1'(z)} - \frac{zq_1'(z)}{q_1(z)} + \bar{b}\sum_{j=1}^{n} j\alpha_j q_1^j(z)\right\} > 0.$$

Thus, an application of Lemma 3 to (4.12) yields our desired result.

In the following, we provide some superordination results for the class $N_p^{\lambda}(\mu,\eta;\alpha,\beta;\phi)$. □

Theorem 12. *Let q_2 be convex univalent in U, $0 < \beta < 1$ and $\alpha \in C$ with $\Re(\alpha) > 0$. Also let*

$$\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \in H[q_2(0),1] \cap Q$$

and

$$(1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}$$

be univalent in U. If

$$q_2(z) + \frac{\alpha z q_2'(z)}{\beta(p+\eta-\lambda)} \prec (1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}, \tag{4.17}$$

then

$$q_2(z) \prec \left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta},$$

and q_2 is the best subordinant.

Proof. Let the function $h(z)$ be defined by (4.2). Then, from (4.3), we have

$$q_2(z) + \frac{\alpha z q_2'(z)}{\beta(p+\eta-\lambda)} \prec (1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} = h(z) + \frac{\alpha z h'(z)}{\beta(p+\eta-\lambda)}. \tag{4.18}$$

Therefore, by means of (4.18) and Lemma 6, we readily get the assertion of Theorem 12. □

Taking $q_2(z) = \frac{1+Az}{1+Bz}$ in Theorem 12, we obtain the following corollary.

Corollary 6. *Let q_2 be convex univalent in U, $-1 \leqslant B < A \leqslant 1$, $0 < \beta < 1$ and $\alpha \in C$ with $\Re(\alpha) > 0$. Also let*

$$0 \neq \left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \in H[q_2(0),1] \cap Q$$

and

$$(1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}$$

be univalent in U. If

$$\frac{1+Az}{1+Bz} + \frac{\alpha(A-B)z}{\beta(p+\eta-\lambda)(1+Bz)^2} \prec (1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta},$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta},$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Theorem 13. *Let $\alpha_j \in C$ $(j = 0, 1, \ldots, n)$, $b \in C^*$, u, v, $k \in C$ such that k, $u + v \neq 0$, and $q_3 \neq 0$ be convex univalent in U. Further assume that*

$$\Re\left(\bar{b}\sum_{j=1}^{n} j\alpha_j q_3^j(z)\right) > 0, \tag{4.19}$$

and $\frac{zq_3''(z)}{q_3'(z)}$ is starlike univalent in U. If

$$0 \neq \left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{k} \in H[q_3(0),1] \cap Q.$$

Let $\Phi(z)$ given by (4.13) be univalent in U and

$$\sum_{j=0}^{n} \alpha_j q_3^j(z) + b\frac{zq_3'(z)}{q_3(z)} \prec \Phi(z), \tag{4.20}$$

then

$$q_3(z) \prec \left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{k},$$

and q_3 is the best subordinant.

Proof. Define the function $p_2(z)$ by

$$p_2(z) = \left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{k} \quad (k, u+v \neq 0). \tag{4.21}$$

Then a simple computation shows that

$$\Phi(z) = \sum_{j=0}^{n} \alpha_j p_2^j(z) + b\frac{zp_2'(z)}{p_2(z)},$$

where $\Phi(z)$ is given by (4.13), then from (4.20), we have

$$\sum_{j=0}^{n} \alpha_j q_3^j(z) + b\frac{zq_3'(z)}{q_3(z)} \prec \sum_{j=0}^{n} \alpha_j p_2^j(z) + b\frac{zp_2'(z)}{p_2(z)}.$$

By setting

$$\vartheta(w) = \sum_{j=0}^{n} \alpha_j w^j \quad \text{and} \quad \varphi(w) = \frac{b}{w}, \tag{4.22}$$

it is easily observe that $\vartheta(w)$ is analytic in C and that $\varphi(w) \neq 0$ is analytic in C^*. Since q_3 is convex univalent in U, if follows that

$$\Re\left(\frac{\vartheta'(q_3(z))}{\varphi(q_3(z))}\right) = \Re\left(\bar{b}\sum_{j=1}^{n} j\alpha_j q_3^j(z)\right) > 0, \tag{4.23}$$

by the hypothesis (4.19) of Theorem 13. Thus, by applying Lemma 5, Our proof of Theorem 13 is completed.

Finally, combining the above mentioned subordination and superordination results, we easily obtain the following sandwich results.

Theorem 14. *Let q_4, q_5 be convex univalent in U, $0 < \beta < 1$ and $\alpha \in C^*$ with $\Re(\alpha) > 0$. Suppose also that q_5 satisfies (4.8) and*

$$0 \neq \left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \in H[q_4(0),1] \cap Q.$$

Let

$$(1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta}$$

be univalent in U. If

$$q_4(z) + \frac{\alpha z q_4'(z)}{\beta(p+\eta-\lambda)} \prec (1+\alpha)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} - \alpha\left(\frac{I_p^{\lambda+1}(\mu,\eta)f(z)}{I_p^{\lambda}(\mu,\eta)f(z)}\right)\left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \prec q_5(z) + \frac{\alpha z q_5'(z)}{\beta(p+\eta-\lambda)},$$

then

$$q_4(z) \prec \left(\frac{z^p}{I_p^{\lambda}(\mu,\eta)f(z)}\right)^{\beta} \prec q_5(z),$$

and q_4, q_5 are, respectively, the best subordinant and the best dominant.

Theorem 15. *Let q_6, q_7 be convex univalent in U, $\alpha_j \in C$ ($j = 0, 1, \ldots, n$), $b \in C^*$, u, v, $k \in C$ such that k, $u + v \neq 0$, and let q_6 satisfies (4.19) and q_7 satisfies (4.11). Further assume that*

$$0 \neq \left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{k} \in H[q_6(0),1] \cap Q,$$

and $\Phi(z)$ defined by (4.13) be univalent in U satisfying

$$\sum_{j=0}^{n}\alpha_j q_6^j(z) + b\frac{zq_6'(z)}{q_6(z)} \prec \Phi(z) \prec \sum_{j=0}^{n}\alpha_j q_7^j(z) + b\frac{zq_7'(z)}{q_7(z)},$$

then

$$q_6(z) \prec \left(\frac{(u+v)z^p}{uI_p^{\lambda+1}(\mu,\eta)f(z) + vI_p^{\lambda}(\mu,\eta)f(z)}\right)^{k} \prec q_7(z),$$

and q_6, q_7 are, respectively, the best subordinant and the best dominant.

We observe that, one can easily restate Theorem 15 for the different choices of λ, μ, η, u, v, k, b and α_j ($j = 0, 1, \ldots, n$). For example, if we take $\eta = 0$, $\alpha_0 = 1$ and $\alpha_j = 0$ ($j = 1, \ldots, n$) in Theorem 15, we get the following result.

Corollary 7. *Let q_6, q_7 be convex univalent in U, $b \in C^*$, u, v, $k \in C$ such that k, $u + v \neq 0$, and let $\frac{zq_i''(z)}{q_i'(z)}$ $(i = 6, 7)$ is starlike univalent in U. Further assume that*

$$0 \neq \left(\frac{(u+v)z^p}{u\Omega_p^{\lambda+1}f(z) + v\Omega_p^{\lambda}f(z)}\right)^{k} \in H[q_6(0),1] \cap Q,$$

and

$$\chi(z) = 1 + bk\left(p - \frac{uz\left(\Omega_p^{\lambda+1}f(z)\right)' + vz\left(\Omega_p^{\lambda}f(z)\right)'}{u\Omega_p^{\lambda+1}f(z) + v\Omega_p^{\lambda}f(z)}\right)$$

be univalent in U satisfying

$$1 + b\frac{zq_6'(z)}{q_6(z)} \prec \chi(z) \prec 1 + b\frac{zq_7'(z)}{q_7(z)},$$

then

$$q_6(z) \prec \left(\frac{(u+v)z^p}{u\Omega_p^{\lambda+1}f(z) + v\Omega_p^{\lambda}f(z)}\right)^{k} \prec q_7(z),$$

and q_6, q_7 are, respectively, the best subordinant and the best dominant.

5. Conclusion

We conclude this paper by remarking that in terms of the generalized operator (1.2) and in view of the function classes defined by (1.4) and (1.5) involving arbitrary coefficients, the main results will lead to additional new results. In fact, by appropriately selecting the functions $\phi(z)$ and $q(z)$, and specializing the parameters p, α, β, γ, λ, μ, η and α_j ($j = 0, 1, \ldots, n$), the results presented in this paper would find further applications for the classes which incorporate generalized forms of linear operators. These considerations can fruitfully be worked out and we skip the details in this regard.

Acknowledgements

The present investigation was partly supported by the Natural Science Foundation of China under Grant 11271045, the Higher School Doctoral Foundation of China under Grant 20100003110004 and the Natural Science Foundation of Inner Mongolia of China under Grant 2010MS0117. The authors are thankful to the referees for their careful reading and making some helpful comments which have essentially improved the presentation of this paper.

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